## حل تشابهی Similarity Solution

$u(y,t)$

متغیر تشابهی $\eta(y,t)$ : $u(\eta(y,t))$

اگر بتوان برای یک معادله دیفرانسیل جزیی متغیر واسطه به نام متغیر تشابهی پیدا کرد به طوری که معادله دیفرانسیل برحسب آن متغیر معمولی شود می‌توان معادله را با روش‌های حل معادلات دیفرانسیل معمولی (نه جزیی) حل کرد.

$$u = yt + a(yt)^2 + \frac{1}{(yt)^3} + \cos b(yt)$$

$$\eta = yt \rightarrow u = \eta + a\eta^2 + \frac{1}{\eta^3} + \cos b\eta$$

مثال:

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

حل:

$$\eta = \frac{y}{\sqrt{t}} \qquad \eta = y t^{-1/2} \rightarrow \frac{\partial \eta}{\partial t} = -\frac{1}{2} y t^{-3/2}, \quad \frac{\partial \eta}{\partial y} = t^{-1/2}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} = \frac{\partial u}{\partial \eta} \cdot -\frac{1}{2} \underbrace{y t^{-3/2}}_{y t^{-\frac{1}{2}} \cdot t^{-1}} = -\frac{1}{2} t^{-1} \eta \frac{\partial u}{\partial \eta}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \eta} \cdot t^{-1/2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}\right) \cdot \frac{\partial \eta}{\partial y}$$

$$= \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial \eta} t^{-1/2}\right) t^{-1/2} = \frac{\partial^2 u}{\partial \eta^2} \cdot t^{-1}$$

جایگذاری در معادله اول : $-\frac{1}{2} t^{-1} \eta \frac{\partial u}{\partial \eta} = \nu t^{-1} \frac{\partial^2 u}{\partial \eta^2} \Rightarrow$

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{\eta}{2\nu} \frac{\partial u}{\partial \eta} = 0 \Rightarrow u'' + \frac{\nu}{2} \eta u' = 0$$ ← مشتق نسبت η

$$\longrightarrow u'' = -\frac{\eta}{2\nu} u' \longrightarrow \frac{u''}{u'} = -\frac{\eta}{2\nu} \xrightarrow{\int d\eta} \ln u' = -\frac{\eta^2}{4\nu} + c_1$$

$$\longrightarrow u' = e^{-\eta^2/4\nu + c_1} \longrightarrow u' = e^{-\eta^2/4\nu} \cdot e^{c_1} \longrightarrow$$

($e^{c_1} = c_2$)

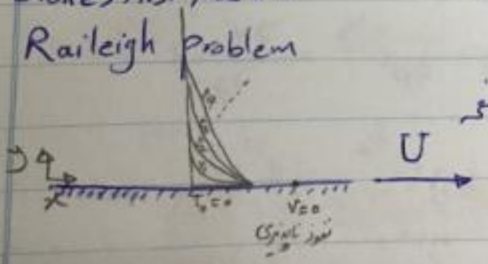
$$u' = c_2 e^{-\eta^2/4\nu} \xrightarrow{\int d\eta} u = c_2 \int e^{-\eta^2/4\nu} d\eta + c_3 \xrightarrow[\text{error function}]{\text{تابع خطا}}$$

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

مثال: مسئله اول استوکس (مسئله ریلی) :

Stoke's first Problem

Raileigh problem




جریان روی یک صفحه بی نهایت در اثر آغاز به حرکت صفحه

فرض‌ها: 1- جریان تراکم ناپذیر

2- جریان دو بعدی (x,y)

3- عدم تغییر سرعت در جهت حرکت x

$\frac{\partial u}{\partial x} = 0$

# Flow near a plate suddenly set in motion

*Rayleigh's problem or Stokes' first problem.*

Consider a semi-infinite incompressible Newtonian liquid of viscosity $\eta$ and density $\rho$, bounded below by a plate at $y=0$ (Fig. 6.20). Initially, both the plate and the liquid are at rest.

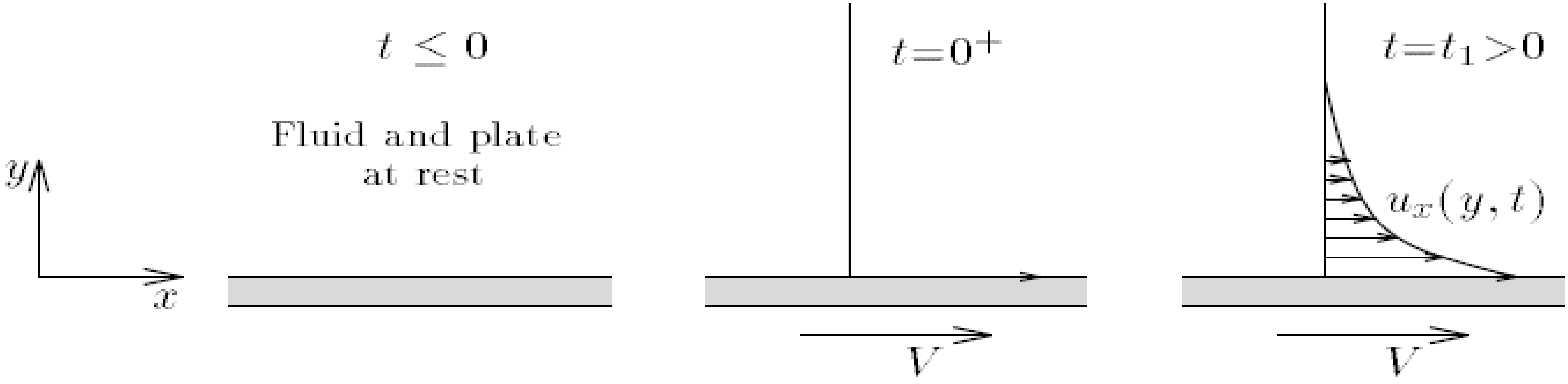



**Figure 6.20.** *Flow near a plate suddenly set in motion.*

The governing equation for $u_x(y,t)$ is homogeneous:

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2} , \tag{6.112}$$

where $\nu \equiv \eta/\rho$ is the kinematic viscosity. Mathematically, Eq. (6.112) is called the *heat* or *diffusion equation*. The boundary and initial conditions are:

$$\left. \begin{array}{lll} u_x = V & \text{at} & y = 0,\ t > 0 \\ u_x = 0 & \text{at} & y \to \infty,\ t \geq 0 \\ u_x = 0 & \text{at} & t = 0,\ 0 \leq y < \infty \end{array} \right\} . \tag{6.113}$$

# Similarity solution:

Let us, however, assume that the existence of a similarity solution and the proper combination of $y$ and $t$ are not known a priori, and assume that the solution is of the form

$$u_x(y,t) = V\, f(\xi)\,, \tag{6.114}$$

where

$$\xi = a\,\frac{y}{t^n}\,, \qquad \text{with} \qquad n > 0\,. \tag{6.115}$$

Here $\xi(y,t)$ is the similarity variable, $a$ is a constant to be determined later so that $\xi$ is dimensionless, and $n$ is a positive number to be chosen so that the original partial differential equation (6.112) can be transformed into an ordinary differential equation with $f$ as the dependent variable and $\xi$ as the independent one.

The boundary condition at $y=0$ is equivalent to

$$f = 1 \quad \text{at} \quad \xi = 0\,, \tag{6.116}$$

whereas the boundary condition at $y \to \infty$ and the initial condition collapse to a single boundary condition for $f$,

$$f = 0 \quad \text{at} \quad \xi \to \infty\,. \tag{6.117}$$

$$u_x(y,t) \;=\; V\, f(\xi)\,, \qquad (6.114)$$

$$\xi \;=\; a\,\frac{y}{t^n}\,, \qquad \text{with} \qquad n > 0\,. \qquad (6.115)$$

Differentiation of Eq. (6.114) using the chain rule gives

$$\frac{\partial u_x}{\partial t} \;=\; -V\; n\,\frac{a\,y}{t^{n+1}}\; f' \;=\; -V\; n\frac{\xi}{t}\; f'\,,$$

$$\frac{\partial u_x}{\partial y} \;=\; V\;\frac{a}{t^n}\; f' \qquad \text{and} \qquad \frac{\partial^2 u_x}{\partial y^2} \;=\; V\;\frac{a^2}{t^{2n}}\; f''\,,$$

where primes denote differentiation with respect to $\xi$. Substitution of the above derivatives into Eq. (6.112) gives the following equation:

$$f'' \;+\; \frac{n\xi}{\nu a^2}\, t^{2n-1}\, f' \;=\; 0\,.$$

By setting $n=1/2$, time is eliminated and the above expression becomes a second-order ordinary differential equation,

$$f'' + \frac{\xi}{2\nu a^2} f' = 0 \quad \text{with} \quad \xi = a \frac{y}{\sqrt{t}} .$$

Taking $a$ equal to $1/\sqrt{\nu}$ makes the similarity variable dimensionless. For convenience in the solution of the differential equation, we set $a=1/(2\sqrt{\nu})$. Hence,

$$\xi = \frac{y}{2\sqrt{\nu\, t}} , \tag{6.118}$$

whereas the resulting ordinary differential equation is

$$f'' + 2\xi\, f' = 0 . \tag{6.119}$$

This equation is subject to the boundary conditions (6.116) and (6.117). By straightforward integration, we obtain

$$f(\xi) = c_1 \int_0^{\xi} e^{-z^2}\, dz + c_2 ,$$

where $z$ is a dummy variable of integration. At $\xi=0$, $f=1$; consequently, $c_2=1$. At $\xi\rightarrow\infty$, $f=0$; therefore,

$$c_1 \int_0^{\infty} e^{-z^2}\, dz + 1 = 0 \quad \text{or} \quad c_1 = -\frac{2}{\sqrt{\pi}} ,$$

and

$$f(\xi) = 1 - \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-z^2}\, dz = 1 - \text{erf}(\xi) , \qquad (6.120)$$

where erf is the *error function*, defined as

$$\mathrm{erf}(\xi) \equiv \frac{2}{\sqrt{\pi}} \int_0^{\xi} e^{-z^2}\, dz \,. \tag{6.121}$$

Values of the error function are tabulated in several math textbooks. It is a monotone increasing function with

$$erf(\mathbf{0}) = \mathbf{0} \quad \text{and} \quad \lim_{\xi \to \infty} \mathrm{erf}(\xi) = 1 \,.$$

Note that the second expression was used when calculating the constant $c_1$. Substituting into Eq. (6.114), we obtain the solution

$$u_x(y,t) = V \left[ 1 - \mathrm{erf}\left( \frac{y}{2\sqrt{\nu\, t}} \right) \right] . \tag{6.122}$$

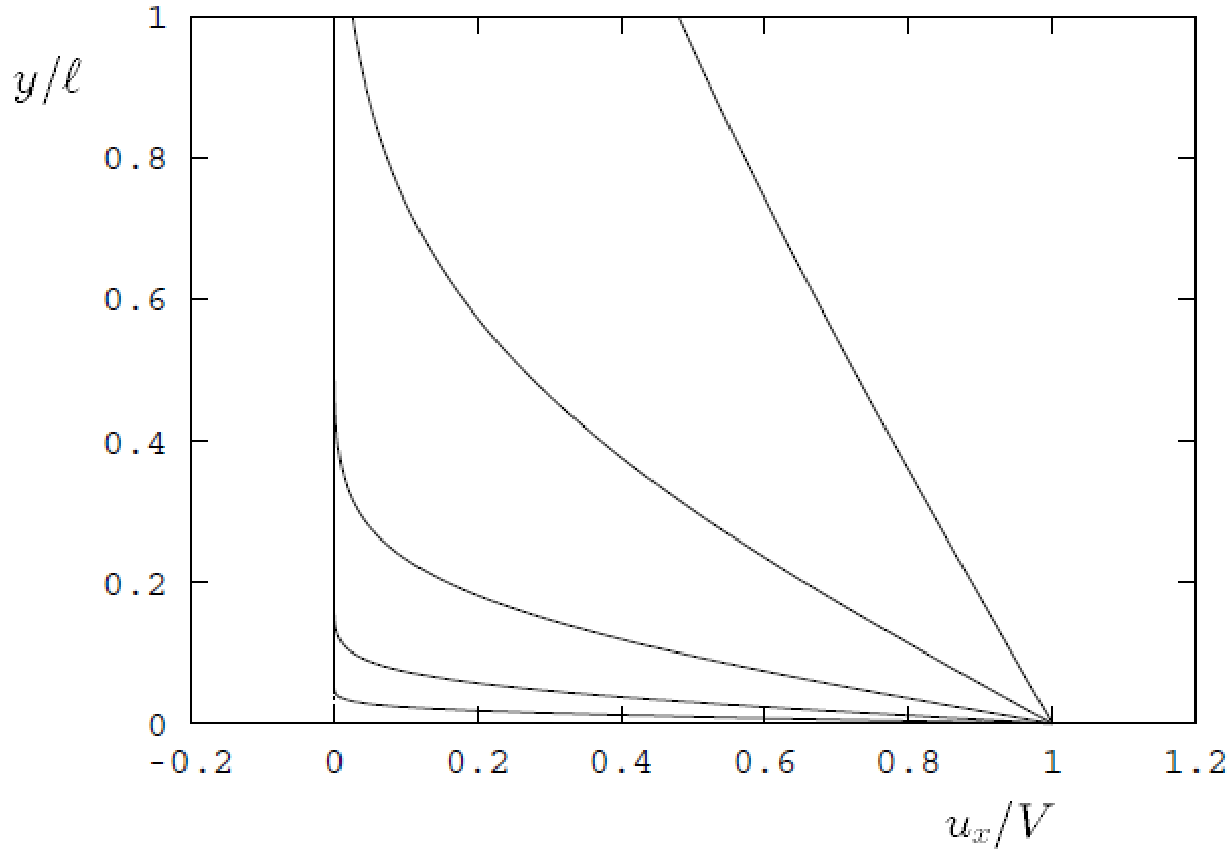



**Figure 6.21.** *Transient flow due to the sudden motion of a plate. Velocity profiles at* $\nu t/\ell^2$*=0.0001, 0.001, 0.01, 0.1 and 1, where* $\ell$ *is an arbitrary length scale.*

A *boundary-layer thickness*, $\delta(t)$, can be defined as the distance from the moving plate at which $u_x/V = 0.01$. This happens when $\xi$ is about 1.8, and thus

$$\delta(t) \; = \; 3.6 \sqrt{\nu t} \; .$$