Transient plane Couette flow

Consider a Newtonian liquid of density ρ and viscosity η bounded by two infinite parallel plates separated by a distance H, as shown in Fig. 6.22. The liquid and the two plates are initially at rest.

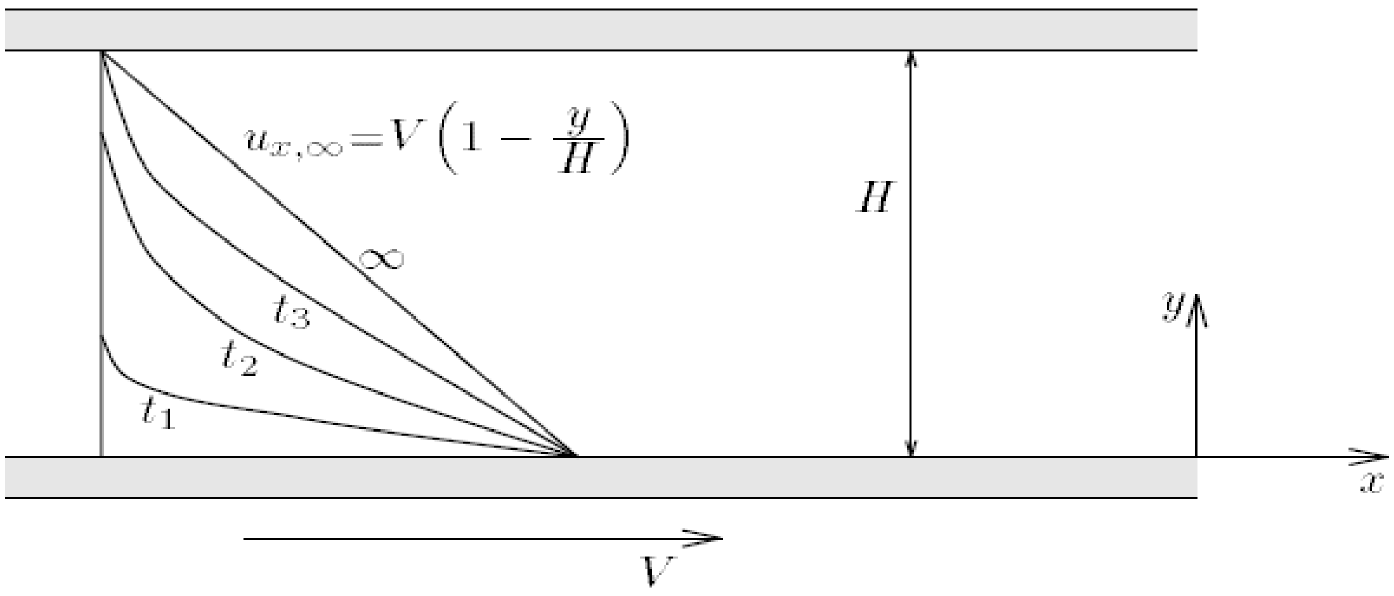

Figure 6.22. *Schematic of the evolution of the velocity in start-up plane Couette flow.*

The governing equation is the same as in the previous example,

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2}, \tag{6.125}$$

with the following boundary and initial conditions:

$$\left.\begin{array}{lll} u_x = V & \text{at} & y = 0,\ t > 0 \\ u_x = 0 & \text{at} & y = H,\ t \geq 0 \\ u_x = 0 & \text{at} & t = 0,\ 0 \leq y \leq H \end{array}\right\} \tag{6.126}$$

Note that, while the governing equation is homogeneous, the boundary conditions are inhomogeneous. Therefore, separation of variables cannot be applied directly. We first have to transform the problem so that the governing equation and the two boundary conditions are homogeneous. This can be achieved by decomposing $u_x(y,t)$ into the steady plane Couette velocity profile, which is expected to prevail at large times, and a transient component:

$$u_x(y,t) = V\left(1 - \frac{y}{H}\right) - u'_x(y,t)\,. \tag{6.127}$$

Substituting into Eqs. (6.125) and (6.126), we obtain the following problem

$$\frac{\partial u'_x}{\partial t} = \nu\,\frac{\partial^2 u'_x}{\partial y^2}\,, \tag{6.128}$$

with

$$\left.\begin{array}{lll} u'_x = 0 & \text{at} & y = 0,\ t > 0 \\ u'_x = 0 & \text{at} & y = H,\ t \geq 0 \\ u'_x = V\left(1 - \frac{y}{H}\right) & \text{at} & t = 0,\ 0 \leq y \leq H \end{array}\right\} \tag{6.129}$$

Note that the new boundary conditions are homogeneous, while the governing equation remains unchanged. Therefore, separation of variables can now be used. The first step is to express $u'_x(y,t)$ in the form

$$u'_x(y,t) \;=\; Y(y)\,T(t)\,. \tag{6.130}$$

Substituting into Eq. (6.128) and separating the functions Y and T, we get

$$\frac{1}{\nu T}\,\frac{dT}{dt} \;=\; \frac{1}{Y}\,\frac{d^2Y}{dY^2}\,.$$

The only way a function of t can be equal to a function of y is for both functions to be equal to the same constant. For convenience, we choose this constant to be $-\alpha^2/H^2$. (One advantage of this choice is that α is dimensionless.) We thus obtain two ordinary differential equations:

$$\frac{dT}{dt} \;+\; \frac{\nu\alpha^2}{H^2}\,T \;=\; 0\,, \tag{6.131}$$

$$\frac{d^2Y}{dy^2} \;+\; \frac{\alpha^2}{H^2}\,Y \;=\; 0\,. \tag{6.132}$$

The solution to Eq. (6.131) is

$$T = c_0 \, e^{-\frac{\nu \alpha^2}{H^2} t} , \tag{6.133}$$

Equation (6.132) is a homogeneous second-order ODE with constant coefficients, and its general solution is

$$Y(y) = c_1 \sin(\frac{\alpha y}{H}) + c_2 \cos(\frac{\alpha y}{H}) . \tag{6.134}$$

Applying Eq. (6.130) to the boundary conditions at y=0 and H, we obtain

$$Y(0) \, T(t) = 0 \quad \text{and} \quad Y(H) \, T(t) = 0 .$$

The case of $T(t)=0$ is excluded, since this corresponds to the steady-state problem. Hence, we get the following boundary conditions for Y:

$$Y(0) = 0 \quad \text{and} \quad Y(H) = 0 \,. \tag{6.135}$$

Applying the boundary condition at $y=0$, we get $c_2=0$. Thus,

$$Y(y) \;=\; c_1 \, \sin(\frac{\alpha y}{H}) \,. \tag{6.136}$$

Applying now the boundary condition at $y=H$, we get

$$\sin(\alpha) \;=\; 0 \,, \tag{6.137}$$

which has infinitely many roots,

$$\alpha_k = k\pi \,, \qquad k = 1, 2, \cdots \tag{6.138}$$

To each of these roots correspond solutions Y_k and T_k. These infinitely many solutions are superimposed by defining

$$u'_x(y,t) = \sum_{k=1}^{\infty} B_k \sin(\frac{\alpha_k y}{H}) e^{-\frac{\nu\alpha_k^2}{H^2}t} = \sum_{k=1}^{\infty} B_k \sin(\frac{k\pi y}{H}) e^{-\frac{k^2\pi^2}{H^2}\nu t}, \quad (6.139)$$

where the constants $B_k = c_{0k} c_{1k}$ are determined from the initial condition. For $t=0$, we get

$$\sum_{k=1}^{\infty} B_k \sin(\frac{k\pi y}{H}) = V\left(1 - \frac{y}{H}\right). \quad (6.140)$$

To isolate B_k, we will take advantage of the orthogonality property

$$\int_0^1 \sin(k\pi x)\,\sin(n\pi x)\,dx = \begin{cases} \frac{1}{2}, & k = n \\ 0, & k \neq n \end{cases} \tag{6.141}$$

BY multiplying both sides of Eq. (6.140) by $\sin(n\pi y/H)\,dy$, and by integrating from 0 to H, we have:

$$\sum_{k=1}^{\infty} B_k \int_0^H \sin\Big(\frac{k\pi\, y}{H}\Big)\,\sin\Big(\frac{n\pi\, y}{H}\Big)\,dy = V \int_0^H \left(1 - \frac{y}{H}\right)\,\sin\Big(\frac{n\pi\, y}{H}\Big)\,dy\,.$$

Setting $\xi = y/H$, we get

$$\sum_{k=1}^{\infty} B_k \int_0^1 \sin(k\pi\xi)\,\sin(n\pi\xi)\,d\xi = V \int_0^1 (1-\xi)\,\sin(n\pi\xi)\,d\xi\,.$$

Due to the orthogonality property (6.141), the only nonzero term on the left hand side is that for $k = n$; hence,

$$B_k \, \frac{1}{2} \;=\; V \int_0^1 (1-\xi)\, \sin(k\pi\xi)\, d\xi \;=\; V \, \frac{1}{k\pi} \qquad \Longrightarrow$$

$$B_k \;=\; \frac{2V}{k\pi} \, . \tag{6.142}$$

Substituting into Eq. (6.139) gives

$$u'_x(y,t) \;=\; \frac{2V}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \, \sin(\frac{k\pi \, y}{H}) \, e^{-\frac{k^2 \pi^2}{H^2} \nu t} \, . \tag{6.143}$$

Finally, for the original dependent variable $u_x(y,t)$ we get

$$u_x(y,t) \;=\; V \left(1 - \frac{y}{H}\right) \;-\; \frac{2V}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \, \sin(\frac{k\pi \, y}{H}) \, e^{-\frac{k^2 \pi^2}{H^2} \nu t} \, . \tag{6.144}$$

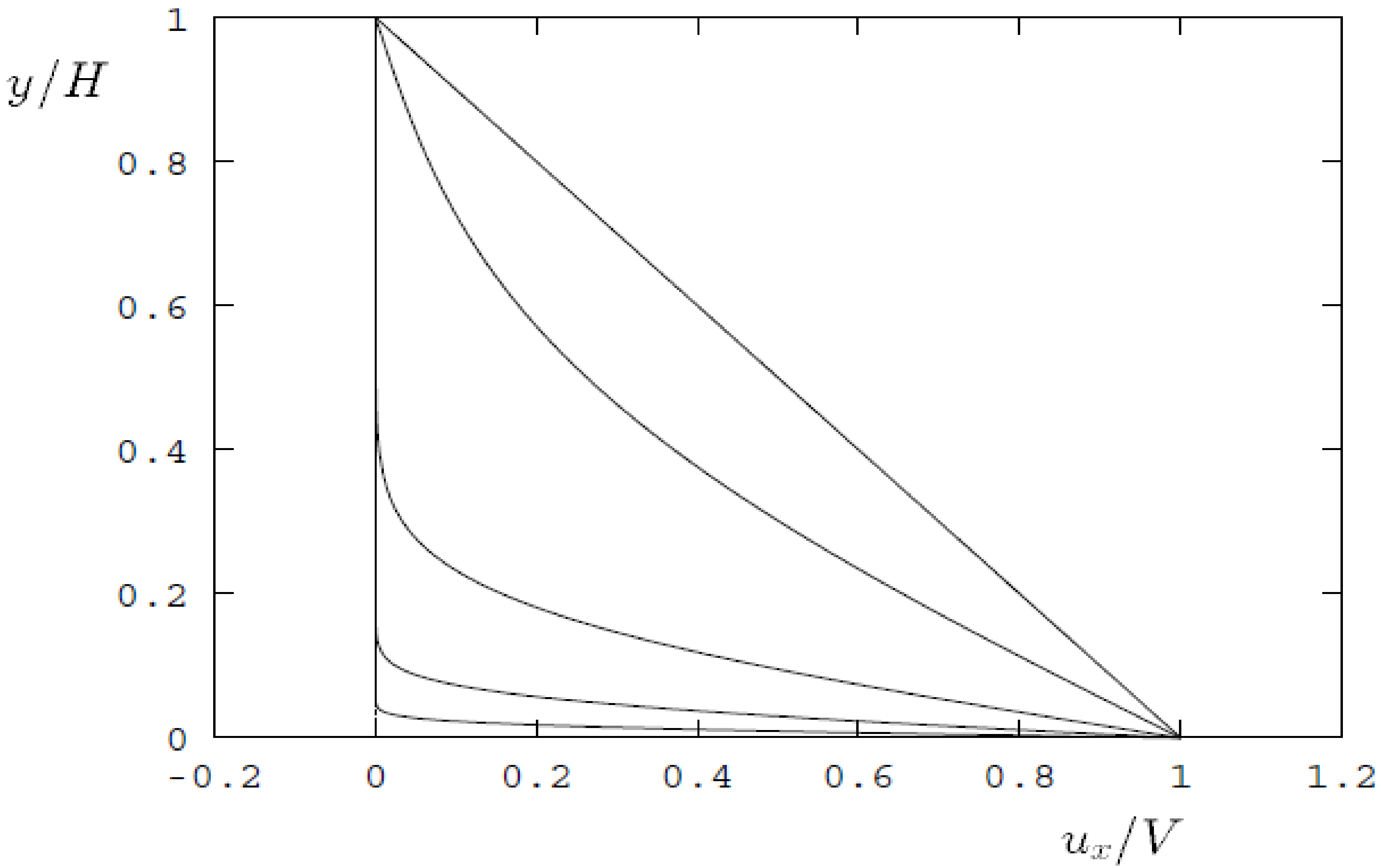

Figure 6.23. *Transient plane Couette flow. Velocity profiles at* $\nu t/H^2 = 0.0001$, *0.001, 0.01, 0.1 and 1.*