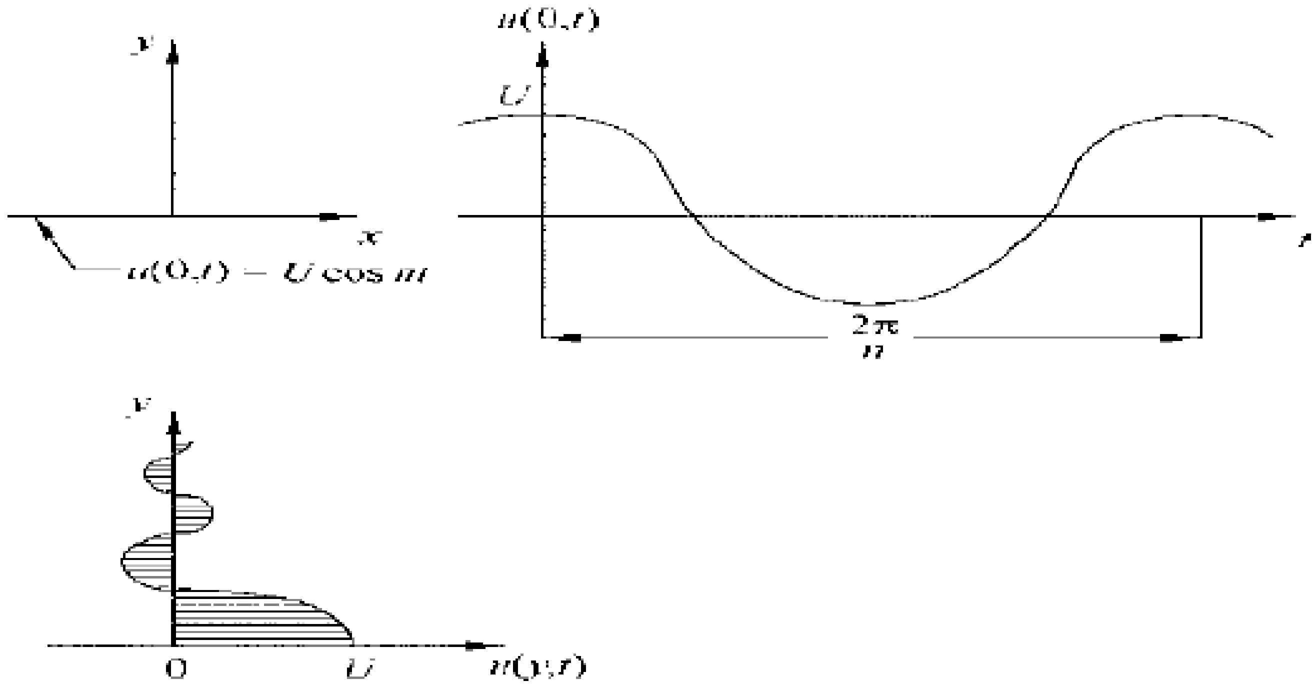
Flow due to an oscillating plate

Stokes problem or Stokes' second problem

Consider flow of a semi-infinite Newtonian liquid, set in motion by an oscillating plate of velocity

$$V = V_0 \cos \omega t , \quad t > 0 . \tag{6.145}$$



The governing equation, the initial condition and the boundary condition at $y \to \infty$ are the same as those of Example 6.6.1. At $y=0$, u_x is now equal to $V_0 \cos \omega t$. Hence, we have the following problem:

$$\frac{\partial u_x}{\partial t} = \nu \frac{\partial^2 u_x}{\partial y^2}, \tag{6.146}$$

with

$$\left. \begin{array}{lll} u_x = V_0 \cos \omega t & \text{at} & y = 0,\ t > 0 \\ u_x \to 0 & \text{at} & y \to \infty,\ t \geq 0 \\ u_x = 0 & \text{at} & t = 0,\ 0 \leq y \leq \infty \end{array} \right\} . \tag{6.147}$$

Since the period of the oscillations of the plate introduces a time scale, no similarity solution exists to this problem. By virtue of Eq. (6.145), it may be expected that u_x will also oscillate in time with the same frequency, but possibly with a phase shift relative to the oscillations of the plate. Thus, we separate the two independent variables by representing the velocity as

$$u_x(y,t) = \mathcal{R}e\left[Y(y)\, e^{i\omega t}\right] , \tag{6.148}$$

where $\mathcal{R}e$ denotes the real part of the expression within the brackets, i is the imaginary unit, and $Y(y)$ is a complex function. Substituting into the governing equation, we have

$$\frac{d^2Y}{dy^2} - \frac{i\omega}{\nu} Y = 0 . \tag{6.149}$$

The general solution of the above equation is

$$Y(y) \;=\; c_1 \,\exp\left\{-\sqrt{\frac{\omega}{2\nu}}\,(1+i)\,y\right\} \;+\; c_2 \,\exp\left\{\sqrt{\frac{\omega}{2\nu}}\,(1+i)\,y\right\} \;.$$

The fact that $u_x{=}0$ at $y{\to}\infty$, dictates that c_2 be zero. Then, the boundary condition at $y{=}0$ requires that $c_1{=}V_0$. Thus,

$$u_x(y,t) \;=\; V_0\,\mathcal{R}e\left[\exp\left\{-\sqrt{\frac{\omega}{2\nu}}\,(1+i)\,y\right\}\;e^{i\omega t}\right]\;, \qquad (6.150)$$

The resulting solution,

$$u_x(y,t) \;=\; V_0\,\exp\left(-\sqrt{\frac{\omega}{2\nu}}\;y\right)\;\cos\left(\omega t-\sqrt{\frac{\omega}{2\nu}}\;y\right)\;, \qquad (6.151)$$