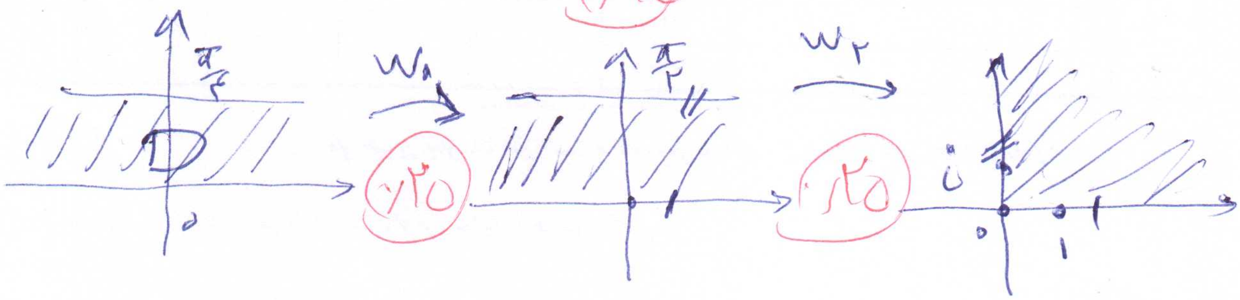
(۱

$$w_1 = 2z \,,\; w_2 = e^{w_1} \Rightarrow w = \frac{iw_2+1}{iw_2-1} = 1 + \frac{2}{iw_2-1}$$ ۰/۲۵

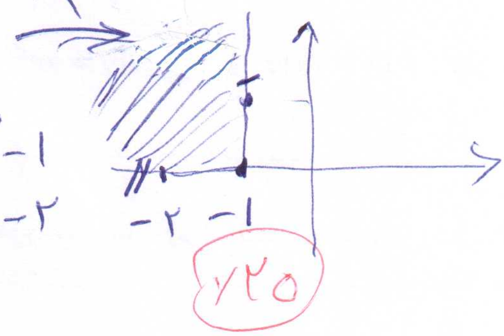
$$w_3 = iw_2 - 1 \,,\; w_4 = \frac{1}{w_3} \Rightarrow w = 2w_4 - 1$$ ۰/۲۵



$w_2 = 0 \rightarrow w_3 = -1$

$w_2 = 1 \rightarrow w_3 = i-1$

$w_2 = i \rightarrow w_3 = -2$ ۰/۲۵



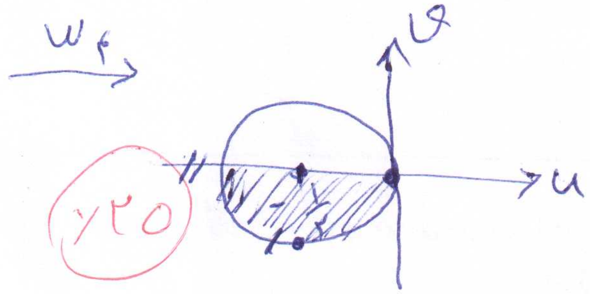
$x = -1 \rightarrow x + 1 = 0$

$\xrightarrow{\frac{1}{z}} u + (u^2 + v^2) = 0$

۰/۲۵ $\rightarrow u^2 + u + \frac{1}{4} - \frac{1}{4} + v^2 = 0$

$\rightarrow (u + \frac{1}{2})^2 + v^2 = \frac{1}{4}$

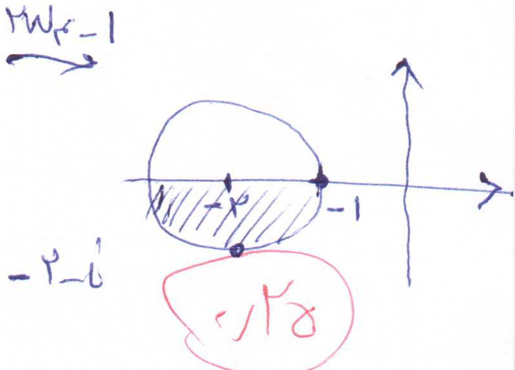
$y = 0 \xrightarrow{\frac{1}{z}} -v = 0 \rightarrow v = 0$



$w_4 = 0 \rightarrow w = -1$

$w_4 = -\frac{1}{2} \rightarrow w = -2$

$w_4 = -\frac{1}{2} - \frac{i}{2} \rightarrow w = -2 - i$ ۰/۲۵



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۲) الف) تابع زوج است پس $b_n = 0$ ۰/۲۵

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |x|\,dx = \frac{2}{\pi}\int_0^{\pi} x\,dx = \frac{2}{\pi}\frac{x^2}{2}\Big|_0^{\pi} = \pi$$ ۰/۲۵ ۰/۲۵

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(\frac{n\pi x}{\pi})\,dx = \frac{2}{\pi}\int_0^{\pi} x \cos(nx)\,dx$$ ۰/۲۵

$dv \rightarrow v = \frac{\sin nx}{n}$

$u \rightarrow du = dx$

$$= \frac{2}{\pi}\left(\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right)\Big|_0^{\pi} = \frac{2}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}\right]$$ ۰/۲۵

$$\Rightarrow a_n = \frac{۲}{\pi n^۲}\left[(-1)^n - 1\right] = \begin{cases} ۰ & n=۲k \\ \dfrac{-۴}{\pi(۲k+1)^۲} & n=۲k+1 \end{cases}$$

۰/۲۵

سری فوریه f

$$= \boxed{\frac{\pi}{۲} + \sum_{k=۰}^{\infty} \frac{-۴}{\pi(۲k+1)^۲}\cos[(۲k+1)x]}$$

۰/۲۵ $F(x)$

ب)

$$1+\frac{1}{۹}+\frac{1}{۲۵}+\cdots = \sum_{k=۰}^{\infty}\frac{1}{(۲k+1)^۲}$$

$$F(۰) = \frac{\pi}{۲} + \sum_{k=۰}^{\infty}\frac{-۴}{\pi}\frac{1}{(۲k+1)^۲}$$

$$\Rightarrow 1+\frac{1}{۹}+\cdots = -\frac{\pi}{۴}\left(F(۰)-\frac{\pi}{۲}\right)$$

۰/۵

چون $x=۰$ نقطه پیوستگی $f(x)=|x|$ است طبق قضیه دیریکله $F(۰)=f(۰)$ ۰/۲۵

پس

$$1+\frac{1}{۹}+\cdots = -\frac{\pi}{۴}\left(f(۰)-\frac{\pi}{۲}\right) = +\frac{\pi^۲}{۸}$$

۰/۲۵

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۳) الف) تابع داده شده فرد است پس $A_w = ۰$ ۰/۵

$$B_w = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin wx\,dx = \frac{1}{\pi}\int_{-۲\pi}^{۲\pi}\sin x \sin wx\,dx$$

۰/۲۵

$$= \frac{۲}{\pi}\int_{۰}^{۲\pi}\sin x\sin wx\,dx$$

$$= \frac{۲}{\pi}\int_{۰}^{۲\pi}\left(\frac{\cos((1-w)x)-\cos((1+w)x)}{۲}\right)dx$$

۰/۲۵

$$= \frac{1}{\pi}\left[\frac{\sin((1-w)x)}{1-w} - \frac{\sin((1+w)x)}{1+w}\right]_{۰}^{۲\pi}$$

$$= \frac{1}{\pi}\left[\frac{\sin((1-w)۲\pi)}{1-w} - \frac{\sin((1+w)۲\pi)}{1+w} - ۰\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(-2\pi w+2\pi)}{1-w} - \frac{\sin(2\pi w+2\pi)}{1+w}\right]$$

$$= -\frac{\sin(2\pi w)}{\pi}\left[\frac{+1}{1-w}+\frac{1}{1+w}\right] = \frac{\sin(2\pi w)}{\pi}\left(\frac{+1+w+1-w}{1-w^2}\right)$$

$$= \frac{-2\sin(2\pi w)}{\pi(1-w^2)}$$ ۰/۲۵

۰/۲۵

انتگرال فوریه $f = \underbrace{-\frac{2}{\pi}\int_0^\infty \frac{\sin(2\pi w)}{(1-w^2)}\sin wx\,dw}_{F(x)}$

ب) اگر در انتگرال فوق قرار دهیم $x=2\pi$ داریم ۰/۲۵

$$F(2\pi) = -\frac{2}{\pi}\int_0^\infty \frac{\sin^2(2\pi w)}{1-w^2}\,dw$$ ۰/۲۵

پس جواب انتگرال خواسته شده:

۰/۲۵

$$\int_0^\infty \frac{\sin^2(2\pi x)}{1-x^2}\,dx = -\frac{\pi}{2}F(2\pi) = -\frac{\pi}{2}f(2\pi)$$

$$= 0$$ ۰/۲۵

۴) ب) با کمک تبدیل فوریه کسینوسی نسبت به x حل می‌کنیم ۰/۲۵

$$\begin{cases} F_c(u_t) = 4F_c(u_{xx}) & ۰/۲۵ \\ F_c(u(x,0)) = 0 & ۰/۲۵ \end{cases} \longrightarrow \begin{cases} \bar{u}_t = 4(-w^2\bar{u} - u_x(0,t)) \\ \bar{u}(w,0) = 0 \end{cases}$$ ۰/۲۵

$$\bar{u}_t + 4w^2\bar{u} = +4 \longrightarrow \mu(t) = e^{\int 4w^2 dt} = e^{4w^2 t}$$

۰/۲۵

$$\longrightarrow \bar{u}(w,t) = e^{-4w^2 t}\left(+4\int e^{4w^2 t}\,dt + C\right) = \frac{+4}{4w^2} + Ce^{-4w^2 t}$$

۰/۲۵ ۰/۲۵

اعمال شرط

$$\bar{u}(w,0) = \frac{1}{w^2} + C = 0 \Rightarrow C = \frac{1}{w^2}$$

۰/۲۵ ۰/۲۵

$$\Rightarrow \bar{u}(w,t) = \frac{e^{-4w^2 t}+1}{w^2}$$

۰/۲۵

$$\Rightarrow u(x,t) = \mathcal{F}_c^{-1}(\bar{u}(w,t)) = \frac{2}{\pi}\int \frac{e^{-4w^2 t}+1}{w^2}\cos wx\,dw$$

۰/۲۵ ۰/۲۵

الف) روش جدا سازی

$$u(x,t) = v(x,t) + r(x) \longrightarrow \begin{cases} u_{tt} = v_{tt} + 0 \\ u_{xx} = v_{xx} + r''(x) \end{cases}$$

۰/۲۵

$$\xrightarrow{\text{معادله}} v_{tt} - v_{xx} - r''(x) = 0 \xrightarrow{\text{باید}} r''(x) = 0 \longrightarrow r'(x) = C_1$$

$$\longrightarrow r(x) = C_1 x + C_2$$

۰/۲۵

$$u(0,t) = v(0,t) + r(0) = v(0,t) + 0C_1 + C_2 = -1$$

$$\xrightarrow{\text{باید}} C_2 = -1$$

۰/۲۵

$$u(\pi,t) = v(\pi,t) + r(\pi) = v(\pi,t) + C_1\pi - 1 = 2\pi - 1$$

$$\xrightarrow{\text{باید}} C_1 = 2 \quad \Rightarrow r(x) = 2x+1$$

۰/۲۵

$$\Rightarrow \boxed{u(x,t) = v(x,t) + 2x+1}$$

معادله همگن حاصل :

$$\Rightarrow v(x,t) = u(x,t) - 2x - 1$$

$$v(x,0) = u(x,0) - 2x - 1 = -4x - 1 \quad , \quad v_t(x,0) = u_t(x,0) = 0$$

$$\begin{cases} v_{tt} - v_{xx} = 0 & (t \geq 0 \ , \ 0 \leq x \leq \pi) \\ v(0,t) = v(\pi,t) = 0 \\ v(x,0) = -4x - 1 \quad v_t(x,0) = 0 \end{cases}$$ ۰/۲۵

$$v(x,t) = X(x)\,T(t) \to v_{xx} = X''T \ , \ v_{tt} = XT''$$

$$\xrightarrow{\text{معادله}} X''T - XT'' = 0 \to \frac{X''}{X} = \frac{T''}{T} = -w^2$$

$$\Rightarrow \begin{cases} X'' + w^2 X = 0 \quad X(0) = X(\pi) = 0 \\ T'' + w^2 T = 0 \end{cases}$$

$$v(0,t) = X(0)\,T(t) = 0 \Rightarrow \begin{cases} X(0) = 0 \ ✓ \\ T(t) = 0 \ ✗ \end{cases}$$

$$v(\pi,t) = X(\pi)\,T(t) = 0 \Rightarrow \begin{cases} X(\pi) = 0 \ ✓ \\ T(t) = 0 \ ✗ \end{cases}$$ ۰/۲۵

حل معادله X :

$$r^2 + w^2 = 0 \to r = \pm iw$$

$$\Rightarrow X(x) = C_1 \cos wx + C_2 \sin wx$$

$$X(0) = C_1 = 0$$ ۰/۲۵

$$X(\pi) = \underbrace{C_1 \cos w\pi}_{0} + C_2 \sin w\pi = 0 \Rightarrow \begin{cases} C_2 = 0 \ ✗ \\ \sin w\pi = 0 \end{cases} \Rightarrow w = k$$

معادله X برای $w=k$ جواب غیر صفر دارد جواب غیر صفر X :

$$X_k(x) = C \sin wx$$ ۰/۲۵

حل معادله T :

$$r^2 + w^2 = 0 \Rightarrow r = \pm iw$$

$$\Rightarrow T_K(t) = a\cos\omega t + b\sin\omega t \quad ۰/۲۵$$

$$\Rightarrow v_K(x,t) = (a\cos kt + b\sin kt)\sin kx$$

$$v(x,t) = \sum_{K=1}^{\infty} (a_K\cos kt + b_K\sin kt)\sin kx \quad ۰/۲۵$$

اعمال شرطهای زمانی:

$$v(x,0) = \sum_{K=1}^{\infty} a_K \sin kx = -4x-1 \qquad x\in(0,\pi)$$

$$\Rightarrow a_K = \frac{2}{\pi}\int_0^{\pi} (-4x-1)\sin kx\, dx \quad ۰/۲۵$$

$$v_t(x,0) = \sum_{K=1}^{\infty} b_K \sin kx = 0 \Rightarrow b_K = 0 \quad ۰/۲۵$$

با استفاده از تبدیل لاپلاس نسبت به t:

$$\begin{cases} \mathcal{L}(u_{tt}) - \mathcal{L}(u_{xx}) = 0 \quad ۰/۲۵ \\ \mathcal{L}(u(0,t)) = \mathcal{L}(-1) \\ \mathcal{L}(u(\pi,t)) = \mathcal{L}(2\pi-1) \quad ۰/۲۵ \end{cases} \Rightarrow \begin{cases} s^2\bar{u} - s\,u(x,0) - u_t(x,0) - \bar{u}_{xx} = 0 \quad ۰/۲۵ \\ \bar{u}(0,s) = -\frac{1}{s} \\ \bar{u}(\pi,s) = \frac{2\pi-1}{s} \quad ۰/۲۵ \end{cases}$$

$$\Rightarrow \bar{u}_{xx} - s^2\bar{u} = +2xs \quad ۰/۲۵ \rightarrow r^2 - s^2 = 0 \rightarrow r = \pm s$$

$$\bar{u}_1 = e^{sx}, \ \bar{u}_2 = e^{-sx} \quad ۰/۲۵ \ , \ \bar{u}_p = Ax + B \rightarrow \bar{u}_{p_{xx}} = 0$$

$$\xrightarrow{\text{معادله}} 0 - s^2Ax - s^2B = +2xs \Rightarrow \begin{cases} -s^2A = +2s \Rightarrow A = -\frac{2}{s} \\ -s^2B = 0 \Rightarrow B = 0 \end{cases} \quad ۰/۲۵$$

$$\Rightarrow \bar{u}(x,s) = c_1e^{sx} + c_2e^{-sx} - \frac{x}{s} \quad ۰/۲۵$$

اعمال شرطها

$$\bar{u}(0,s) = c_1 + c_2 = -\frac{1}{s} \quad ۰/۲۵$$

$$\bar{U}(\pi,s) = c_1 e^{\pi s} + c_2 e^{-\pi s} - \frac{\pi}{s} = \frac{2\pi-1}{s}$$

۰/۲۵

$$-e^{\pi s}\begin{cases} c_1 + c_2 = -\frac{1}{s} \\ c_1 e^{\pi s} + c_2 e^{-\pi s} = \frac{3\pi-1}{s} \end{cases}$$

$$c_2\left(e^{-\pi s} - e^{\pi s}\right) = \frac{+e^{\pi s} + 3\pi - 1}{s} \Rightarrow c_2 = \frac{e^{\pi s} + 3\pi - 1}{-2s\sinh(\pi s)}$$

$$c_1 = -\frac{1}{s} - c_2 = \frac{e^{-\pi s} - e^{\pi s} + e^{\pi s} + 3\pi - 1}{+2s\sinh(\pi s)} = \frac{e^{-\pi s} + 3\pi - 1}{2s\sinh(\pi s)}$$

$$\Rightarrow \bar{u}(x,s) = \frac{-1}{2s\sinh(\pi s)}\left[\left(e^{\pi s} + 3\pi - 1\right)e^{sx} + \left(e^{-\pi s} + 3\pi - 1\right)e^{-sx}\right]$$

۰/۲۵

$$-\frac{x}{s} = \frac{-\cosh((\pi+s)x) - (3\pi-1)\cosh(sx) - x}{s\sinh(\pi s)}$$

$$\Rightarrow u(x,t) = L^{-1}\left(\bar{u}(x,s)\right)$$

۰/۲۵