(1

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\underbrace{(\pi-|x|)}_{\text{زوج}}\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi-x)\,dx$$

۰/۲۵

$$= \frac{1}{\pi}\left(\pi x-\frac{x^2}{2}\right)\Big|_0^{\pi} = \frac{1}{\pi}\left(\pi^2-\frac{\pi^2}{2}\right) = \boxed{\frac{\pi}{2}}$$

۰/۲۵

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\underbrace{(\pi-|x|)\cos(nx)}_{\text{زوج}}\,dx = \frac{2}{\pi}\int_0^{\pi}\underbrace{(\pi-x)}_{u\to du=-dx}\overbrace{\cos(nx)dx}^{dv\to v=\frac{\sin(nx)}{n}}$$

۰/۲۵

$$= \frac{2}{\pi}\left((\pi-x)\frac{\sin(nx)}{n}-\frac{\cos(nx)}{n^2}\right)\Big|_0^{\pi} = \frac{-2}{\pi n^2}\left((-1)^n - 1\right)$$

۰/۲۵

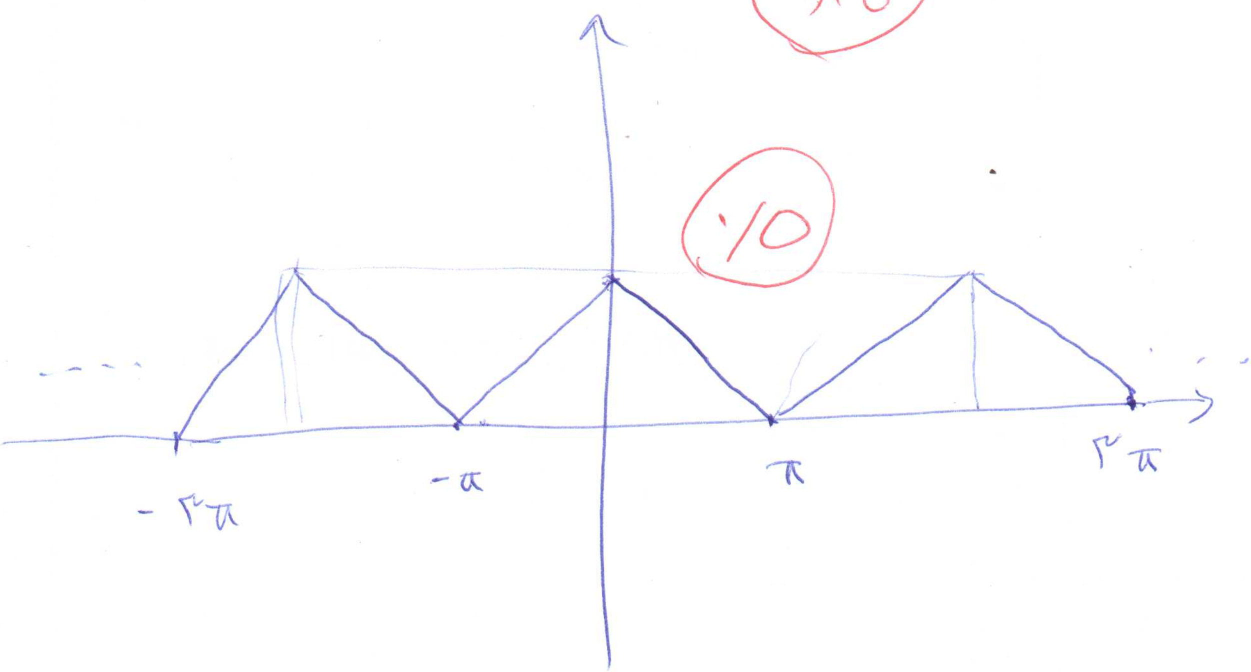
$$= \begin{cases} 0 & n=2k \\ \dfrac{4}{(2k+1)^2\pi} & n=2k+1 \end{cases}$$

$b_n = 0$ چون زوج ۰/۲۵

$$\Rightarrow \text{سری فوریه} = \frac{\pi}{2} + \sum_{k=0}^{\infty}\frac{4}{(2k+1)^2\pi}\cos((2k+1)x)$$

۰/۲۵

۰/۵



۲) الف) تبدیل فوریه

$$\bar{f}(w)=F(f(x))=\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\,dx = \int_{-1}^{1}(1-x^2)\,e^{-iwx}\,dx$$

۰/۲۵ ۰/۲۵

$1-x^2$	e^{-iwx}
$-2x$	$-\dfrac{e^{-iwx}}{iw}$
-2	$-\dfrac{e^{-iwx}}{w^2}$
0	$\dfrac{e^{-iwx}}{iw^3}$

$$=\left[\frac{(x^2-1)}{iw}-\frac{2x}{w^2}-\frac{2}{iw^3}\right]e^{-iwx}\Big|_{-1}^{1}$$

۱/۵

$$=\left[0-\frac{2}{w^2}-\frac{2}{iw^3}\right]e^{-iw}-\left[0+\frac{2}{w^2}-\frac{2}{iw^3}\right]e^{iw}$$

$$=-\frac{2}{w^2}\left(e^{iw}+e^{-iw}\right)-\frac{2}{iw^3}\left(e^{-iw}-e^{iw}\right)$$

$$=-\frac{4}{w^2}\cos w+\frac{4}{w^3}\sin w$$

۱/۵

ب)

$$f(x)=F^{-1}(\bar{f}(w))=\frac{1}{2\pi}\int_{-\infty}^{\infty}\bar{f}(w)\,e^{iwx}\,dw$$

$$=+\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{\sin w-w\cos w}{w^3}\,e^{iwx}\,dw$$

$$=\frac{2}{\pi}\left(\int_{0}^{\infty}\frac{\sin w-w\cos w}{w^3}\,e^{iwx}\,dw+\int_{-\infty}^{0}\frac{\sin w-w\cos w}{w^3}\,e^{iwx}\,dw\right)$$

$$w\to -w$$

$$=\frac{2}{\pi}\left(\int_{0}^{\infty}\frac{\sin w-w\cos w}{w^3}\,e^{iwx}\,dw-\int_{0}^{\infty}\frac{-\sin w+w\cos w}{-w^3}\,e^{-iwx}\,(-dw)\right)$$

$$=\frac{4}{\pi}\int_{0}^{\infty}\frac{\sin w-w\cos w}{w^3}\underbrace{\left(\frac{e^{iwx}+e^{-iwx}}{2}\right)}_{\cos wx}dw$$

۱/۵

۲) الف) اگر به جای تبدیل فوریه انتگرال فوریه حساب شود.

$$A_w = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos wx\,dx = \frac{1}{\pi}\int_{-1}^{1}(1-x^2)\cos wx\,dx$$

۰/۲۵ (زوج) ۰/۲۵

$$= \frac{2}{\pi}\int_{0}^{1}(1-x^2)\cos wx\,dx = \frac{2}{\pi}\left[\frac{(1-x^2)\sin wx}{w} - \frac{2x\cos wx}{w^2}\right.$$

$1-x^2$	$-2x$	-2	0
$\cos wx$	$\frac{\sin wx}{w}$	$-\frac{\cos wx}{w^2}$	$-\frac{\sin wx}{w^3}$

۰/۲۵

$$\left. + \frac{2\sin wx}{w^3}\right]\Bigg|_0^1$$

$$\Rightarrow A_w = \frac{2}{\pi}\left[0 - \frac{2\cos w}{w^2} + \frac{2\sin w}{w^3} - 0\right] = \frac{4}{\pi}\left(\frac{\sin w - w\cos w}{w}\right)$$

۰/۲۵

$$B_w = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin wx\,dx = \frac{1}{\pi}\int_{-1}^{1}(1-x^2)\sin wx\,dx = 0$$

۰/۲۵ فرد ۰/۲۵

$$\Rightarrow \text{انتگرال فوریه} = \frac{4}{\pi}\underbrace{\int_0^{\infty}\frac{\sin w - w\cos w}{w}\cos wx\,dw}_{F(x)}$$

۰/۵

ب)

$$\int_0^{\infty}\frac{\sin x - x\cos x}{x}\cos\left(\frac{x}{2}\right)dx = \frac{\pi}{4}F\left(\frac{1}{2}\right) = \frac{\pi}{4}\left(1-\frac{1}{4}\right)$$

۰/۵ ۰/۵

$$= \frac{3\pi}{16}$$

۳) الف) روش جداسازی : منبع مستقل از زمان است

$$u(x,t)=v(x,t)+r(x) \longrightarrow \begin{cases} u_{xx}=v_{xx}+r''(x) \\ u_{tt}=v_{tt} \end{cases}$$

۰/۲۵

$$\xrightarrow{\text{معادله}} v_{xx}+r''(x)-v_{tt}=\sin x \xrightarrow{\text{باید}} r''(x)=\sin x \xrightarrow{\text{دوبار انتگرال}} r(x)=-\sin x+c_1 x+c_2$$

۰/۲۵

$$u(0,t)=v(0,t)+r(0)=1 \Rightarrow r(0)=1 \Rightarrow r(0)=0+0+c_2=1$$
$$\Rightarrow \boxed{c_2=1}$$

۰/۲۵

$$u(\pi,t)=v(\pi,t)+r(\pi)=1 \Rightarrow r(\pi)=1 \Rightarrow r(\pi)=0+c_1\pi+1=1$$
$$\Rightarrow \boxed{c_1=0}$$

$$\Rightarrow r(x)=1-\sin x \longrightarrow \boxed{u(x,t)=v(x,t)+1-\sin x}$$

۰/۲۵

$$\Rightarrow v(x,0)=u(x,0)-1+\sin x=1-1+\sin x$$
$$v_t(x,0)=u_t(x,0)=0$$

معادله همگن حاصل

$$\begin{cases} v_{xx}-v_{tt}=0 \qquad (0\le x\le\pi\ ,\ t\ge 0) \\ v(0,t)=v(\pi,t)=0 \\ v(x,0)=\sin x \qquad v_t(x,0)=0 \end{cases}$$

$$v(x,t)=X(x)T(t) \longrightarrow \begin{cases} v_{xx}=X''T \\ v_{tt}=XT'' \end{cases} \rightarrow X''T-XT''=0 \Rightarrow \frac{X''}{X}=\frac{T''}{T}=-w^2$$

۰/۲۵

$$\rightarrow \begin{cases} X''+w^2X=0 \longrightarrow r^2+w^2=0 \rightarrow r=\pm iw \\ T''+w^2T=0 \end{cases}$$

۰/۲۵

$$\Rightarrow X(x)=c_1\cos wx+c_2\sin wx$$

$$v(0,t)=X(0)T(t)=0 \rightarrow X(0)=0 \longrightarrow X(0)=c_1=0$$

۰/۲۵

$$v(\pi,t)=X(\pi)T(t)=0 \rightarrow X(\pi)=0 \rightarrow X(\pi)=c_1\cos(\pi w)+c_2\sin(\pi w)$$

$\rightarrow C_2 \sin(\omega\pi) = 0 \begin{cases} C_2 = 0 & \times \\ \sin\omega\pi = 0 \rightarrow \omega = K \end{cases}$

$\Rightarrow X_K(x) = C\sin(Kx)$ ٪۲۵

حل معادله T :

$T'' + K^2 T = 0 \Rightarrow r^2 + K^2 = 0 \Rightarrow r = \pm iK$ ٪۲۵

$\Rightarrow T_K(t) = a\cos(Kt) + b\sin(Kt)$

$$\Rightarrow v(x,t) = \sum_{K=1}^{\infty} [a_K\cos(Kt) + b_K\sin(Kt)]\sin(Kx)$$ ٪۲۵

$$v(x,0) = \sum_{K=1}^{\infty} a_K \sin(Kx) = \sin x \rightarrow a_1 = 1, \quad a_K = 0 \quad K > 1$$

$$v_t(x,0) = \sum_{K=1}^{\infty} K b_K \sin(Kx) = 0 \rightarrow b_K = 0$$ ٪۲۵

$$\Rightarrow \boxed{u(x,t) = \cos(\cancel{K}t)\sin(\cancel{K}x) + 1 - \sin x}$$

٪۲۵

ب) تبدیل فوریه نسبت به x : ٪۲۵

معادله $* \; F(u_{xx}) = F(u_t) \Rightarrow -\omega^2\bar{u} = \bar{u}_t \Rightarrow \frac{\bar{u}_t}{\bar{u}} = -\omega^2$ ٪۲۵ ٪۲۵ ٪۲۵

$\xrightarrow{\text{انتگرال}} \ln(\bar{u}) = -\omega^2 t + C_\omega \Rightarrow \bar{u}(\omega,t) = C_\omega e^{-\omega^2 t}$ (1) ٪۲۵ ٪۲۵

شرط $* \; F(u(x,0)) = F(f(x)) \Rightarrow \bar{u}(\omega,0) = \bar{f}(\omega)$ (2) ٪۲۵ ٪۲۵

$(1) \longrightarrow \bar{u}(\omega,0) = C_\omega \xrightarrow{(2)} C_\omega = \bar{f}(\omega)$ ٪۲۵

$$\bar f(\omega) = \int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, dx = \int_{-1}^{0} \underbrace{(1+x)}_{u}\, \underbrace{e^{-i\omega x}\, dx}_{dv} + \int_{0}^{1} \underbrace{(1-x)}_{u}\, \underbrace{e^{-i\omega x}\, dx}_{dv}$$

$$du = dx \qquad v = \frac{e^{-i\omega x}}{-i\omega} \qquad du = -dx \qquad v = \frac{e^{-i\omega x}}{-i\omega}$$

$$\Rightarrow \bar f(\omega) = \left(\frac{(1+x)\, e^{-i\omega x}}{-i\omega} + \frac{e^{-i\omega x}}{\omega^2}\right)\Bigg|_{-1}^{0} + \left(\frac{(1-x)\, e^{-i\omega x}}{-i\omega} + \frac{e^{-i\omega x}}{\omega^2}\right)\Bigg|_{0}^{1}$$

$$= \left(\frac{1}{\omega^2} - \frac{1}{i\omega} - \frac{e^{i\omega}}{\omega^2}\right) + \left(0 + \frac{e^{-i\omega}}{\omega^2} + \frac{1}{i\omega} - \frac{1}{\omega^2}\right)$$

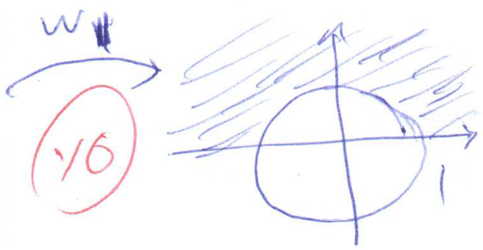
/۲۵

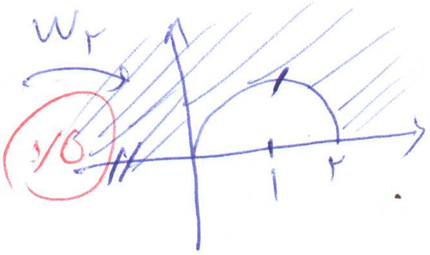
$$= \frac{-2i}{\omega^2} \sin\omega$$

$$\Rightarrow \bar u(\omega, t) = \frac{-2i \sin\omega}{\omega}\, e^{-\omega t} \Rightarrow u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{-2i \sin\omega}{\omega}\, e^{-\omega t}\, e^{i\omega x}\, d\omega$$

/۲۵ /۲۵

$$f(z) = 1 + \frac{-2}{z^2+1} \to w_1 = z^2,\ w_2 = z+1,\ w_3 = \frac{1}{z},\ w_4 = -2z,\ w_5 = z+1$$

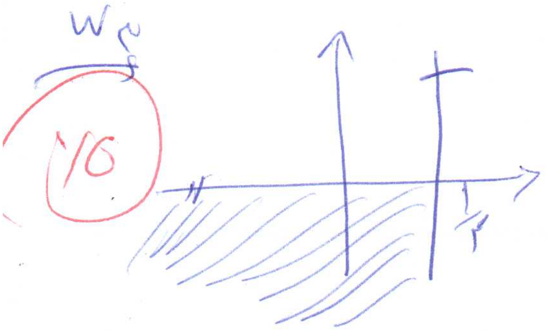




$$(x-1)^2 + y^2 = 1 \Rightarrow \underset{A}{(x^2+y^2)} - \underset{B}{2x} + \underset{C}{0}\, y + \underset{D}{0} = 0 \xrightarrow{\frac{1}{z}} 1 - 2u = 0 \to u = \frac{1}{2}$$

/۵

$$x = 0 \to u = 0$$



$\xrightarrow{w_4}$

$1 - 21 = 2$

$\mathrm{Arg}(-2) = \pi$

/۵