پاسخ تکلیف سری اول درس پردازش سیگنال‌های دیجیتالی

(Digital Signal Processing)

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حل 1 :

(a) Since $\cos(\pi n)$ only takes on values of +1 or -1, this transformation outputs the current value of $x[n]$ multiplied by either ± 1. $T(x[n]) = (-1)^n x[n]$.

- Hence, it is stable, because it doesn't change the magnitude of $x[n]$ and hence satisfies bounded-in/bounded-out stability.
- It is causal, because each output depends only on the current value of $x[n]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = \cos(\pi n)x_1[n]$, and $y_2[n] = T(x_2[n]) = \cos(\pi n)x_2[n]$. Now

$$T(ax_1[n] + bx_2[n]) = \cos(\pi n)(ax_1[n] + bx_2[n]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = (-1)^n x[n]$, then $T(x[n-1]) = (-1)^n x[n-1] \neq y[n-1]$.

(b) This transformation simply "samples" $x[n]$ at location which can be expressed as k^2.

- The system is stable, since if $x[n]$ is bounded, $x[n^2]$ is also bounded.
- It is not causal. For example, $Tx[4] = x[16]$.
- It is linear. Let $y_1[n] = T(x_1[n]) = x_1[n^2]$, and $y_2[n] = T(x_2[n]) = x_2[n^2]$. Now

$$T(ax_1[n] + bx_2[n]) = ax_1[n^2] + bx_2[n^2]) = ay_1[n] + by_2[n]$$

- It is not time-invariant. If $y[n] = T(x[n]) = x[n^2]$, then $T(x[n-1]) = x[n^2-1] \neq y[n-1]$.

(c) First notice that

$$\sum_{k=0}^{\infty} \delta[n-k] = u[n]$$

So $T(x[n]) = x[n]u[n]$. This transformation is therefore stable, causal, linear, but not time-invariant.
To see that it is not time invariant, notice that $T(\delta[n]) = \delta[n]$, but $T(\delta[n+1]) = 0$.

(d) $T(x[n]) = \sum_{k=n-1}^{\infty} x[k]$

- This is not stable. For example, $T(u[n]) = \infty$ for all n ¿ 1.
- It is not causal, since it sums *forward* in time.
- It is linear, since

$$\sum_{k=n-1}^{\infty} ax_1[k] + bx_2[k] = a \sum_{k=n-1}^{\infty} x_1[k] + b \sum_{k=n-1}^{\infty} x_2[k]$$

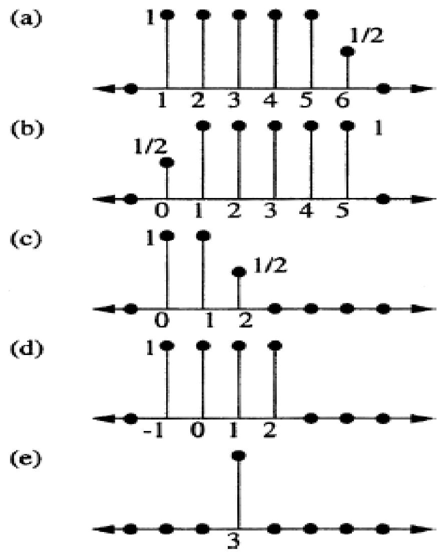
- It is time-invariant. Let

$$y[n] = T(x[n]) = \sum_{k=n-1}^{\infty} x[k],$$

then

$$T(x[n-n_0]) = \sum_{k=n-n_0-1}^{\infty} x[k] = y[n-n_0]$$

حل2:



حل3:

(a) The difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

Taking the Fourier transform of both sides,

$$Y(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega}] = X(e^{j\omega})[1 + 2e^{-j\omega} + e^{-j2\omega}].$$

Hence, the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \end{aligned}$$

(b) A system with frequency response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}} \\ &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \end{aligned}$$

cross multiplying,

$$Y(e^{j\omega})[1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}] = X(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}],$$

and the inverse transform gives

$$y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3].$$

حل 4:

(a) First the frequency response:

$$Y(e^{j\omega}) - \frac{5}{6}e^{-j\omega}Y(e^{j\omega}) + \frac{1}{6}e^{-2j\omega}Y(e^{j\omega}) = \frac{1}{3}e^{-2j\omega}X(e^{j\omega})$$

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{\frac{1}{3}e^{-2j\omega}}{1-\frac{5}{6}e^{-j\omega}+\frac{1}{6}e^{-2j\omega}} \end{aligned}$$

Now we take the inverse Fourier transform to find the impulse response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{-2}{1-\frac{1}{3}e^{-j\omega}} + \frac{2}{1-\frac{1}{2}e^{-j\omega}} \\ h[n] &= -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n] \end{aligned}$$

For the step response $s[n]$:

$$\begin{aligned} s[n] &= \sum_{k=-\infty}^{\infty} h[k]u[n-k] \\ &= \sum_{k=-\infty}^{n} h[k] \\ &= -2\frac{1-(1/3)^{n+1}}{1-1/3}u[n] + 2\frac{1-(1/2)^{n+1}}{1-1/2}u[n] \\ &= (1+(\frac{1}{3})^n - 2(\frac{1}{2})^n)u[n] \end{aligned}$$

(b) The homogeneous solution $y_h[n]$ solves the difference equation when $x[n] = 0$. It is in the form $y_h[n] = \sum A(c)^n$, where the c's solve the quadratic equation

$$c^2 - \frac{5}{6}c + \frac{1}{6} = 0$$

So for $c = 1/2$ and $c = 1/3$, the general form for the homogeneous solution is:

$$y_h[n] = A_1(\frac{1}{2})^n + A_2(\frac{1}{3})^n$$

(c) The total solution is the sum of the homogeneous and particular solutions, with the particular solution being the impulse response found in part (a):

$$\begin{aligned} y[n] &= y_h[n] + y_p[n] \\ &= A_1(\frac{1}{2})^n + A_2(\frac{1}{3})^n + -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n] \end{aligned}$$

Now we use the constraint $y[0] = y[1] = 1$ to solve for A_1 and A_2:

$$\begin{aligned} y[0] &= A_1 + A_2 - 2 + 2 = 1 \\ y[1] &= A_1/2 + A_2/3 - 2/3 + 1 = 1 \\ A_1 + A_2 &= 1 \\ A_1/2 + A_2/3 &= 2/3 \end{aligned}$$

With $A_1 = 2$ and $A_2 = -1$ solving the simultaneous equations, we find that the impulse response is

$$y[n] = 2(\frac{1}{2})^n - (\frac{1}{3})^n + -2(\frac{1}{3})^n u[n] + 2(\frac{1}{2})^n u[n]$$

حل 5:

Eigenfunctions of LTI systems are of the form α^n, so functions (a), (b), and (e) are eigenfunctions. Notice that part (d), $\cos(\omega_0 n) = .5(e^{j\omega_0 n} + e^{-j\omega_0 n})$ is a sum of two α^n functions, and is therefore not an eigenfunction itself.

حل 6:

$h[n]$ is stable if it is absolutely summable.

(a) Not stable because $h[n]$ goes to ∞ as n goes to ∞.

(b) Stable, because $h[n]$ is non-zero only for $0 \leq n \leq 9$.

(c) Stable.

$$\sum_n |h[n]| = \sum_{n=-\infty}^{-1} 3^n = \sum_{n=1}^{\infty} (1/3)^n = 1/2 < \infty$$

(d) Not stable. Notice that

$$\sum_{n=0}^{5} |\sin(\pi n/3)| = 2\sqrt{3}$$

and summing $|h[n]|$ over all positive n therefore grows to ∞.

(e) Stable. Notice that $|h[n]|$ is upperbounded by $(3/4)^{|n|}$, which is absolutely summable.

(f) Stable.

$$h[n] = \begin{cases} 2, & ,-5 \leq n \leq -1 \\ 1, & ,0 \leq n \leq 4 \\ 0, & ,\text{otherwise} \end{cases}$$

So $\sum |h[n]| = 15$.

حل 7:

(a) Taking the difference equation $y[n] = (1/a)y[n-1] + x[n-1]$ and assuming $h[0] = 0$ for $n < 0$:

$$\begin{aligned} h[0] &= 0 \\ h[1] &= 1 \\ h[2] &= 1/a \\ h[3] &= (1/a)^2 \\ &\vdots \\ h[n] &= (1/a)^{n-1}u[n-1] \end{aligned}$$

(b) $h[n]$ is absolutely summable if $|1/a| < 1$ or if $|a| > 1$

حل8:

$x[n]$ is periodic with period N if $x[n] = x[n+N]$ for some integer N.

(a) $x[n]$ is periodic with period 5:

$$e^{j(\frac{2\pi}{5}n)} = e^{j(\frac{2\pi}{5})(n+N)} = e^{j(\frac{2\pi}{5}n+2\pi k)}$$

$$\Longrightarrow 2\pi k = \frac{2\pi}{5}N, \text{for integers } k, N$$

Making $k = 1$ and $N = 5$ shows that $x[n]$ has period 5.

(b) $x[n]$ is periodic with period 38. Since the sin function has period of 2π:

$$x[n+38] = \sin(\pi(n+38)/19) = \sin(\pi n/19 + 2\pi) = x[n]$$

(c) This is not periodic because the linear term n is not periodic.

(d) This is again not periodic. $e^{j\omega}$ is periodic over period 2π, so we have to find k, N such that

$$x[n+N] = e^{j(n+N)} = e^{j(n+2\pi k)}$$

Since we can make k and N integers at the same time, $x[n]$ is not periodic.