## پاسخ تکلیف سری سوم درس پردازش سیگنال‌های دیجیتالی

(Digital Signal Processing)

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حل 1:

**(a)**

$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n u[n]\right] = \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n z^{-n} = \sum_{n=0}^{\infty}\left(\frac{1}{2z}\right)^n = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| > \frac{1}{2}$$

**(b)**

$$\begin{aligned}\mathcal{Z}\left[-\left(\frac{1}{2}\right)^n u(-n-1)\right] &= -\sum_{n=-\infty}^{-1}\left(\frac{1}{2}\right)^n z^{-n} = -\sum_{n=1}^{\infty}(2z)^n \\ &= -\frac{2z}{1-2z} = \frac{1}{1-\frac{1}{2}z^{-1}} \qquad |z| < \frac{1}{2}\end{aligned}$$

**(c)**

$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n u[-n]\right] = \sum_{n=-\infty}^{0}(2z)^n = \frac{1}{1-2z} \qquad |z| < \frac{1}{2}$$

**(g)**

$$\mathcal{Z}\left[\left(\frac{1}{2}\right)^n (u[n]-u[n-10])\right] = \sum_{n=0}^{9}\left(\frac{1}{2z}\right)^n = \frac{1-(2z)^{-10}}{1-(2z)^{-1}} \qquad |z| > 0$$

حل 2:

$$x[n] \text{ causal } \Rightarrow X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

which means this summation will include *no* positive powers of $z$. This means that the closed form of $X(z)$ must converge at $z = \infty$, i.e., $z = \infty$ must be in the ROC of $X(z)$, or $\lim_{z\to\infty} X(z) \neq \infty$.

**(a)**

$$\lim_{z\to\infty}\frac{(1-z^{-1})^2}{(1-\frac{1}{2}z^{-1})} = 1 \qquad \text{could be causal}$$

**(c)**

$$\lim_{z\to\infty}\frac{(z-\frac{1}{4})^5}{(z-\frac{1}{2})^6} = 0 \qquad \text{could be causal}$$

حل 3:

In both cases, the ROC of $H(z)$ has to be chosen such that ROC($Y(z)$) includes the intersection of ROC($H(z)$) and ROC($X(z)$).

(a)

$$H(z) = \frac{1 - \frac{3}{4}z^{-1}}{1 + \frac{2}{3}z^{-1}}$$

The ROC is $|z| > \frac{2}{3}$.

(b)

$$H(z) = \frac{1}{1 - \frac{1}{6}z^{-1}}$$

The ROC is $|z| > \frac{1}{6}$.

حل 4:

(a)

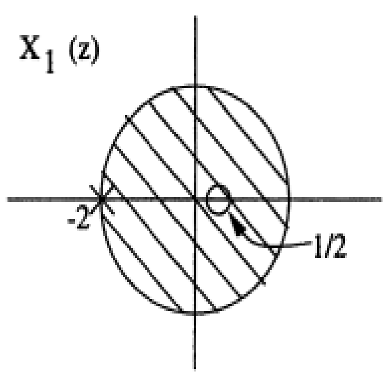
$$X_1(z) = \frac{1 - \frac{1}{2}z^{-1}}{1 + 2z^{-1}}$$

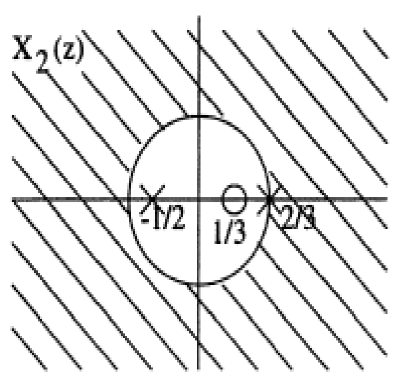
The pole is at -2, and the zero is at 1/2.

(b)

$$X_2(z) = \frac{1 - \frac{1}{3}z^{-1}}{(1 + \frac{1}{2}z^{-1})(1 - \frac{2}{3}z^{-1})}$$

The poles are at -1/2 and 2/3, and the zero is at 1/3. Since $x_2[n]$ is causal, the ROC is extends from the outermost pole: $|z| > 2/3$.



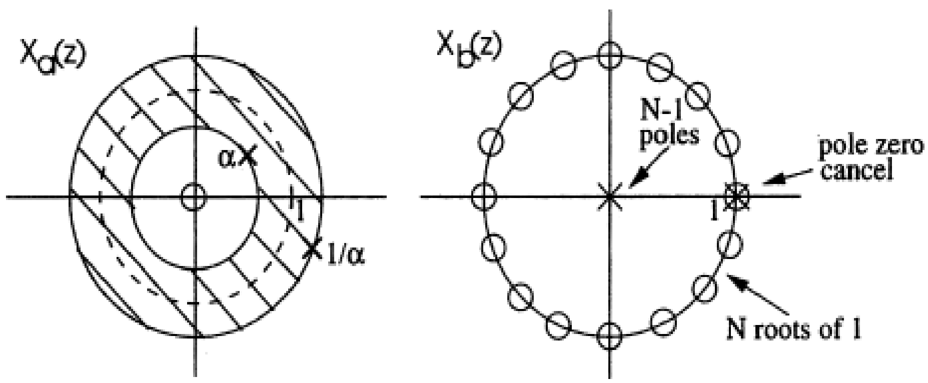



حل 5:

(a)

$$x_a[n] = \alpha^{|n|} \qquad 0 < |\alpha| < 1$$

$$\begin{aligned}
X_a(z) &= \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \sum_{n=1}^{\infty} \alpha^n z^n + \sum_{n=0}^{\infty} \alpha^n z^{-n} \\
&= \frac{\alpha z}{1-\alpha z} + \frac{1}{1-\alpha z^{-1}} = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)}, \qquad |\alpha| < |z| < \frac{1}{|\alpha|}
\end{aligned}$$




(b)

$$x_b = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & N \le n \\ 0, & n < 0 \end{cases} \Rightarrow X_b(z) = \sum_{n=0}^{N-1} z^{-n} = \frac{1-z^{-N}}{1-z^{-1}} = \frac{z^N - 1}{z^{N-1}(z-1)} \quad z \neq 0$$

حل 6:

(a)

$$x[n] = u[-n-1] + \left(\frac{1}{2}\right)^n u[n]$$

$$\Rightarrow \qquad X(z) = \frac{-1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \qquad \frac{1}{2} < |z| < 1$$

Now to find $H(z)$ we simply use $H(z) = Y(z)/X(z)$; i.e.,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{-\frac{1}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1+z^{-1})} \cdot \frac{(1-z^{-1})(1-\frac{1}{2}z^{-1})}{-\frac{1}{2}z^{-1}} = \frac{1-z^{-1}}{1+z^{-1}}$$

$H(z)$ causal $\Rightarrow$ ROC $|z| > 1$.

(b) Since one of the poles of $X(z)$, which limited the ROC of $X(z)$ to be less than 1, is cancelled by the zero of $H(z)$, the ROC of $Y(z)$ is the region in the $z$-plane that satisfies the remaining two constraints $|z| > \frac{1}{2}$ and $|z| > 1$. Hence $Y(z)$ converges on $|z| > 1$.

(c)

$$Y(z) = \frac{-\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{3}}{1+z^{-1}} \qquad |z| > 1$$

Therefore,

$$y[n] = -\frac{1}{3}\left(\frac{1}{2}\right)^n u[n] + \frac{1}{3}(-1)^n u[n]$$

حل 7:

We solve this problem by finding the system function $H(z)$ of the system, and then looking at the different impulse responses which can result from our choice of the ROC.

Taking the z-transform of the difference equation, we get

$$Y(z)(1-\frac{5}{2}z^{-1}+z^{-2}) = X(z)(1-z^{-1}),$$

and thus

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} = \frac{1-z^{-1}}{1-\frac{5}{2}z^{-1}+z^{-2}} \\ &= \frac{1-z^{-1}}{(1-2z^{-1})(1-\frac{1}{2}z^{-1})} \\ &= \frac{2/3}{1-2z^{-1}} + \frac{1/3}{1-\frac{1}{2}z^{-1}} \end{aligned}$$

If the ROC is

(a) $|z| < \frac{1}{2}$:

$$h[n] = -\frac{2}{3}2^n u[-n-1] - \frac{1}{3}(\frac{1}{2})^n u[-n-1]$$
$$\Longrightarrow \qquad h[0] = 0.$$

(b) $\frac{1}{2} < |z| < 2$:

$$h[n] = -\frac{2}{3}2^n u[-n-1] + \frac{1}{3}(\frac{1}{2})^n u[n]$$
$$\Longrightarrow \qquad h[0] = \frac{1}{3}.$$

(c) $|z| > 2$:

$$h[n] = \frac{2}{3}2^n u[n] + \frac{1}{3}(\frac{1}{2})^n u[n]$$
$$\Longrightarrow \qquad h[0] = 1.$$

(d) $|z| > 2$ or $|z| < \frac{1}{2}$:

$$h[n] = \frac{2}{3}2^n u[n] - \frac{1}{3}(\frac{1}{2})^n u[n-1]$$
$$\Longrightarrow \qquad h[0] = \frac{2}{3}.$$

حل 8:

(a)

$$\begin{aligned} H(z) &= \frac{1+2^{z}-1+z^{-2}}{(1+\frac{1}{2}z^{-1})(1-z^{-1})} \\ &= -2+\frac{\frac{1}{3}}{1+\frac{1}{2}z^{-1}}+\frac{\frac{8}{3}}{1-z^{-1}} \end{aligned}$$

Taking the inverse z-transform:

$$h[n] = -2\delta[n] + \frac{1}{3}(-\frac{1}{2})^{n}u[n] + \frac{8}{3}u[n].$$

**(b)**

Given

$$H(z)=\frac{1+2z^{-1}+z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)\left(1-z^{-1}\right)},$$

$z=2$ is inside the ROC. Therefore,

$$\begin{aligned} y[n] &= H(z)\big|_{z=2}\,2^{n} \\ &= \frac{18}{5}2^{n}. \end{aligned}$$

حل 9:

$$\begin{aligned} H(z) &= \frac{1-z^{3}}{1-z^{4}} = z^{-1}\left(\frac{1-z^{-3}}{1-z^{-4}}\right) \qquad |z|>1 \\ u[n] &\Leftrightarrow \frac{1}{1-z^{-1}} = \frac{z}{z-1} \qquad |z|>1 \\ U(z)H(z40 &= \frac{z^{-1}-z^{-4}}{(1-z^{-4})(1-z^{-1})} \\ &= \frac{z^{-1}}{1-z^{-1}} - \frac{z^{-4}}{1-z^{-4}} \qquad |z|>1 \\ u[n]*h[n] &= u[n-1] - \sum_{k=0}^{\infty}\delta[n-4-4k] \end{aligned}$$

حل 10:

$$X(z) = \frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{1}{4}}{1-2z^{-1}}$$

has poles at $z=\frac{1}{2}$ and $z=2$.

Since the unit circle is in the region of convergence $X(z)$ and $x[n]$ have both a causal and an anticausal part. The causal part is "outside" the pole at $\frac{1}{2}$. The anticausal part is "inside" the pole at 2, therefore, $x[0]$ is the sum of the two parts

$$x[0] = \lim_{z\to\infty}\frac{\frac{1}{3}}{1-\frac{1}{2}z^{-1}} + \lim_{z\to 0}\frac{\frac{1}{4}z}{z-2} = \frac{1}{3}+0=\frac{1}{3}$$