

# Digital Signal Processing, Fall 2006

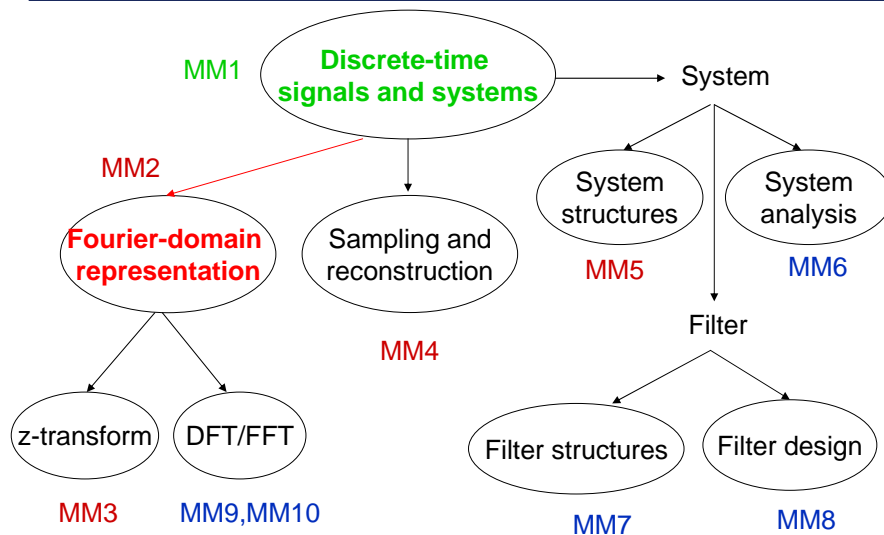
- E-Study -

## Lecture 2: Fourier transforms and frequency response

Zheng-Hua Tan

Department of Electronic Systems  
Aalborg University, Denmark  
[zt@kom.aau.dk](mailto:zt@kom.aau.dk)

## Course at a glance



## Part I: System impulse response

- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response

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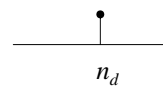


## FIR systems – reflected in the $h[n]$

- Ideal delay

$$y[n] = x[n - n_d], \quad -\infty < n < \infty$$

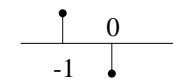
➔  $h[n] = \delta[n - n_d], \quad n_d \text{ a positive integer.}$



- Forward difference

$$y[n] = x[n + 1] - x[n]$$

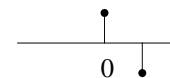
➔  $h[n] = \delta[n + 1] - \delta[n]$



- Backward difference

$$y[n] = x[n] - x[n - 1]$$

➔  $h[n] = \delta[n] - \delta[n - 1]$



- Finite-duration impulse response (FIR) system

- The impulse response has only a finite number of nonzero samples.

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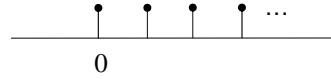


## IIR systems – reflected in the $h[n]$

- Accumulator

$$y[n] = \sum_{k=-\infty}^n x[k]$$

$$\rightarrow h[n] = \sum_{k=-\infty}^n \delta[k] = u[n]$$



- Infinite-duration impulse response (IIR) system

- The impulse response is infinite in duration.

- Stability  $S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$

- FIR systems always are stable, if each of  $h[n]$  values is finite in magnitude.

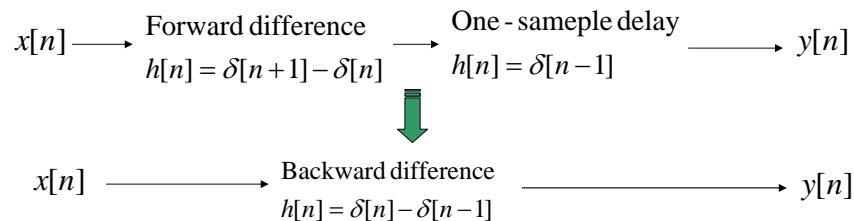
- IIR systems can be stable, e.g.  $h[n] = a^n u[n]$  with  $|a| < 1$

$$S = \sum_0^{\infty} |a|^n = 1/(1-|a|) < \infty$$

## Cascading systems

- Causality  $h[n]=0, n < 0$

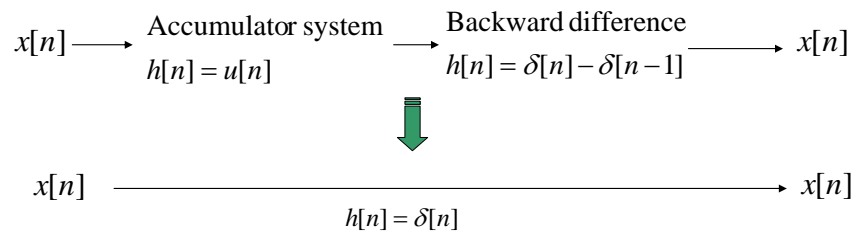
- Ideal delay  $h[n] = \delta[n - n_d]$



- Any noncausal FIR system can be made cause by cascading it with a sufficiently long delay!

## Cascading systems

- Accumulator + Backward difference system



Inverse system:

$$h[n] * h_i[n] = h_i[n] * h[n] = \delta[n]$$

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## Part II: LCCD equations

- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response

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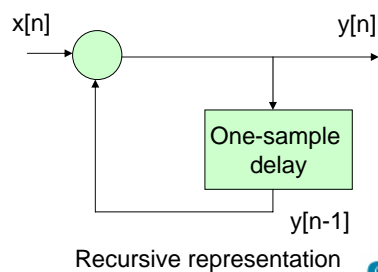
## LCCD equations

- An important class of LTI systems: input and output satisfy an Nth-order LCCD equations

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

- Difference equation representation of the accumulator

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^n x[k] \\ y[n-1] &= \sum_{k=-\infty}^{n-1} x[k] \\ y[n] &= x[n] + \sum_{k=-\infty}^{n-1} x[k] = x[n] + y[n-1] \\ y[n] - y[n-1] &= x[n] \end{aligned}$$



## Part III: Fourier transforms

- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response
  - Frequency-domain representation of discrete-time signals and systems
  - Symmetry properties of the Fourier transform
  - Fourier transform theorems

## Signal representations

- A sum of scaled, delayed **impulse**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n-k] \quad \Rightarrow \quad y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k]$$

- **Sinusoidal and complex exponential sequences**

- Sinusoidal input  $\rightarrow$  sinusoidal response with the same frequency and with amplitude and phase determined by the system  $x[n] = A \cos(\omega_0 n + \phi)$

- Complex exponential sequences are eigenfunctions of LTI systems.  $x[n] = e^{j\omega n}$

$\rightarrow$  signal representation based on sinusoids or complex exponentials

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## Eigenfunctions

Complex exponentials as input to system  $h[n]$

$$x[n] = e^{j\omega n}, \quad -\infty < n < \infty$$

$$y[n] = T\{e^{j\omega n}\} = \sum_{k=-\infty}^{\infty} h[k] e^{j\omega(n-k)}$$

$$= \left( \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k} \right) e^{j\omega n}$$

$$Af = \lambda f$$

$A$  – linear operator

Define

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\omega k}$$

Then

$$y[n] = H(e^{j\omega}) e^{j\omega n}$$

Eigenvalue

Eigenfunction

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## Eigenvalue – called frequency response

- Frequency response is generally complex

$$H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})$$

$$= |H(e^{j\omega})| e^{j\angle H(e^{j\omega})} \quad \text{describes changes in magnitude and phase}$$

- Frequency response of the ideal delay system

$$y[n] = x[n - n_d] \rightarrow h[n] = \delta[n - n_d]$$

- Method 1: using the eigenfunction

$$y[n] = T\{e^{j\omega n}\} = e^{j\omega(n - n_d)} = e^{-j\omega n_d} e^{j\omega n}$$

$$H(e^{j\omega}) = e^{-j\omega n_d}$$

$$H_R(e^{j\omega}) = \cos(\omega n_d),$$

$$H_I(e^{j\omega}) = -\sin(\omega n_d).$$

- Method 2: using the impulse response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_d] e^{-j\omega n} = e^{-j\omega n_d}$$

$$|H(e^{j\omega})| = 1,$$

$$\angle H(e^{j\omega}) = -\omega n_d.$$

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## Frequency response

- The frequency response of **discrete-time** LTI systems is always a periodic function of the frequency variable  $\omega$  with period  $2\pi$ .

$$H(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n} e^{-j2\pi n} = H(e^{j\omega})$$

- Only specify over the interval  $-\pi < \omega \leq \pi$
- The 'low frequencies' are close to 0
- The 'high frequencies' are close to  $\pm\pi$

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## Ideal frequency-selective filters

- For which the frequency response is unity over a certain range of frequencies, and is zero at the remaining frequencies.
  - Ideal low-pass filter: passes only low and rejects high

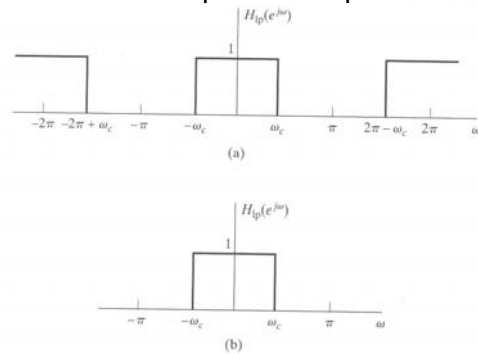


Figure 2.17 Ideal lowpass filter showing (a) periodicity of the frequency response and (b) one period of the periodic frequency response.

## Ideal frequency-selective filters

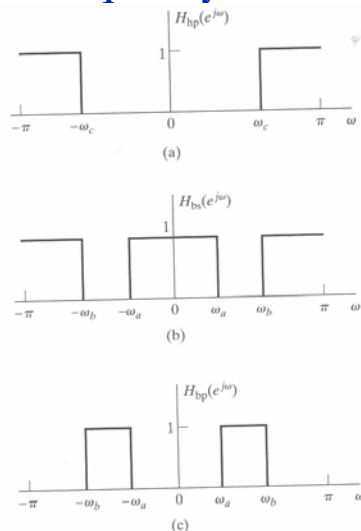


Figure 2.18 Ideal frequency-selective filters. (a) Highpass filter. (b) Bandstop filter. (c) Bandpass filter. In each case, the frequency response is periodic with period  $2\pi$ . Only one period is shown.



## Sinusoidal response of LTI systems

- Sinusoidal input  $\rightarrow$  sinusoidal response with the same frequency and with amplitude and phase determined by the system

$$\begin{aligned}x[n] &= A \cos(\omega_0 n + \phi) \\ &= \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n} \\ y[n] &= \frac{A}{2} e^{j\phi} H(e^{j\omega_0}) e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} H(e^{-j\omega_0}) e^{-j\omega_0 n}\end{aligned}$$

if  $h[n]$  is real, then

$$H(e^{-j\omega_0}) = H^*(e^{j\omega_0}) = |H(e^{j\omega_0})| e^{-\angle H(e^{j\omega_0})} = |H(e^{j\omega_0})| e^{-\theta}$$

$$\text{then } y[n] = A |H(e^{j\omega_0})| \cos(\omega_0 n + \phi + \theta)$$

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## Signal representation

- More than sinusoids, a broad class of signals can be represented as a linear combination of complex exponentials:

$$\begin{aligned}x[n] &= \sum_k \alpha_k e^{j\omega_k n} \\ \therefore y[n] &= \sum_k \alpha_k H(e^{j\omega_k}) e^{j\omega_k n}\end{aligned}$$

- If  $x[n]$  can be represented as a superposition of complex exponentials, output  $y[n]$  can be computed by using the frequency response, which is similar to the function of impulse response.

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## Frequency-domain representation of $x[n]$

- By Fourier transforms  $\rightarrow$  Fourier representation:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Inverse Fourier transform  
 $\omega$  is continuous-time variable

where  $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

Fourier transform  
 $n$  is discrete-time variable

represent  $x[n]$  as

a superposition of infinitesimally small complex sinusoids

$$\frac{1}{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- In general, Fourier transform is complex

$$\begin{aligned} X(e^{j\omega}) &= X_R(e^{j\omega}) + jX_I(e^{j\omega}) \\ &= |X(e^{j\omega})| e^{j\angle X(e^{j\omega})} \end{aligned}$$

Fourier spectrum  
Spectrum  
Magnitude spectrum  
Amplitude spectrum  
Phase spectrum

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## Frequency and impulse responses

- Are a Fourier transform pair

Recall  $H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$

$$\therefore h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega$$

- Fourier transform is periodic with period  $2\pi$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

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## Sufficient condition for Fourier transform

- Condition for the convergence of the infinite sum

$$\begin{aligned} |X(e^{j\omega})| &= \left| \sum_{-\infty}^{\infty} x[n]e^{-j\omega n} \right| \\ &\leq \sum_{-\infty}^{\infty} |x[n]| |e^{-j\omega n}| \\ &\leq \sum_{-\infty}^{\infty} |x[n]| < \infty \end{aligned}$$

- $X[n]$  is absolutely summable, then its Fourier transform exists (sufficient condition).

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## Example: ideal lowpass filter

- Frequency response  $H_{lp}(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \omega_c < |\omega| \leq \pi \end{cases}$

$$h_{lp}[n] = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- Noncausal.
- Not absolutely summable.

$$H_{lp}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

$$H_M(e^{j\omega}) = \sum_{n=-M}^M \frac{\sin \omega_c n}{\pi n} e^{-j\omega n}$$

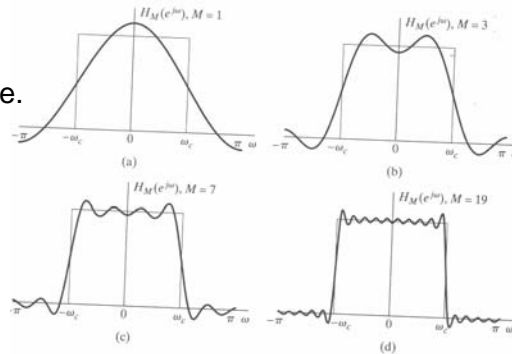


Figure 2.21 Convergence of the Fourier transform. The oscillatory behavior at  $\omega = \omega_c$  is often called the Gibbs phenomenon.

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## Example: Fourier transform of a constant

- Constant sequence

$$x[n] = 1$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

$$\text{where } X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- Its Fourier transform is defined as the periodic impulse train

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} 2\pi \delta(\omega + 2\pi r)$$

## Part III: Fourier transforms

- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response
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  - Fourier transform theorems

## Symmetry properties of Fourier transform

- Conjugate-symmetric sequence  $x_e[n] = x_e^*[-n]$
- Conjugate-antisymmetric sequence  $x_o[n] = -x_o^*[-n]$
- Any  $x[n]$ ,  $x[n] = \frac{1}{2}(x[n] + x^*[-n]) + \frac{1}{2}(x[n] - x^*[-n])$

$$\text{define } x_e[n] = \frac{1}{2}(x[n] + x^*[-n]) = x_e^*[-n]$$

$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) = -x_o^*[-n]$$

$$\therefore x[n] = x_e[n] + x_o[n]$$

$$X(e^{j\omega}) = X_e(e^{j\omega}) + X_o(e^{j\omega})$$

where

$$X_e(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) + X^*(e^{-j\omega})) = X_e^*(e^{-j\omega})$$

$$X_o(e^{j\omega}) = \frac{1}{2}(X(e^{j\omega}) - X^*(e^{-j\omega})) = -X_o^*(e^{-j\omega})$$

**TABLE 2.1** SYMMETRY PROPERTIES OF THE FOURIER TRANSFORM

Sequence $x[n]$	Fourier Transform $X(e^{j\omega})$
1. $x^*[n]$	$X^*(e^{-j\omega})$
2. $x^*[-n]$	$X^*(e^{j\omega})$
3. $\mathcal{R}\{x[n]\}$	$X_e(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$ )
4. $j\mathcal{I}\{x[n]\}$	$X_o(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$ )
5. $x_e[n]$ (conjugate-symmetric part of $x[n]$ )	$X_R(e^{j\omega}) = \mathcal{R}\{X(e^{j\omega})\}$
6. $x_o[n]$ (conjugate-antisymmetric part of $x[n]$ )	$jX_I(e^{j\omega}) = j\mathcal{I}\{X(e^{j\omega})\}$
<i>The following properties apply only when <math>x[n]</math> is real:</i>	
7. Any real $x[n]$	$X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric)
8. Any real $x[n]$	$X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even)
9. Any real $x[n]$	$X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd)
10. Any real $x[n]$	$ X(e^{j\omega})  =  X(e^{-j\omega}) $ (magnitude is even)
11. Any real $x[n]$	$\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd)
12. $x_e[n]$ (even part of $x[n]$ )	$X_R(e^{j\omega})$
13. $x_o[n]$ (odd part of $x[n]$ )	$jX_I(e^{j\omega})$

## Part III: Fourier transforms

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- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response
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## Linearity of the Fourier Transform

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$$x_1[n] \stackrel{F}{\leftrightarrow} X_1(e^{j\omega})$$

$$x_2[n] \stackrel{F}{\leftrightarrow} X_2(e^{j\omega})$$

$$ax_1[n] + bx_2[n] \stackrel{F}{\leftrightarrow} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

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## Time shifting and frequency shifting

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$$x[n] \stackrel{F}{\leftrightarrow} X(e^{j\omega})$$

$$x[n - n_d] \stackrel{F}{\leftrightarrow} e^{-j\omega n_d} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \stackrel{F}{\leftrightarrow} X(e^{j(\omega - \omega_0)})$$

## Time reversal

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$$x[n] \stackrel{F}{\leftrightarrow} X(e^{j\omega})$$

$$x[-n] \stackrel{F}{\leftrightarrow} X(e^{-j\omega})$$

if  $x[n]$  is real.

$$x[-n] \stackrel{F}{\leftrightarrow} X^*(e^{j\omega})$$

## Differentiation in frequency

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$$x[n] \stackrel{F}{\leftrightarrow} X(e^{j\omega})$$

$$nx[n] \stackrel{F}{\leftrightarrow} j \frac{dX(e^{j\omega})}{d\omega}$$

## Parseval's theorem

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$$x[n] \stackrel{F}{\leftrightarrow} X(e^{j\omega})$$

$$E = \sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$



## The convolution theorem

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$$x[n] \stackrel{F}{\leftrightarrow} X(e^{j\omega})$$

$$h[n] \stackrel{F}{\leftrightarrow} H(e^{j\omega})$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

## The modulation or Windowing theorem

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$$x[n] \stackrel{F}{\leftrightarrow} X(e^{j\omega})$$

$$w[n] \stackrel{F}{\leftrightarrow} W(e^{j\omega})$$

$$y[n] = x[n]w[n]$$

$$Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})d\theta$$

**TABLE 2.2** FOURIER TRANSFORM THEOREMS

Sequence	Fourier Transform
$x[n]$	$X(e^{j\omega})$
$y[n]$	$Y(e^{j\omega})$
1. $ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
2. $x[n - n_d]$ ( $n_d$ an integer)	$e^{-j\omega n_d} X(e^{j\omega})$
3. $e^{j\omega_0 n} x[n]$	$X(e^{j(\omega - \omega_0)})$
4. $x[-n]$	$X(e^{-j\omega})$ $X^*(e^{j\omega})$ if $x[n]$ real.
5. $nx[n]$	$j \frac{dX(e^{j\omega})}{d\omega}$
6. $x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
7. $x[n]y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\theta})Y(e^{j(\omega - \theta)})d\theta$
Parseval's theorem:	
8. $\sum_{n=-\infty}^{\infty}  x[n] ^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi}  X(e^{j\omega}) ^2 d\omega$	
9. $\sum_{n=-\infty}^{\infty} x[n]y^*[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$	



**TABLE 2.3** FOURIER TRANSFORM PAIRS

Sequence	Fourier Transform
1. $\delta[n]$	1
2. $\delta[n - n_0]$	$e^{-j\omega n_0}$
3. 1 ( $-\infty < n < \infty$ )	$\sum_{k=-\infty}^{\infty} 2\pi \delta(\omega + 2\pi k)$
4. $a^n u[n]$ ( $ a  < 1$ )	$\frac{1}{1 - ae^{-j\omega}}$
5. $u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi \delta(\omega + 2\pi k)$
6. $(n + 1)a^n u[n]$ ( $ a  < 1$ )	$\frac{1}{(1 - ae^{-j\omega})^2}$
7. $\frac{r^n \sin \omega_p (n + 1)}{\sin \omega_p} u[n]$ ( $ r  < 1$ )	$\frac{1}{1 - 2r \cos \omega_p e^{-j\omega} + r^2 e^{-j2\omega}}$
8. $\frac{\sin \omega_c n}{\pi n}$	$X(e^{j\omega}) = \begin{cases} 1, &  \omega  < \omega_c \\ 0, & \omega_c <  \omega  \leq \pi \end{cases}$
9. $x[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin[\omega(M + 1)/2]}{\sin(\omega/2)} e^{-j\omega M/2}$
10. $e^{j\omega_0 n}$	$\sum_{k=-\infty}^{\infty} 2\pi \delta(\omega - \omega_0 + 2\pi k)$
11. $\cos(\omega_0 n + \phi)$	$\sum_{k=-\infty}^{\infty} [\pi e^{j\phi} \delta(\omega - \omega_0 + 2\pi k) + \pi e^{-j\phi} \delta(\omega + \omega_0 + 2\pi k)]$



## Example

- Determining a Fourier transform using Tables 2.2 and 2.3

$$\underline{x[n] = a^n u[n-5]}$$

$$x_1[n] = a^n u[n] \xleftrightarrow{F} X_1(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}$$

$$x_2[n] = x_1[n-5] \quad (\text{i.e.} = a^{n-5} u[n-5])$$

$$X_2(e^{j\omega}) = e^{-j5\omega} X_1(e^{j\omega}) = \frac{e^{-j5\omega}}{1 - ae^{-j\omega}}$$

$$x[n] = a^5 x_2[n] \quad (\text{i.e.} = a^n u[n-5])$$

$$X(e^{j\omega}) = \frac{a^5 e^{-j5\omega}}{1 - ae^{-j\omega}}$$

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## Summary

- System impulse response
- Linear constant-coefficient difference equations
- Fourier transforms and frequency response
  - Frequency-domain representation of discrete-time signals and systems
  - Symmetry properties of the Fourier transform
  - Fourier transform theorems

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# Course at a glance

