

# Digital Signal Processing, Fall 2006

## Lecture 5: System analysis

Zheng-Hua Tan

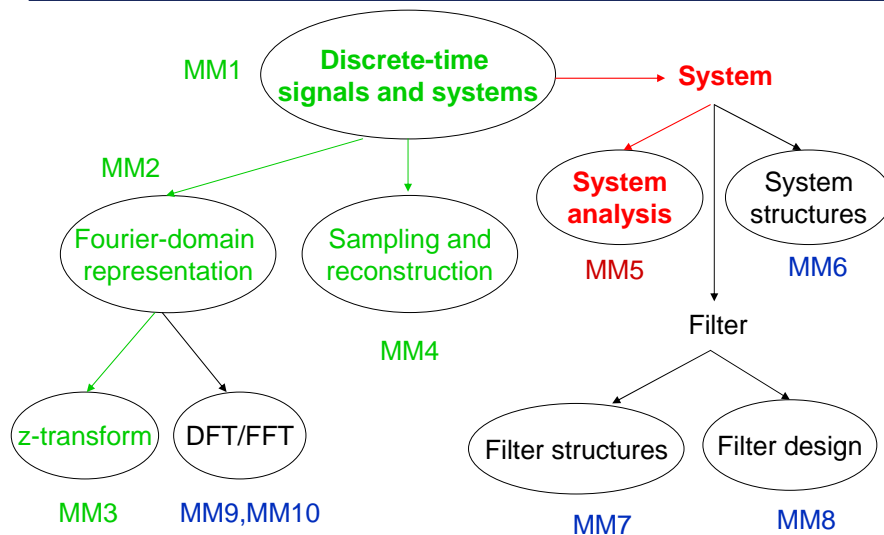
Department of Electronic Systems  
Aalborg University, Denmark  
[zt@kom.aau.dk](mailto:zt@kom.aau.dk)

1

Digital Signal Processing, V, Zheng-Hua Tan, 2006



## Course at a glance



2

Digital Signal Processing, V, Zheng-Hua Tan, 2006



## System analysis

---

- Three domains
  - Time domain: impulse response, convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- Frequency domain: **frequency response**

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- z-transform: **system function**

$$Y(z) = X(z)H(z)$$

- **LTI system** is completely characterized by ...

3

Digital Signal Processing, V, Zheng-Hua Tan, 2006



## Part I: Frequency response

---

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

4

Digital Signal Processing, V, Zheng-Hua Tan, 2006



## Frequency response

- Relationship btw Fourier transforms of input and output

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

- In polar form
  - Magnitude → magnitude response, gain, distortion

$$|Y(e^{j\omega})| = |X(e^{j\omega})| \cdot |H(e^{j\omega})|$$

- Phase → phase response, phase shift, distortion

$$\angle Y(e^{j\omega}) = \angle X(e^{j\omega}) + \angle H(e^{j\omega})$$

## Ideal lowpass filter – an example

- Frequency response

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

- Frequency selective filter

- Impulse response

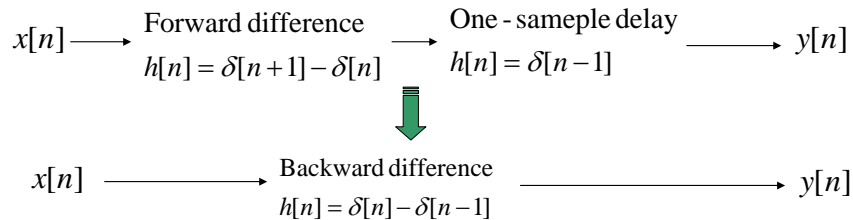
$$h_p[n] = \frac{\sin \omega_c n}{\pi n}, \quad -\infty < n < \infty$$

- Noncausal, cannot be implemented!  $h[n] = 0, \quad n < 0$
- How to make a noncausal system causal?

## Make noncausal system causal

- Cascading systems

- Ideal delay  $h[n] = \delta[n - n_d]$



- In general, any noncausal FIR system can be made cause by cascading it with a sufficiently long delay!
- But ideal lowpass filter is an IIR system!

## Phase distortion and delay

- Ideal delay system

$$h_{id}[n] = \delta[n - n_d]$$

Delay distortion

$$H_{id}(e^{j\omega}) = e^{-j\omega n_d}$$

$$|H_{id}(e^{j\omega})| = 1$$

$$\angle H_{id}(e^{j\omega}) = -\omega n_d, |\omega| < \pi$$

Linear phase distortion

- Ideal lowpass filter with linear phase

$$H_{lp}(e^{j\omega}) = \begin{cases} e^{-j\omega n_d}, & |\omega| < \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$

Ideal lowpass filter is always noncausal!

$$h_{lp}[n] = \frac{\sin \omega_c (n - n_d)}{\pi (n - n_d)}, \quad -\infty < n < \infty$$

## Group delay

- A measure of the linearity of the phase
- Concerning the phase distortion on a **narrowband** signal

$$x[n] = s[n] \cos(\omega_0 n)$$



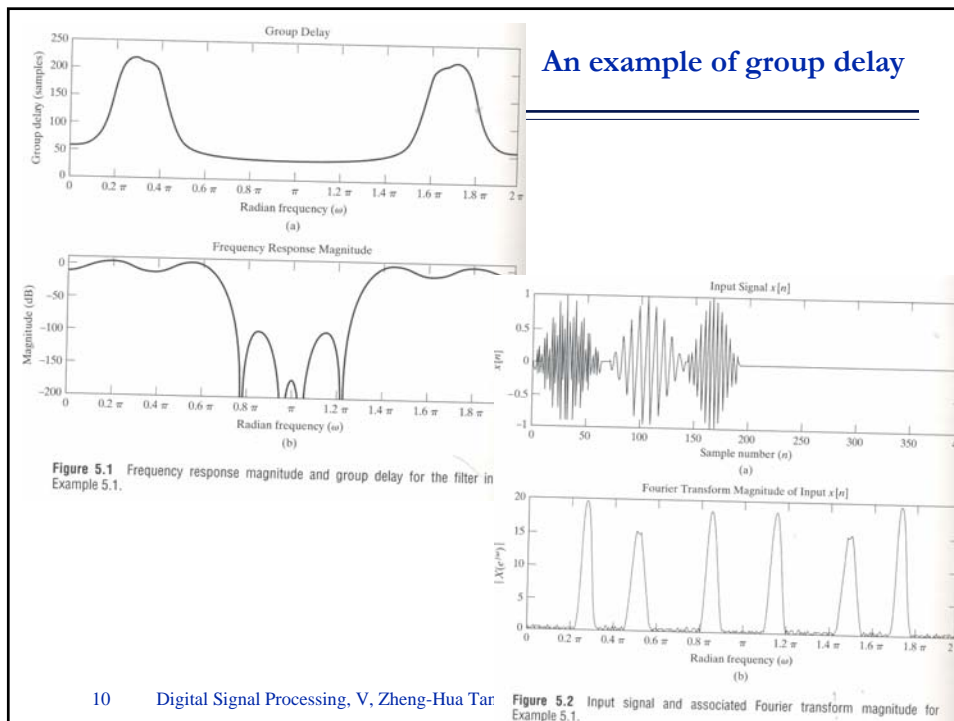
- For this input with spectrum only around  $w_0$ , phase effect can be approximated around  $w_0$  as the **linear approximation** (though in reality maybe nonlinear)

$$\angle H(e^{j\omega}) \approx -\omega n_d - \phi_0$$

and the output is approximately

$$y[n] \approx |H(e^{j\omega_0})| s[n - n_d] \cos(\omega_0(n - n_d) - \phi_0)$$

- Group delay  $\tau = \text{grd}[H(e^{j\omega})] = -\frac{d}{d\omega} \{\arg[H(e^{j\omega})]\}$



## An example of group delay

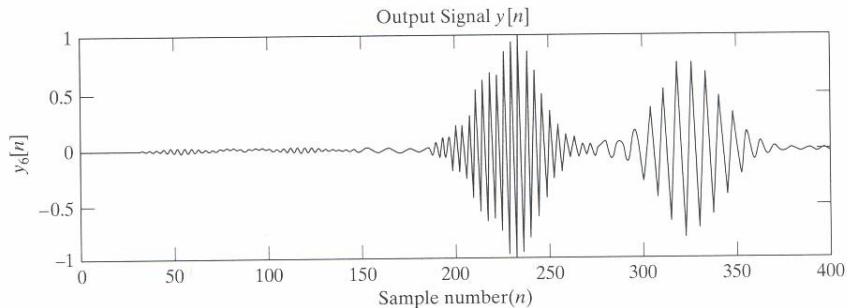


Figure 5.3 Output signal for Example 5.1.

Since the filter has considerable attenuation at  $\omega = 0.85\pi$ , the pulse at that frequency is not clearly present in the output. Also, since the group delay at  $\omega = 0.25\pi$  is approximately 200 samples and at  $\omega = 0.5\pi$  is approximately 50 samples, the second pulse in  $x[n]$  will be delayed by about 200 samples and the third pulse by 50 samples, as we see is the case in Figure 5.3.

## Part II: System functions

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

## System function of LCCDE systems

- Linear constant-coefficient difference equation

$$\sum_{k=0}^N a_k y[n-k] = \sum_{m=0}^M b_m x[n-m]$$

- z-transform format

$$\sum_{k=0}^N a_k z^{-k} Y(z) = \sum_{m=0}^M b_m z^{-m} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{m=0}^M b_m z^{-m}}{\sum_{k=0}^N a_k z^{-k}}$$

$$= \frac{b_0 \prod_{m=1}^M (1 - c_m z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})}$$

$(1 - c_m z^{-1})$  in the numerator

a zero at  $z = c_m$  a pole at  $z = 0$

$(1 - d_k z^{-1})$  in the denominator

a zero at  $z = 0$  a pole at  $z = d_k$

## Stability and causality

- Stable
  - $h[n]$  absolutely summable
  - $H(z)$  has a ROC including the unit circle
- Causal
  - $h[n]$  right side sequence
  - $H(z)$  has a ROC being outside the outermost pole

## Inverse systems

- Many systems have inverses, specially systems with rational system functions

$$G(z) = H(z)H_i(z) = 1$$

$$H_i(z) = \frac{1}{H(z)}$$

$$g[n] = h[n] * h_i[n] = \delta[n]$$

$$H(z) = \left(\frac{b_0}{a_0}\right) \frac{\prod_{m=1}^M (1 - c_m z^{-1})}{\prod_{k=1}^N (1 - d_k z^{-1})}$$

$$H_i(z) = \left(\frac{a_0}{b_0}\right) \frac{\prod_{k=1}^N (1 - d_k z^{-1})}{\prod_{m=1}^M (1 - c_m z^{-1})}$$

- Poles become zeros and vice versa.
- ROC: must have overlap btw the two for the sake of  $G(z)$ .

## Example

$$H(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}, |z| > 0.9$$

$$H_i(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}} = \frac{1}{1 - 0.5z^{-1}} - \frac{0.9z^{-1}}{1 - 0.5z^{-1}}$$

So,  $|z| > 0.5$

$$h_i[n] = (0.5)^n u[n] - 0.9(0.5)^{n-1} u[n-1]$$



## Part III: Magnitude and phase

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

## Relationship btw magnitude and phase

- In particular, for systems with rational system functions, there is constraint btw magnitude and phase

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{j\angle H(e^{j\omega})}$$

- Consider the square of the magnitude

$$|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H(z)H^*(1/z^*) \Big|_{z=e^{j\omega}}$$

$$H(z) = \frac{b_0 \prod_{m=1}^M (1 - c_m z^{-1})}{a_0 \prod_{k=1}^N (1 - d_k z^{-1})} \quad H^*(1/z^*) = \frac{b_0 \prod_{m=1}^M (1 - c_m^* z)}{a_0 \prod_{k=1}^N (1 - d_k^* z)}$$

$$C(z) = H(z)H^*(1/z^*) = \frac{b_0^2 \prod_{m=1}^M (1 - c_m z^{-1})(1 - c_m^* z)}{a_0^2 \prod_{k=1}^N (1 - d_k z^{-1})(1 - d_k^* z)}$$

### Example 5.11 Systems with the Same $C(z)$

Consider two stable systems with system functions

$$H_1(z) = \frac{2(1 - z^{-1})(1 + 0.5z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})} \quad (5.83)$$

and

$$H_2(z) = \frac{(1 - z^{-1})(1 + 2z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}. \quad (5.84)$$

The pole-zero plots for these systems are shown in Figures 5.19(a) and 5.19(b), respectively.

Now,

$$\begin{aligned} C_1(z) &= H_1(z)H_1^*(1/z^*) \\ &= \frac{2(1 - z^{-1})(1 + 0.5z^{-1})2(1 - z)(1 + 0.5z)}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z)(1 - 0.8e^{j\pi/4}z)} \end{aligned} \quad (5.85)$$

and

$$\begin{aligned} C_2(z) &= H_2(z)H_2^*(1/z^*) \\ &= \frac{(1 - z^{-1})(1 + 2z^{-1})(1 - z)(1 + 2z)}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z)(1 - 0.8e^{j\pi/4}z)}. \end{aligned} \quad (5.86)$$



### An example

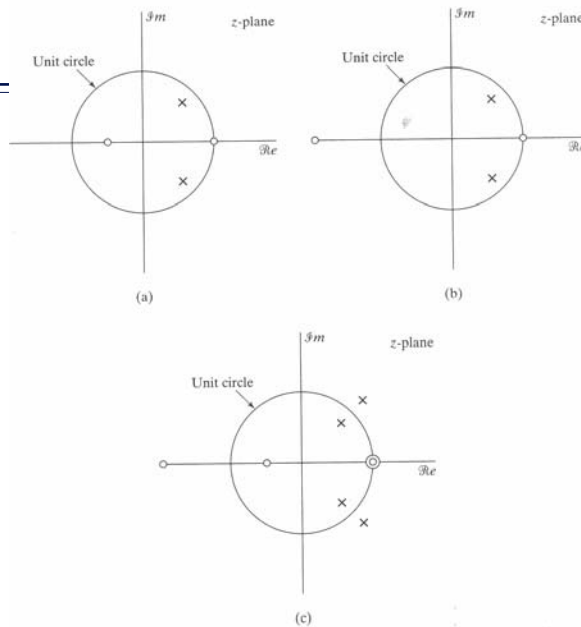


Figure 5.19 Pole-zero plots for two system functions and their common magnitude-squared function. (a)  $H_1(z)$ . (b)  $H_2(z)$ . (c)  $C_1(z)$ ,  $C_2(z)$ .



## Part VI: All-pass systems

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

## All-pass systems

- Consider the following stable system function

$$H_{ap}(z) = \frac{z^{-1} - a^*}{1 - az^{-1}}$$

$$\begin{aligned} H_{ap}(e^{j\omega}) &= \frac{e^{-j\omega} - a^*}{1 - ae^{-j\omega}} \\ &= e^{-j\omega} \frac{1 - a^* e^{j\omega}}{1 - ae^{-j\omega}} \end{aligned}$$

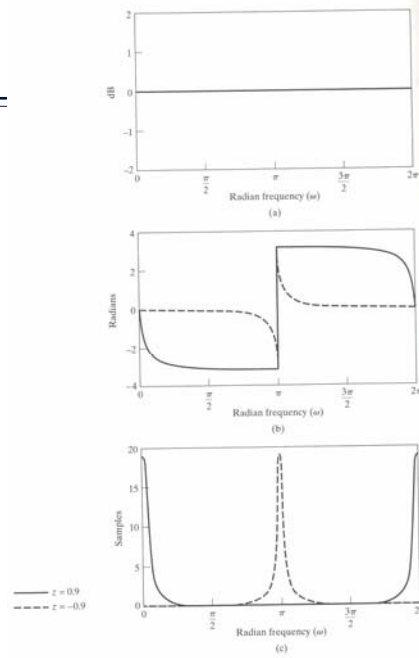
- $|H_{ap}(e^{j\omega})| = 1$  all-pass system: for which the frequency response magnitude is a constant.

- General form

$$H_{ap}(z) = A \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k^*)(z^{-1} - e_k)}{(1 - e_k z^{-1})(1 - e_k^* z^{-1})}$$

## An example

### P275 Example 5.13, First-order all-pass system

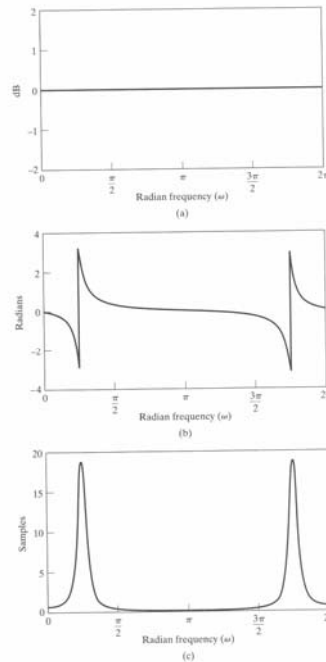


23 Digital Signal Processing, V, Zheng-Hua Tan

Figure 5.22 Frequency response for all-pass filters with real poles at  $z = 0.9$  (solid line) and  $z = -0.9$  (dashed line). (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

## An example

### Second-order all-pass system



24 Digital Signal Processing, V, Zheng-Hua Tan, 20

Figure 5.23 Frequency response of second-order all-pass system with poles at  $z = 0.9e^{j\pi/4}$ . (a) Log magnitude. (b) Phase (principal value). (c) Group delay.

## Part V: Minimum-phase systems

---

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

## Minimum-phase systems

---

- Magnitude does not uniquely characterize the system
  - Stable and causal  $\rightarrow$  poles inside unit circle, no restriction on zeros
  - **Zeros** are also inside unit circle  $\rightarrow$  inverse system is also stable and causal (in many situations, we need inverse systems!)
  - $\rightarrow$  such systems are called minimum-phase systems (explanation to follow): **are stable and causal and have stable and causal inverses**

## Minimum-phase and all-pass decomposition

Any rational system function can be expressed as:

$$H(z) = H_{\min}(z)H_{ap}(z)$$

Suppose  $H(z)$  has one zero outside the unit circle at  $z = 1/c^*$ ,  $|c| < 1$

$$\begin{aligned} H(z) &= H_1(z)(z^{-1} - c^*) \\ &= \underbrace{H_1(z)(1 - cz^{-1})}_{\text{minimum-phase}} \underbrace{\frac{z^{-1} - c^*}{1 - cz^{-1}}}_{\text{all-pass}} \end{aligned}$$

## Frequency response compensation

When the distortion system is not minimum-phase system:

$$H_d(z) = H_{d\min}(z)H_{ap}(z) \quad H_c(z) = \frac{1}{H_{d\min}(z)}$$

$$G(z) = H_d(z)H_c(z) = H_{ap}(z)$$

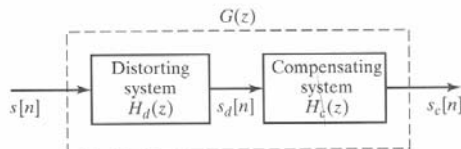


Figure 5.25 Illustration of distortion compensation by linear filtering.

Frequency response magnitude is compensated  
Phase response is the phase of the all-pass

## Properties of minimum-phase systems

---

- From minimum-phase and all-pass decomposition

$$H(z) = H_{\min}(z)H_{ap}(z)$$

$$\arg[H(e^{j\omega})] = \arg[H_{\min}(e^{j\omega})] + \arg[H_{ap}(e^{j\omega})]$$

- From previous figures, the **continuous-phase** curve of an all-pass system is negative for  $0 \leq \omega \leq \pi$
- So change from minimum-phase to nonminimum-phase (+all-pass phase) always decreases the continuous phase or increases the negative of the phase (called the phase-lag function). Minimum-phase is more precisely called minimum phase-lag system

## Part VI: Linear-phase systems

---

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

## Design a system with non-zero phase

- System design sometimes desires
  - Constant frequency response magnitude
  - Zero phase, when not possible
    - accept phase distortion, in particular **linear phase** since it only introduce time shift
    - Nonlinear phase will change the shape of the input signal though having constant magnitude response

## Ideal delay

$$H_{id}(e^{j\omega}) = e^{-j\omega\alpha}, |\omega| < \pi$$

$$|H_{id}(e^{j\omega})| = 1$$

$$\angle H_{id}(e^{j\omega}) = -\omega\alpha, |\omega| < \pi$$

$$\text{grd}[H_{id}(e^{j\omega})] = \alpha$$

$$h_{id}[n] = \frac{\sin \pi(n - \alpha)}{\pi(n - \alpha)}$$

when  $\alpha = n_d$

$$h_{id}[n] = \delta[n - n_d]$$

Ideal lowpass with linear phase

$$h_{lp}[n] = \frac{\sin \omega_c(n - n_d)}{\pi(n - n_d)}$$



## Generalized linear phase

---

- Linear phase filters

$$H(e^{j\omega}) = |H(e^{j\omega})| e^{-j\omega\alpha}$$

- Generalized linear phase filters

$$H(e^{j\omega}) = A(e^{j\omega}) e^{-j\omega\alpha + j\beta}$$

$A(e^{j\omega})$  is a real function of  $\omega$ ,

$\alpha$  and  $\beta$  are real constants

## Summary

---

- Frequency response
- System functions
- Relationship between magnitude and phase
- All-pass systems
- Minimum-phase systems
- Linear systems with generalized linear phase

## Course at a glance

