Digital Signal Processing, Fall 2006

Lecture 9: The Discrete Fourier Transform

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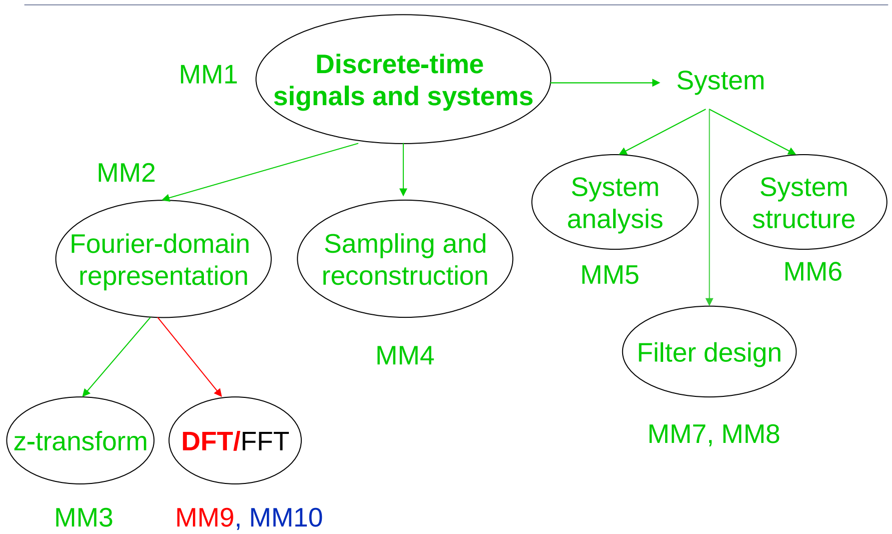
Course at a glance



The discrete-time Fourier transform (DTFT)

- The DTFT is useful for the theoretical analysis of signals and systems.
- But, according to its definition

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

computation of DTFT by computer has several problems:
  - The summation over n is infinite
  - The independent variable w is continuous

The discrete Fourier transform (DFT)

- In many cases, only finite duration is of concern
  - The signal itself is finite duration
  - Only a segment is of interest at a time
  - Signal is periodic and thus only finite unique values
- For finite duration sequences, an alternative Fourier representation is DFT
  - The summation over n is finite
  - DFT itself is a sequence, rather than a function of a continuous variable
  - Therefore, DFT is computable and important for the implementation of DSP systems
  - DFT corresponds to samples of the Fourier transform

Part I: The discrete Fourier series

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

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The discrete Fourier series

- A periodic sequence with period N

$$\tilde{x}[n] = \tilde{x}[n + rN]$$

- Periodic sequence can be represented by a Fourier series, i.e. a sum of complex exponential sequences with frequencies being integer multiples of the fundamental frequency $(2\pi / N)$ associated with the $\tilde{x}[n]$

$$\tilde{x}[n] = \frac{1}{N}\sum_{k} \tilde{X}[k] e^{j(2\pi/N)kn}$$

The frequency of the periodic sequence.

  - Only N unique harmonically related complex exponentials since

$$e^{j(2\pi/N)(k+mN)n} = e^{j(2\pi/N)kn} e^{j2\pi mn} = e^{j(2\pi/N)kn}$$

  - so $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$

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The Fourier series coefficients

- The coefficients

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$$

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)kn}$$

- The sequence is periodic with period N

$$\tilde{X}[k+N] = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j(2\pi/N)(k+N)n} = \tilde{X}[k]$$

- For convenience, define $W_N = e^{-j(2\pi/N)}$

Synthesis equation $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$

Analysis equation $\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$

Very similar equations → duality

DFS of a periodic impulse train

- Periodic impulse train

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$$

- The discrete Fourier series coefficients

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = 1$$

- By using synthesis equation, an alternative representation of $\tilde{x}[n]$ is

$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-kn} = \frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi/N)kn}$$

Part II: The Fourier transform of periodic signals

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

The Fourier transform of periodic signals

- Fourier transform of complex exponentials

$$x[n] = \sum_k a_k e^{j\omega_k n}, \qquad -\infty < n < \infty$$

$$X(e^{j\omega}) = \sum_{r=-\infty}^{\infty} \sum_k 2\pi a_k \delta(\omega - \omega_k + 2\pi r)$$

- Fourier transform of $\tilde{x}[n]$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{j(2\pi/N)kn}$$

$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \tilde{X}[k] \delta(\omega - \frac{2\pi k}{N})$$

$\tilde{X}(e^{j\omega})$ has the required periodicity with period 2π

Fourier transform of a periodic impulse train

- Periodic impulse train $\tilde{p}[n] = \sum_{r=-\infty}^{\infty} \delta[n - rN]$
  - The discrete Fourier series coefficients $$\tilde{P}[k] = \sum_{n=0}^{N-1} \delta[n] W_N^{kn} = 1$$
  - Fourier transform $$\tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \delta(\omega - \frac{2\pi k}{N})$$
- Finite duration signal $x[n]$ ($x[n] = 0$ outside of $[0, N-1]$)
  - Construct $\tilde{x}[n]$ $$\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_{r=-\infty}^{\infty} \delta(n - rN) = \sum_{r=-\infty}^{\infty} x(n - rN)$$
  - Its Fourier transform $$\tilde{X}(e^{j\omega}) = X(e^{j\omega}) \tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X(e^{j(2\pi/N)k}) \delta(\omega - \frac{2\pi k}{N})$$

The Fourier transform of periodic signals

- Compare
$$\tilde{X}(e^{j\omega}) = X(e^{j\omega}) \tilde{P}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} X(e^{j(2\pi/N)k}) \delta(\omega - \frac{2\pi k}{N})$$
$$\tilde{X}(e^{j\omega}) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{N} \tilde{X}[k] \delta(\omega - \frac{2\pi k}{N})$$
→ First represent it as Fourier series and then calculate Fourier transform
- Conclude that
$$\tilde{X}[k] = X(e^{j(2\pi/N)k}) = X(e^{j\omega})|_{\omega=(2\pi/N)k}$$

i.e. the DFS coefficients of $\tilde{x}[n]$ are ↔ samples of the Fourier transform of the one period of $\tilde{x}[n]$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Part III: Sampling the Fourier transform

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

Sampling the Fourier transform

- An aperiodic sequence and its Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \leftrightarrow x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

- Sampling the Fourier transform

$$\tilde{X}[k] = X(e^{j\omega})|_{\omega=(2\pi/N)k} = X(e^{j(2\pi/N)k})$$

  - generates a periodic sequence in k with period N since the Fourier transform is periodic in ω with period 2π

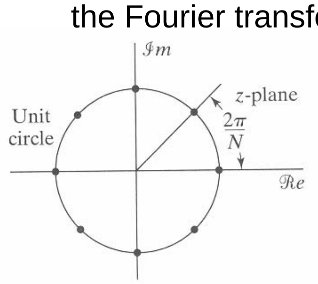

Figure 8.7 Points on the unit circle at which $X(z)$ is sampled to obtain the periodic sequence $\tilde{X}[k]$ ($N = 8$).

Sampling the Fourier transform

- Now we want to see if the sampling sequence $\tilde{X}[k]$ is the sequence of DFS coefficients of a sequence $\tilde{x}[n]$ this can be done by using the synthesis equation

$$\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$$
$$=\frac{1}{N}\sum_{k=0}^{N-1}[\sum_{m=-\infty}^{\infty}x[m]e^{-j(2\pi/N)km}]W_N^{-kn}$$
$$=\sum_{m=-\infty}^{\infty}x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1}W_N^{-k(n-m)}\right]=\sum_{m=-\infty}^{\infty}x[m]\tilde{p}[n-m]$$
$$=\sum_{r=-\infty}^{\infty}x[n-rN]$$
$$=\tilde{x}[n]$$

A periodic sequence resulting from aperiodic convolution

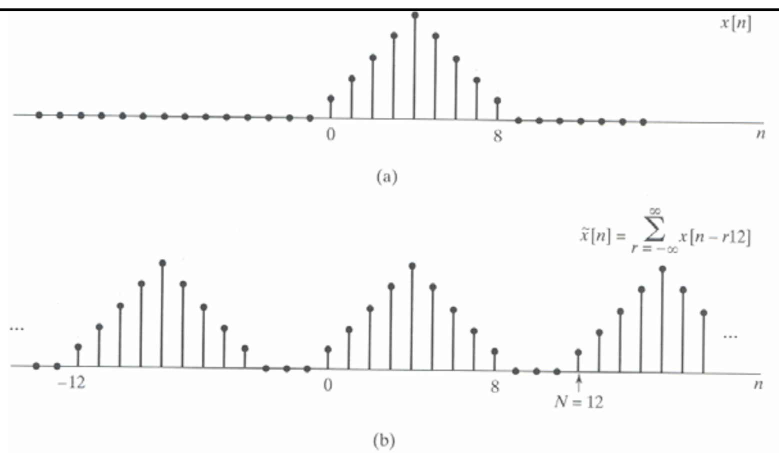

Figure 8.8 (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ with $N = 12$.

- In this case, the Fourier series coefficients for a periodic sequence are samples of the Fourier transform of one period

Examples

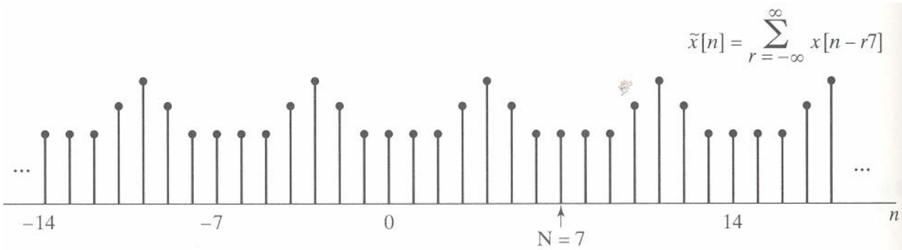

Figure 8.9 Periodic sequence $\tilde{x}[n]$ corresponding to sampling the Fourier transform of $x[n]$ in Figure 8.8(a) with $N = 7$.

- In this case, still the Fourier series coefficients for $\tilde{x}[n]$ are samples of the Fourier transform of $x[n]$. But, one period of $\tilde{x}[n]$ is no longer identical to $x[n]$
- This is just sampling in the frequency domain as compared in the time domain discussed before.

Sampling in the frequency domain

- The relationship between $x[n]$ and one period of $\tilde{x}[n]$ in the undersampled case is considered a form of time domain aliasing.
- Time domain aliasing can be avoided only if $x[n]$ has finite length, just as frequency domain aliasing can be avoided only for signals being bandlimited.
- If $x[n]$ has finite length and we take a sufficient number of equally spaced samples of its Fourier transform (specifically, a number greater than or equal to the length of $x[n]$), then the Fourier transform is recoverable from these samples, equivalently $x[n]$ is recoverable from $\tilde{x}[n]$.

Sampling in the frequency domain

- Recovering $x[n]$

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

i.e. recovering $x[n]$ does not require to know its Fourier transform at all frequencies

- Application: represent finite length sequence by using Fourier series (coefficients) → DFT

$$x[n] \to \tilde{x}[n] \to DFS, \tilde{X}[k] \to \tilde{x}[n] \to x[n]$$

Sampling the Fourier transform

- Fourier transform $X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$$

- Discrete-time Fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Discrete Fourier transform

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{j(2\pi/N)kn}$$

Part IV: The DFT

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

The discrete Fourier transform

- Consider a finite length sequence $x[n]$ of length N samples (if smaller than N, appending zeros)
  - Construct a periodic sequence

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n - rN]$$

  Assuming no overlap btw $x[n - rN]$

$$\tilde{x}[n] = x[(n \text{ modulo } N)] = x[((n))_N]$$

  - Recover the finite length sequence

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

- To maintain a duality btw the time and frequency domains, choose one period of $\tilde{X}[k]$ as the DFT

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

The DFT

- Periodic sequence and DFS coefficients

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn}$$

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

- Since summations are calculated btw 0 and (N-1)

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \le k \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Generally

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

The DFT

- A finite or periodic sequence has only N unique values, $x[n]$ for $0<=n<N$
- Spectrum is completely defined by N distinct frequency samples
- DFT: uniform sampling of DTFT spectrum

The DFT of a rectangular pulse

Example 8.7 pp.561

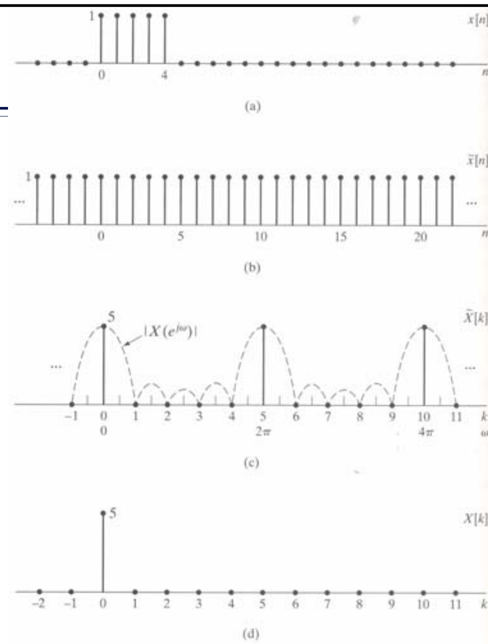

Figure 8.10 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 5$. (c) Fourier series coefficients $\tilde{X}[k]$ for $\tilde{x}[n]$. To emphasize that the Fourier series coefficients are samples of the Fourier transform, $|X(e^{j\omega})|$ is also shown. (d) DFT of $x[n]$.

The DFT of a rectangular pul

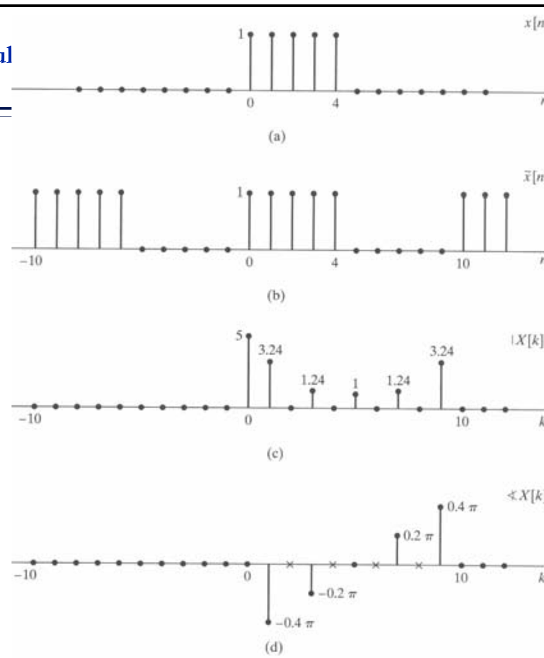

Figure 8.11 Illustration of the DFT. (a) Finite-length sequence $x[n]$. (b) Periodic sequence $\tilde{x}[n]$ formed from $x[n]$ with period $N = 10$. (c) DFT magnitude. (d) DFT phase. (x's indicate indeterminate values.)

Part V: Properties of the DFT

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

Properties of the DFT – linearity

Linearity

$$ax_1[n]+bx_2[n]\overset{DFT}{\leftrightarrow} aX_1[k]+bX_2[k]$$

The lengths of sequences and their DFTs are all equal to the maximum of the lengths of $x_1[n]$ and $x_2[n]$

Circular shift of a sequence

- Given

$$x[n] \overset{DFT}{\leftrightarrow} X[k]$$

$$x_1[n] \overset{DFT}{\leftrightarrow} X_1[k] = e^{-j(2\pi k/N)m} X[k]$$

- Then

$$x_1[n] = \begin{cases} \tilde{x}_1[n] = \tilde{x}[n-m] = x[((n-m))_N], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

an example

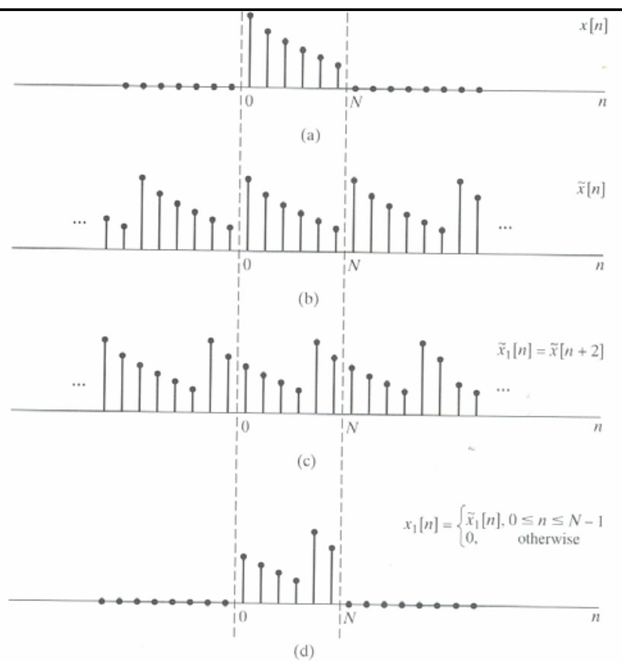

Figure 8.12 Circular shift of a finite-length sequence; i.e., the effect in the time domain of multiplying the DFT of the sequence by a linear phase factor.

Duality

$$x[n] \overset{DFT}{\leftrightarrow} X[k]$$

$$X[n] \overset{DFT}{\leftrightarrow} Nx[((-k))_N], \quad 0 \le k \le N-1$$

Circular convolution

$$x_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m], \quad 0 \le n \le N-1$$

$$= \sum_{m=0}^{N-1} x_1[((m))_N]x_2[((n-m))_N], \quad 0 \le n \le N-1$$

$$= \sum_{m=0}^{N-1} x_1[m]x_2[((n-m))_N], \quad 0 \le n \le N-1$$

- In linear convolution, one sequence is multiplied by a time –reversed and linearly shifted version of the other. For convolution here, the second sequence is circularly time reversed and circularly shifted. So it is called an N-point circular convolution

$$x_3[n] = x_1[n] \circledN x_2[n]$$

Circular convolution with a delayed impulse

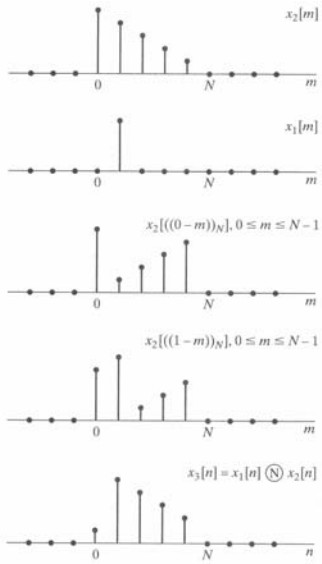

Figure 8.14 Circular convolution of a finite-length sequence $x_2[n]$ with a single delayed impulse, $x_1[n] = \delta[n-1]$.

$$x_1[n] = \delta[n - n_0]$$
$$X_1[k] = W_N^{kn_0}$$
$$X_3[k] = W_N^{kn_0} X_2[k]$$

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Summary of properties of the DFT

TABLE 8.2

Finite-Length Sequence (Length N)	N-point DFT (Length N)
1. $x[n]$	$X[k]$
2. $x_1[n], x_2[n]$	$X_1[k], X_2[k]$
3. $ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
4. $X[n]$	$Nx[((-k))_N]$
5. $x[((n-m))_N]$	$W_N^{km} X[k]$
6. $W_N^{-\ell n} x[n]$	$X[((k-\ell))_N]$
7. $\sum_{m=0}^{N-1} x_1(m) x_2[((n-m))_N]$	$X_1[k]X_2[k]$
8. $x_1[n]x_2[n]$	$\frac{1}{N}\sum_{\ell=0}^{N-1} X_1(\ell) X_2[((k-\ell))_N]$
9. $x^*[n]$	$X^*[((-k))_N]$
10. $x^*[((-n))_N]$	$X^*[k]$
11. $\mathcal{R}e\{x[n]\}$	$X_{ep}[k] = \frac{1}{2}\{X[((k))_N] + X^*[((-k))_N]\}$
12. $j\mathcal{J}m\{x[n]\}$	$X_{op}[k] = \frac{1}{2}\{X[((k))_N] - X^*[((-k))_N]\}$
13. $x_{ep}[n] = \frac{1}{2}\{x[n] + x^*[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
14. $x_{op}[n] = \frac{1}{2}\{x[n] - x^*[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$
Properties 15–17 apply only when $x[n]$ is real.	
15. Symmetry properties	$\begin{cases} X[k] = X^*[((-k))_N] \\ \mathcal{R}e\{X[k]\} = \mathcal{R}e\{X[((-k))_N]\} \\ \mathcal{J}m\{X[k]\} = -\mathcal{J}m\{X[((-k))_N]\} \\ \lvert X[k]\rvert = \lvert X[((-k))_N]\rvert \\ \sphericalangle\{X[k]\} = -\sphericalangle\{X[((-k))_N]\} \end{cases}$
16. $x_{ep}[n] = \frac{1}{2}\{x[n] + x[((-n))_N]\}$	$\mathcal{R}e\{X[k]\}$
17. $x_{op}[n] = \frac{1}{2}\{x[n] - x[((-n))_N]\}$	$j\mathcal{J}m\{X[k]\}$

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Part VI: Linear convolution of the DFT

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
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- Linear convolution using the DFT

Linear convolution using the DFT

- Procedure
  - Compute the N-point DFTs $X_1[k]$ and $X_2[k]$ of two sequences $x_1[n]$ and $x_2[n]$, respectively
  - Compute the product of $X_3[k] = X_1[k]X_2[k]$ for $0 \le k \le N-1$
  - Compute the sequence $x_3[n] = x_1[n] \textcircled{N} x_2[n]$ as the inverse DFT of $X_3[k]$
- As we know, the multiplication of DFTs corresponds to a circular convolution of the sequences. To obtain a linear convolution, we must ensure that circular convolution has the effect of linear convolution.

Linear convolution of two finite-length sequences

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

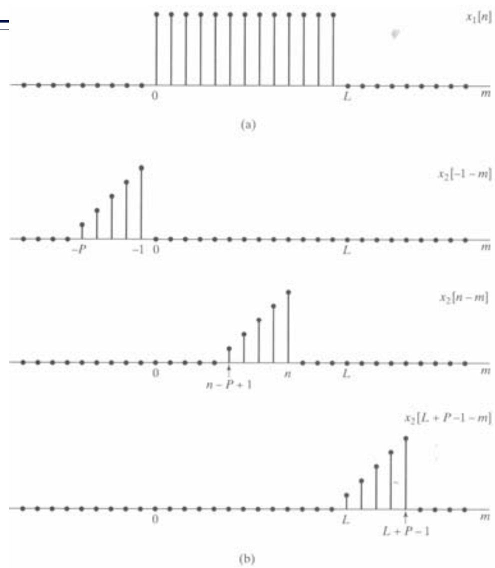

Figure 8.17 Example of linear convolution of two finite-length sequences showing that the result is such that $x_3[n] = 0$ for $n \le -1$ and for $n \ge L + P - 1$. (a) Finite-length sequence $x_1[n]$. (b) $x_2[n-m]$ for several values of n.

Circular convolution as linear convolution with alaising

Fourier transform of $x_3[n]: X_3(e^{j\omega}) = X_1(e^{j\omega})X_2(e^{j\omega})$

Define a DFT: $X_3[k] = X_3(e^{j(2\pi k/N)}), \quad 0 \le k \le N-1$

Also $X_3[k] = X_1(e^{j(2\pi k/N)})X_2(e^{j(2\pi k/N)}), \quad 0 \le k \le N-1$

So, $X_3[k] = X_1[k]X_2[k]$

the inverse DFT of $X_3[k]$:

$$x_{3p}[n] = \begin{cases} \sum_{r=-\infty}^{\infty} x_3[n-rN], & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$$x_{3p}[n] = x_1[n] \textcircled{N} x_2[n]$$

The circular convolution corresponding to $X_1[k]X_2[k]$ **is identical to the linear convolution corresponding to** $X_1(e^{j\omega})X_2(e^{j\omega})$ **if the length of DFTs satisfies** $N \ge L + P - 1$

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Circular convolution as linear convolution with aliasing

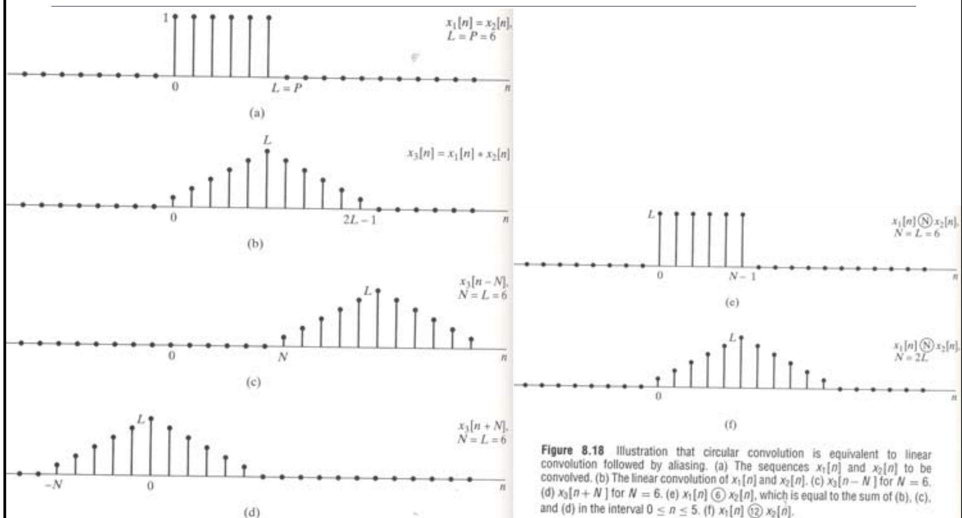

Figure 8.18 Illustration that circular convolution is equivalent to linear convolution followed by aliasing. (a) The sequences $x_1[n]$ and $x_2[n]$ to be convolved. (b) The linear convolution of $x_1[n]$ and $x_2[n]$. (c) $x_3[n-N]$ for $N = 6$. (d) $x_3[n+N]$ for $N = 6$. (e) $x_1[n]$ ⑥ $x_2[n]$, which is equal to the sum of (b), (c), and (d) in the interval $0 \le n \le 5$. (f) $x_1[n]$ ⑫ $x_2[n]$.

Summary

- The discrete Fourier series
- The Fourier transform of periodic signals
- Sampling the Fourier transform
- The discrete Fourier transform
- Properties of the DFT
- Linear convolution using the DFT

Course at a glance