Journal of
Linear and Topological Algebra
Vol. 07*, No.* 02*,* 2018*,* 75- 85

Numerical solution of a type of weakly singular nonlinear Volterra integral equation by Tau Method

H. Laeli Dastjerdi[a], M. Nili Ahmadabadi[a,*]

[a]*Department of Mathematics, Najafabad Branch, Islamic Azad University, Najafabad, Iran.*

Received 27 December 2016; Accepted 14 June 2018.
Communicated by Hamidreza Rahimi

Abstract. In this paper, a matrix based method is considered for the solution of a class of nonlinear Volterra integral equations with a kernel of the general form $s^{\beta}(t-s)^{-\alpha}G(y(s))$ based on the Tau method. In this method, a transformation of the independent variable is first introduced in order to obtain a new equation with smoother solution. Error analysis of this method is also presented. Some numerical examples are provided to illustrate the accuracy and computational efficiency of the method.

Keywords: Weakly singular, Volterra integral equations, nonlinear equations, Lighthill equation, Tau method.

2010 AMS Subject Classification: 45D05

1. Introduction

In this paper, we consider an operational approach based on the Tau Method for nonlinear Volterra integral equations of the form

$$u(t) = f(t) - \int_0^t \frac{s^{\beta}}{(t-s)^{\alpha}} G(u(s))ds, \quad 0 < \alpha < 1, \quad \beta > 0, \quad t \in [0, T], \tag{1}$$

where α, β are positive real constants. The kernel of these equations, $s^{\beta}(t-s)^{-\alpha}G(y(s))$, with $\alpha \in (0,1)$ and $\beta > 0$ possesses two types of singularities (depending on the value of β) and their solutions are typically not regular.

*Corresponding author.
E-mail address: hojatld@gmail.com (H. Laeli Dastjerdi); M. Nili Ahmadabadi (mneely59@hotmail.com).

Print ISSN: 2252-0201
Online ISSN: 2345-5934

In [13], we have considered the equation

$$u(x) = f(x) + \int_0^x \frac{G(u(t))}{(x-t)^\alpha} dt, \qquad 0 < \alpha < 1. \tag{2}$$

As the following situations suggest, (2) does not cover all applications of (1). Here s^β, as part of the integrand, plays a crucial role.

In the study of the temperature distribution of the surface of a projectile moving through a laminar layer, Lighthill encountered the following equation [11].

$$F(z)^4 = -\frac{1}{2\sqrt{z}} \int_0^z \frac{F'(s)}{(z^{\frac{3}{2}} - s^{\frac{3}{2}})^{\frac{1}{3}}} ds, \tag{3}$$

where

$$F(0) = 1, \qquad F(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{4}$$

Franco et al. applied an Abel type inversion formula [14] to the above equation and obtained the results of [5, 6].

$$F(z) = 1 - \frac{3\sqrt{3}}{2\pi} \int_0^z \frac{xF(x)^4}{(z^{\frac{3}{2}} - x^{\frac{3}{2}})^{\frac{2}{3}}} dx, \qquad z \in [0,1]. \tag{5}$$

After setting $z = t^{\frac{2}{3}}$ and $x = s^{\frac{2}{3}}$ in (5) one gets

$$y(t) = 1 - \frac{\sqrt{3}}{\pi} \int_0^t \frac{s^{\frac{1}{3}} y(s)^4}{(t-s)^{\frac{2}{3}}} ds, \tag{6}$$

where $y(t) = F(t^{\frac{2}{3}})$. This is a special case of (1). It can be shown that (6) has a unique continuous solution $y(t)$ for $t \in [0,1]$ (see e.g.[12]). Some other situations where (1) arises have been discussed in [10].

The numerical solvability of the (1) has been pursued by several authors. Diogo et al. in [3] applied a collocation with graded mesh. In [16] a hybrid collocation method introduced for (1). Furthermore a Nystrom type method has been concerned after a smoothing transformation [2]. Recently [1] Jacobi spectral collocation method is proposed for the solution of (1).

In 1981, Ortiz and Samara [15] proposed an operational technique for the numerical solution of nonlinear ordinary differential equations with some supplementary conditions based on the standard Tau method. The Tau method has been applied to a wide class of ordinary differential equations (ODEs), partial differential equations, integral equations and integro-differential equations [4, 7, 8]. In this paper, we are interested in the numerical Tau approximation of the second kind weakly singular non linear Volterra-Hammerstein integral equations (1).

The organization of this paper is as follows: In section 2, we review some preliminaries and we present two kinds of orthogonal polynomials. A matrix based method for (1) is described in section 3. In section 4, error analysis of the method is given. Finally, in section 5, some numerical experiments are reported to clarify the efficiency of the method.

2. Preliminaries

In this section, we present some preliminaries and notations used in the paper. For any integrable functions $u(t)$ and $v(t)$ on $[0,T]$ we define the scalar product $\langle , \rangle$ and the 2-norm by:

$$\langle u(t), v(t)\rangle_w = \int_0^T u(t)v(t)w(t)dt, \quad \|u(t)\|_w^2 = \langle u(t), u(t)\rangle_w,$$

where $w(x)$ is a weight function. By $L_w^2[0,T]$, we mean the space of all functions $f : [0,T] \to \mathbb{R}$ with $\|f\|_w^2 < \infty$ and $\{\phi_k(t)\}_{k=0}^{\infty}$ is a given set of arbitrary polynomial basis which are orthogonal with respect to the weight function $w(x)$ on $[0,T]$.

2.1 *Shifted Chebyshev and Legendre polynomials*

The Chebyshev polynomials are defined on $[-1,1]$ by the formula

$$\begin{cases} T_0(x) = 1, \quad T_1(x) = x, \\ T_{i+1}(x) = 2xT_i(x) - T_{i-1}(x), \quad i = 1, 2, \cdots . \end{cases} \tag{7}$$

The matrix representation of Chebyshev polynomials is

$$[T_0(x), T_1(x), \ldots]^T = \mathbf{T}X_t, \tag{8}$$

where $\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ -1 & 0 & 2 & 0 & \ldots \\ 0 & -3 & 0 & 4 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$ and shifted Chebyshev polynomials are defined on $[a,b]$ as

$$\begin{cases} T_0^*(x) = 1, \quad T_1(x) = \frac{2x-(b+a)}{b-a}, \\ T_{i+1}^*(x) = 2(\frac{2x-(b+a)}{b-a})T_i^*(x) - T_{i-1}^*(x), \quad i = 1, 2, \cdots . \end{cases} \tag{9}$$

Lemma 2.1 [9] Let $\mathbf{T}, \mathbf{T}^*$ are coefficient matrices of Chebyshev polynomials and shifted Chebyshev polynomials respectively. Then we have $\mathbf{T}^* = \mathbf{TQ}$, where

$$\mathbf{Q}_{i,j} = \begin{cases} \binom{i}{j} v^{i-j}w^j, \quad i \geqslant j, \ i,j = 0, 1, 2, \ldots . \\ 0, \quad i < j, \end{cases} \tag{10}$$

with $v = \frac{2}{b-a}, w = \frac{a+b}{a-b}$.

Moreover, the Legendre polynomials on $[-1,1]$ are defined as

$$\begin{cases} P_0(x) = 1, \quad P_1(x) = x, \\ P_{i+1}(x) = \frac{2i+1}{i+1} x P_i(x) - \frac{i}{i+1} P_{i-1}(x), \quad i = 1, 2, ..., \end{cases} \tag{11}$$

and shifted Legendre polynomials on $[a, b]$ are defined as

$$\begin{cases} P_0^*(x) = 1, \quad P_1^*(x) = \frac{2x-(a+b)}{b-a}, \\ P_{i+1}^*(x) = \frac{2i+1}{i+1} (\frac{2x-(a+b)}{b-a}) P_i^*(x) - \frac{i}{i+1} P_{i-1}^*(x), \quad i = 1, 2, ... \end{cases} \tag{12}$$

It can be simply shown that $\mathbf{P}^* = \mathbf{PC}$ where $\mathbf{P}$ and $\mathbf{P}^*$ are coefficient matrices of Legendre polynomials and shifted Legendre polynomials, respectively.

3. The Tau method

In this section, the Tau method is applied for solving (1). First a variable transformation is used on the original equation so that a new equation with smoother solution is obtained. Let us introduce the change of variables $t = s^q$ with $q \geqslant 2$ an integer. Then Eq. (1) is transformed into the following integral equation

$$\hat{u}(t) = \hat{f}(t) - q \int_0^t \frac{s^{\beta q+q-1}}{(t^q - s^q)^\alpha} G(\hat{u}(s)) ds, \qquad t \in [0, T^{\frac{1}{q}}], \tag{13}$$

where $\hat{u}(t) = u(t^q)$ and $\hat{f}(t) = f(t^q)$. For the sake of simplicity suppose $G(\hat{u}(s)) = \hat{u}^p(s)$.

Let $\{\phi_i(t)\}_{i=0}^\infty$ be a given set of arbitrary polynomial basis which are orthogonal with respect to the weight function $w(t)$ on $[0, T]$. Now let $\hat{u}(t) = \sum_{i=0}^{\infty} u_i \phi_i(t)$ be orthogonal series expansion of the exact solution of (13). We define $u_n(t)$ as a Tau approximation of the exact solution $\hat{u}(t)$ as follows:

$$u_n(t) = \sum_{i=0}^{n} u_i \phi_i(t) = \mathbf{u} \Phi X_t,$$

where $\mathbf{u} = [u_0, u_1, ...u_n, ...]$, Φ is a nonsingular triangular matrix and $X_t = [1, t, t^2, ...]^T$. For approximation of nonlinear functions we have the following lemma:

Lemma 3.1 [7] Let $v(t) = \sum_{i=0}^{\infty} v_i \phi_i(t) = \mathbf{v} \Phi X_t$ be a polynomial, where $\mathbf{v} = [v_0, v_1, ...]$, $X_t = [1, t, t^2, ...]^T$ and $\Phi = [\varphi_{i,j}]_{i,j=0}^\infty$ is a non-singular lower triangular matrix. Then for any natural number $p \in \mathbb{N}$, we have

$$v^p(t) = \mathbf{v} \Phi U^{p-1} X_t$$

and U is an infinite upper triangular Toeplitz matrix with the following structure:

$$U = \begin{bmatrix} \mathbf{v}\Phi_0 & \mathbf{v}\Phi_1 & \mathbf{v}\Phi_2 & \cdots \\ 0 & \mathbf{v}\Phi_0 & \mathbf{v}\Phi_1 & \cdots \\ 0 & 0 & \mathbf{v}\Phi_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

with $\Phi_i = [\varphi_{i,j}]_{j=0}^{\infty}$.

Introduce $\tilde{\mathbf{u}} = \mathbf{u}\Phi$ with $\tilde{\mathbf{u}} = [\tilde{u}_0, \tilde{u}_1, ..., \tilde{u}_n]$. Then the matrix U can be represented as an upper triangular Toeplitz form

$$U = \begin{bmatrix} \tilde{u}_0 & \tilde{u}_1 & \tilde{u}_2 & ... \\ 0 & \tilde{u}_0 & \tilde{u}_1 & ... \\ 0 & 0 & \tilde{u}_0 & ... \\ & & \vdots & \end{bmatrix}.$$

It is easy to show that the upper triangular Toeplitz matrix U^p is as follows[7]:

$$U^p = \begin{bmatrix} (\tilde{u}_0)^p & c_0(\tilde{u}_0)^{p-1}\tilde{u}_1 & c_1(\tilde{u}_0)^{p-2}\tilde{u}_1^2 + c_2(\tilde{u}_0)^{p-1}\tilde{u}_2 & \\ 0 & (\tilde{u}_0)^p & c_0(\tilde{u}_0)^{p-1}\tilde{u}_1 & ... \\ 0 & 0 & (\tilde{u}_0)^p & ... \\ & & \vdots & \end{bmatrix},$$

where $c_0, c_1, ...$ are constants.

In order to apply the Tau method for (13) suppose $\hat{f}(t) = \sum_{i=0}^{\infty} f_i\phi_i(t) = \mathbf{f}\Phi X_t$. Using Lemma 3.1 and substituting $u(t)$ in (13), we have

$$\mathbf{u}\Phi X_t = \mathbf{f}\Phi X_t - q\int_0^t \frac{s^{\beta q+q-1}}{(t^q - s^q)^\alpha}(\mathbf{u}\Phi X_s)^p ds.$$

Therefore

$$\mathbf{u}\Phi X_t = \mathbf{f}\Phi X_t - q\mathbf{u}\Phi U^{p-1}\int_0^t \frac{s^{\beta q+q-1}}{(t^q - s^q)^\alpha}X_s ds$$

or

$$\mathbf{u}\Phi X_t = \mathbf{f}\Phi X_t - q\mathbf{u}\Phi U^{p-1}\left[\int_0^t \frac{s^{\beta q+q-1+m}}{(t^q - s^q)^\alpha}ds\right]_{m=0}^{\infty}. \quad (14)$$

It is easy to show that

$$\int_0^t \frac{s^\gamma}{(t^\sigma - s^\sigma)^\alpha}ds = \frac{1}{\sigma}B(1-\alpha, \frac{1+\gamma}{\sigma})t^{1+\gamma-\sigma\alpha},$$

where $B(x, y)$ is the well known Beta function defined by

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{15}$$

Therefore, from (14) we have

$$\mathbf{u}\Phi X_t = \mathbf{f}\Phi X_t - \mathbf{u}\Phi U^{p-1}\left[B(1-\alpha, \frac{\beta q + q + m}{q})t^{\beta q+q+m-q\alpha}\right]_{m=0}^{\infty}.$$

Now we can write

$$t^{\beta q+q-q\alpha+m} = \sum_{i=0}^{\infty} c_{m,i}\phi_i(t) = \mathbf{c}_m \Phi X_t,$$

where $\mathbf{c}_m = [c_{m,0}, c_{m,1}, ...]$. So we have

$$\left[t^{\beta q+q-q\alpha+m}\right]_{m=0}^{\infty} = [c_m \Phi X_t]_{m=0}^{\infty} = C\Phi X_t,$$

where $C = [c_0, c_1, ...]^T$. Therefore

$$\int_0^t \frac{s^{\beta q+q-1}}{(t^q - s^q)^\alpha}\hat{u}^p(s)ds = \mathbf{u}\Phi U^{p-1}MC\Phi X_t,$$

where M is an infinite diagonal matrix with diagonal elements $M_{i,i} = B(1-\alpha, \frac{\beta q+q+i}{q})$. Thus, from (14) we have

$$\mathbf{u}\Phi X_t = \mathbf{f}\Phi X_t - \mathbf{u}\Phi U^{p-1}MC\Phi X_t \tag{16}$$

or

$$\tilde{\mathbf{u}} = \tilde{\mathbf{f}} - \tilde{\mathbf{u}}U^{p-1}MC\Phi. \tag{17}$$

Set $Q = I + U^{p-1}MC\Phi$ in which I represents the identity matrix then (17) can be written as

$$\tilde{\mathbf{u}}Q = \tilde{\mathbf{f}}. \tag{18}$$

Using a finite truncated series of (18) including the first (n + 1) terms and solving the resulting system of equations, we can obtain the vector $\tilde{\mathbf{u}}$ and so the approximate solution $\hat{u}_n(x)$ will be determined.

4. Error analysis

In this section, we are going to provide the convergence analysis of the proposed method. First, let us bring some Gronwall-type inequalities.

Theorem 4.1 [17] Suppose $\beta > 0$, $a(x)$ is a nonnegative locally integrable function defined on $0 \leqslant x < T \leqslant \infty$ and $g(x)$ is a nonnegative, nondecreasing continuous function defined on $0 \leqslant x < T$, $g(x) \leqslant M$, and suppose $u(x)$ is a nonnegative and locally integrable on $0 \leqslant x < T$ with

$$u(x) \leqslant a(x) + g(x) \int_0^x \frac{u(t)}{(x-t)^{1-\beta}} dt. \tag{19}$$

Then we have

$$u(x) \leqslant a(x) + \int_0^x \left[\sum_{n=1}^{\infty} \frac{(g(x)\Gamma(\beta))^n}{\Gamma(n\beta)} (x-t)^{n\beta-1} a(t) \right] dt, \quad 0 \leqslant x < T. \tag{20}$$

Corollary 4.2 [17] Suppose $b \geqslant 0$, $\beta > 0$ and $a(x)$ is a nonnegative locally integrable function defined on $0 \leqslant x < T \leqslant \infty$, and suppose $u(x)$ is a nonnegative and locally integrable on $0 \leqslant x < T$ with

$$u(x) \leqslant a(x) + b \int_0^x \frac{u(t)}{(x-t)^{1-\beta}} dt. \tag{21}$$

Then we have

$$u(x) \leqslant a(x) + \int_0^x \left[\sum_{n=1}^{\infty} \frac{(b\Gamma(\beta))^n}{\Gamma(n\beta)} (x-t)^{n\beta-1} a(t) \right] dt, \quad 0 \leqslant x < T. \tag{22}$$

Corollary 4.3 [17] Under the hypothesis of Theorem 4.1, let $a(x)$ be a nondecreasing function on $[0, T)$. Then we have

$$u(x) \leqslant a(x) E_\beta \left(g(x)\Gamma(\beta) x^\beta \right), \tag{23}$$

where E_β is the Mittag-Leffler function defined by $E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}$.

Next, we prove the convergence of the proposed algorithm by obtaining a posteriori error bound.

Theorem 4.4 Suppose $\sup_{0<t<T} |\hat{f}(t) - \hat{f}_n(t)| < M_{1_n}$ and $e_n(t) = \hat{u}(t) - \hat{u}_n(t)$. Under hypothesis of Theorem 4.1 we have the following error bound

$$|e_n(t)| \leqslant a(t) E_{1-\alpha}(K M_3 \Gamma(1-\alpha) t^{1-\alpha}),$$

in which $a(t) = K M_2 \int_0^t \frac{u^{n+1}(s)}{(t-s)^\alpha} ds + M_{1_n}$, $K = \max_{t,s \in [0,T]} |\hat{k}(t,s)|$ and M_2 is a constant.

Proof. We have

$$\hat{u}_n(t) = \hat{f}_n(t) - q \int_0^t \frac{s^{\beta q + q - 1} G_n(u_n(s))}{(t^q - s^q)^\alpha} ds \tag{24}$$

and

$$G_n(x) = \sum_{i=0}^{n} \frac{G^i(0)}{i!} x^i, \qquad G(x) - G_n(x) = \frac{G^{(n+1)}(\xi)}{(n+1)!} x^{n+1}.$$

Using the formula

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}),$$

we can rewrite equation (24) as

$$\hat{u}_n(t) = \hat{f}_n(t) - q \int_0^t \frac{\hat{k}(t,s) G_n(u_n(s))}{(t-s)^\alpha} ds, \tag{25}$$

where

$$\hat{k}(t,s) = \frac{s^{\beta q + q - 1}}{(t^{q-1} + t^{q-2}s + \cdots + ts^{q-2} + s^{q-1})^\alpha}.$$

So

$$|\hat{u}(t) - \hat{u}_n(t)| \leqslant |\hat{f}(t) - \hat{f}_n(t)| + \int_0^t \tfrac{\hat{k}(t,s)}{(t-s)^\alpha} |G(u(s)) - G_n(u_n(s))| ds$$

$$\leqslant M_{1_n} + \int_0^t \tfrac{\hat{k}(t,s)}{(t-s)^\alpha} (|G(u(s)) - G_n(u(s))| + |G_n(u(s)) - G_n(u_n(s))|) ds.$$

Suppose $G(x)$ has a bounded $(n+1)$-th derivative, i.e., $G^{(n+1)}(x) < M_2$, for some constant M_2. Since $G_n(x)$ is a polynomial it is a Lipschitz function, i.e., $|G_n(x_2) - G_n(x_1)| < M_3|x_2 - x_1|$. Therefore we have

$$\begin{aligned} |e_n(t)| &\leqslant M_{1_n} + \int_0^t \tfrac{\hat{k}(t,s)}{(t-s)^\alpha} (M_3|e_n(s)| + \tfrac{M_2}{(n+1)!} u^{n+1}(s)) ds, \\ &\leqslant KM_3 \int_0^t \tfrac{|e_n(s)|}{(t-s)^\alpha} ds + a(t), \end{aligned}$$

in which $a(t) = KM_2 \int_0^t \frac{u^{n+1}(s)}{(t-s)^\alpha} ds + M_{1_n}$ and $K = \max_{t,s \in [0,T]} |\hat{k}(t,s)|$. Since $0 < \alpha < 1$, $a(t)$ is non-decreasing. Therefore according to corollary 4.3 we have

$$|e_n(t)| \leqslant a(t) E_{1-\alpha}(KM_3 \Gamma(1-\alpha) t^{1-\alpha}),$$

which concludes the proof. ■

Note that $e_n(t) \to 0$ as $n \to \infty$.

5. Numerical experiments

In this section, two test problems are studied using the proposed method. We consider $||e|| = \max_{t \in [0,1]} |\hat{u}_n(t) - u(t)|$ as our error criterion. All calculations have been done by Maple 16.

***Example* 5.1** Consider the following integral equation:

$$u(z) = \sqrt{z} + B(\frac{1}{4}, \frac{9}{4})z^{\frac{3}{2}} - \int_0^z \frac{x^{\frac{1}{4}} u(x)^2}{(z-x)^{\frac{3}{4}}} dx, \qquad z \in [0,1], \tag{26}$$

the exact solution of which is $u(z) = \sqrt{z}$. Using transformation $x = s^2$ and $z = t^2$, (26) becomes

$$\hat{u}(t) = t + B(\frac{1}{4}, \frac{9}{4})t^3 - 2\int_0^t \frac{s^{\frac{3}{2}} \hat{u}(s)^2}{(t^2 - s^2)^{\frac{3}{4}}} ds, \qquad t \in [0,1]$$

with $\hat{u}(t) = u(t^2) = t$. For numerical implementation of the proposed method we take $p = 2$ and $n = 4$. So the following matrices in the case of Legendre basis function will be obtained:

$$\tilde{\mathbf{u}} = [\tilde{u}_0, \tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4],$$

$$\tilde{\mathbf{f}} = \left[0, 1, 0, B(\frac{1}{4}, \frac{9}{4}), 0\right]^T,$$

$$\Phi = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{-1}{2} & 0 & \frac{3}{2} & 0 & 0 \\ 0 & \frac{-3}{2} & 0 & \frac{5}{2} & 0 \\ \frac{3}{8} & 0 & \frac{-15}{4} & 0 & \frac{35}{8} \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{2}{3} & 0 & 0 \\ 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ \frac{1}{5} & 0 & \frac{4}{7} & 0 & \frac{8}{35} \\ 0 & \frac{3}{7} & 0 & \frac{4}{9} & 0 \end{bmatrix},$$

$$M = diag\left[B(\frac{1}{4}, \frac{5}{4}), B(\frac{1}{4}, \frac{7}{4}), B(\frac{1}{4}, \frac{9}{4}), B(\frac{1}{4}, \frac{11}{4}), B(\frac{1}{4}, \frac{13}{4})\right],$$

and

$$U = \begin{bmatrix} \tilde{u}_0 & \tilde{u}_1 & \tilde{u}_2 & \tilde{u}_3 & \tilde{u}_4 \\ 0 & \tilde{u}_0 & \tilde{u}_1 & \tilde{u}_2 & \tilde{u}_3 \\ 0 & 0 & \tilde{u}_0 & \tilde{u}_1 & \tilde{u}_2 \\ 0 & 0 & 0 & \tilde{u}_0 & \tilde{u}_1 \\ 0 & 0 & 0 & 0 & \tilde{u}_0 \end{bmatrix}.$$

We obtain the following non-linear system of algebraic equations

$$\begin{aligned}
&\tilde{u}_0 = 0,\\
&\tilde{u}_0^2 B(\tfrac{1}{4},\tfrac{5}{4}) - \tfrac{10}{21}\,\tilde{u}_4\tilde{u}_0 B\left(\tfrac{1}{4},\tfrac{13}{4}\right) + \tilde{u}_1 - \tfrac{10}{21}\,\tilde{u}_3\tilde{u}_1 B\left(\tfrac{1}{4},\tfrac{13}{4}\right) - \tfrac{5}{21}\,\tilde{u}_2^2 B\left(\tfrac{1}{4},\tfrac{13}{4}\right) = 1,\\
&\tfrac{3}{2}\,\tilde{u}_0\tilde{u}_1\pi\sqrt{2} + \tilde{u}_2 = 0,\\
&2\,\tilde{u}_0\tilde{u}_2 B\left(\tfrac{1}{4},\tfrac{9}{4}\right) + \tfrac{20}{9}\,\tilde{u}_4\tilde{u}_0 B\left(\tfrac{1}{4},\tfrac{13}{4}\right) +\\
&\tilde{u}_1^2 B\left(\tfrac{1}{4},\tfrac{9}{4}\right) + \tfrac{20}{9}\,\tilde{u}_3\tilde{u}_1 B\left(\tfrac{1}{4},\tfrac{13}{4}\right) + \tfrac{10}{9}\,\tilde{u}_2^2 B\left(\tfrac{1}{4},\tfrac{13}{4}\right) + \tilde{u}_3 = B(\tfrac{1}{4},\tfrac{9}{4})\\
&\tfrac{21}{16}\,\tilde{u}_0\tilde{u}_3\pi\sqrt{2} + \tfrac{21}{16}\,\tilde{u}_1\tilde{u}_2\pi\sqrt{2} + \tilde{u}_4 = 0,
\end{aligned}$$

with the solution $\tilde{u}_0 = 0, \tilde{u}_1 = 1, \tilde{u}_2 = 0, \tilde{u}_3 = 0, \tilde{u}_4 = 0$. Thus $\hat{u}_n(t) = t$ which is the exact solution of the equation.

***Example* 5.2** [2]

$$u(z) = 1 - \frac{\sqrt{3}}{\pi}\int_0^z \frac{x^{\frac{1}{4}}y(x)^2}{(z-x)^{\frac{3}{4}}}dx, \qquad x \in [0,1]. \tag{27}$$

Setting $x = s^2$ and $z = t^2$, (27) can be written as

$$\hat{u}(t) = 1 - \frac{2\sqrt{3}}{\pi}\int_0^t \frac{s^{\frac{3}{2}}\hat{u}(s)^2}{(t^2-s^2)^{\frac{3}{4}}}ds,$$

the exact solution of which is not known. In order to be able to analyse this problem we consider $\hat{u}_{10}(t)$ as the appropriate solution and compare our numerical findings with it. Table 1 shows the maximum absolute error with different n in comparison with $\hat{u}_{10}(t)$.

Table 1. $||e||$ for Example 2 with different values of n

n	$\|\|e\|\|$
1	1.5×10^{-1}
3	2.5×10^{-2}
5	4×10^{-3}
7	6×10^{-4}

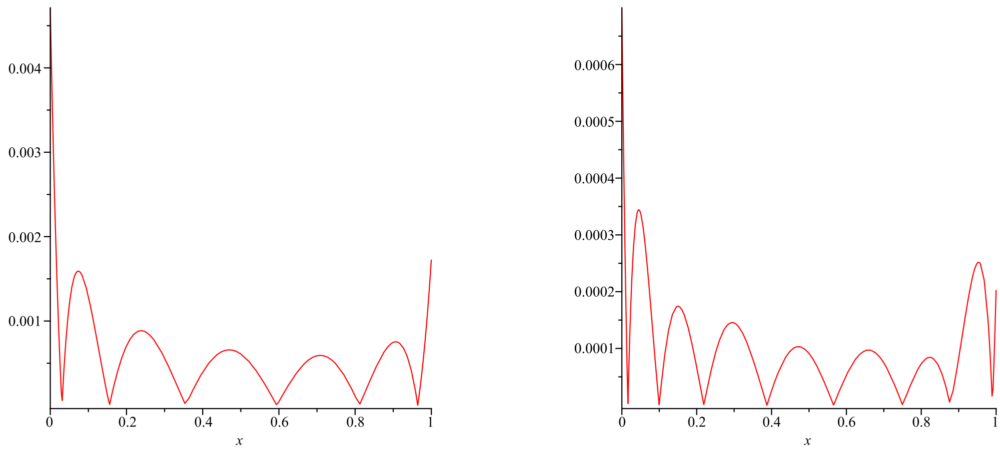

Figure 1. Absolute error for Example 2 with $n = 5$ and $n = 7$ (Left: $n = 5$, Right: $n = 7$).

6. Conclusion

We have considered a kind of nonlinear weakly singular Volterra integral equation of Hammerstein type which is encountered in important practical applications. A matrix based method with a smoothing policy has been successfully applied. Convergence analysis of the proposed method is provided and numerical results approve its applicability, efficiency and reliability.

References

[1] S. S. Allaei, T. Diogo, M. Rebelo, The jacobi collocation method for a class of nonlinear Volterra integral equations with weakly singular kernel, J. Sci. Comput. DOI 10.1007/s10915-016-0213-x.
[2] P. Baratella, A Nystrom interpolant for some weakly singular nonlinear Volterra integral equations, J. Comput. Appl. Math. 237 (2013), 542-555.
[3] T. Diogo, J. Ma, M. Rebelo, Fully discretized collocation methods for nonlinear singular Volterra integral equations, J. Comput. Appl. Math. 247 (2013), 84-101.
[4] G. Ebadi, M. Y. Rahimi-Ardabili, S. Shahmorad, Numerical solution of the nonlinear Volterra integro-differential equations by the Tau method, Appl. Math. Comput. 188 (2) (2007), 1580-1586.
[5] N. M. B. Franco, S. Mckee, A family of high order product integration methods for an integral equation of Lighthill, Int. J. Comput. Math. 18 (1985), 173-184.
[6] N. B. Franco, S. McKee, J. Dixon, A numerical solution of Lighthill's equation for the surface temperature distribution of a projectile, Mat. Aplic. Comp. 12 (1983), 257-271.
[7] F. Ghoreishi, M. Hadizadeh, Numerical computation of the Tau approximation for the Volterra-Hammerstein integral equations, Numer. Algor. 52 (2009), 541-559.
[8] S. M. Hossieni, S. Shahmorad, Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial bases, Appl. Math. Model. 27 (2003), 145-154.
[9] S. Karimi Vanani, F. Soleymani, Tau approximate solution of weakly singular Volterra integral equations, Math. Comput. Model. 57 (2013), 494-502.
[10] I. J. Kumar, On the asymptotic solution of a nonlinear Volterra integral equation, Proc. Roy. Soc. Lond. A. 324 (1971), 45-61.
[11] J. M. Lighthill, Contributions to the theory of the heat transfer trough a laminar boundary layer, Proc. Roy. Soc. London. 202 (A) (1950), 359-377.
[12] R. Miller, Nonlinear Volterra Integral Equations, W.A. Benjamin, California, 1971.
[13] M. Nili Ahmadabadi, H. Laeli Dastjerdi, Tau approximation method for the weakly singular Volterra-Hammerstein integral equations, Appl. Math. Comput. 285 (2016), 241-247.
[14] B. Noble, The numerical solution of nonlinear integral equations and related topics, Anselone, 1964.
[15] E. L. Ortiz, H. Samara, An operational approach to the Tau method for the numerical solution of nonlinear differential equations, Computing. 27 (1981), 15-25.
[16] M. Rebelo, T. Diogo, A hybrid collocation method for a nonlinear Volterra integral equation with weakly singular kernel, J. Comput. Appl. Math. 234 (2010), 2859-2869.
[17] H. Ye, J. Gao, Y. Ding, A generalized Gronwall inequality and its application to a fractional differential equation, J. Math. Anal. Appl. 328 (2007), 1075-1081.