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APPLICATION OF CHEBYSHEV APPROXIMATION IN THE PROCESS OF VARIATIONAL ITERATION METHOD FOR SOLVING DIFFERENTIAL-ALGEBRAIC EQUATIONS

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Abstract- In this paper, we use Chebyshev approximations in the process of He's variational iteration method for finding the solution of differential-algebraic equations. This allows us to make integration at each of the iterations possible and at the same time, obtain a good accuracy in a reasonable number of iterations. Numerical results show that using Chebyshev approximation is much more efficient than using Taylor approximation which is more popular.

First, an index reduction technique is implemented for semi-explicit differential-algebraic equations, then the obtained problem is solved by He's variational iteration method. The scheme is tested for some high index differential-algebraic equations and the results demonstrate reliability and efficiency of the proposed method.

Key words- Chebyshev approximation, Variational iteration method, Differential algebraic equations

1. INTRODUCTION

Many physical problems are governed by a system of differential-algebraic equations (DAEs), and finding the solution of these equations has been the subject of many investigations in recent years.

In 1999, the variational iteration method (VIM) was proposed by He [1-3]. This method is now widely used by many researchers to study linear and nonlinear problems. The method introduces a reliable and efficient process for a wide variety of scientific and engineering applications. It is based on Lagrange multiplier and it has the merits of simplicity and easy execution. Unlike the traditional numerical methods, VIM needs no discretization, linearization, transformation or perturbation. The method gives rapidly convergent successive approximations of the exact solution if such a solution exists; otherwise a few approximations can be used for numerical purposes. The VIM was successfully applied to autonomous ordinary and partial differential equations [1–14]. Recently in [15] the VIM has been implemented for finding the solution of differential-algebraic equations. To avoid tedious integration, in the latter paper the Taylor approximation is used.

In this paper, we are going to use the Chebyshev approximation instead of Taylor approximation. Examples show that this procedure is much more efficient.

It is well known that the eigenfunctions of certain singular Sturm-Liouville problems allow the approximation of functions $C^{\infty}[a,b]$ where truncation error approaches zero faster than any negative power of the number of basic functions used in the approximation, as that number (order of truncation ν) tends to infinity [16]. This phenomenon is usually referred to as " spectral accuracy ". In this work, we are using first kind orthogonal Chebyshev polynomials $\{T_k\}_{k=0}^{+\infty}$ which are eigenfunctions of singular Sturm-Liouville problem:

$$\left(\sqrt{1-x^2}\,T'(x)\right)' + \frac{k^2}{\sqrt{1-x^2}}\,T_k(x) = 0$$

2. DAEs AND REDUCING INDEX

It is well known that the index of DAEs is a measure of the degree of singularity of the system and also widely regarded as an indication of certain difficulties for numerical methods. So, DAEs can be difficult to solve when they have a higher index, i.e., an index greater than 1 [17]. In this case, an alternative treatment is the use of index reduction methods [17-19].

In this section, we briefly review the reducing index method for DAEs which is mentioned in [18]. Consider the linear (or linearized) semi-explicit DAEs:

$$X^{(m)} = \sum_{j=1}^{m} A_j X^{(j-1)} + BY + q, \tag{1a}$$

$$0 = CX + r, \tag{1b}$$

where A_j, B and C are smooth functions of t, $t_0 \le t \le t_f$, $A_j(t) \in R^{n\times n}$, $j = 1,\dots,m$, $B(t) \in R^{n\times k}$, $C(t) \in R^{k\times n}$, $n \ge 2$, $1 \le k \le n$ and CB is nonsingular (DAE has index $m+1$) except possibly at a finite number of isolated points of t. The inhomogenities are $q(t) \in R^n$ and $r(t) \in R$. Now suppose that CB is non-singular, from (1a), we can write

$$Y = (CB)^{-1} C\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right], \quad t \in [t_0, t_f] \tag{2}$$

Substituting (2) into (1a) implies that

$$\left[I - B(CB)^{-1}C\right]\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right] = 0$$

So, problem (1) transforms to the over-determined system:

$$\left[I - B(CB)^{-1}C\right]\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right] = 0, \tag{3}$$

$$CX + r = 0, \quad t \in [t_0, t_f]$$

Now system (3) can be transformed to a full-rank DAE system with n equations and n unknowns with index m [18]. Here, for simplicity, we consider problem (1) when $m = 1$ (problem has index 2).

Theorem 1: Consider problem (1) with index 2, $n = 2$ and $k = 1$. This problem is equivalent to the following index -1 DAE system:

$$E_1X' + E_0X = \hat{q} \tag{4}$$

such that

$$E_0 = \begin{bmatrix} b_1a_{21} - b_2a_{11} & b_1a_{22} - b_2a_{12} \\ c_1 & c_2 \end{bmatrix}, \quad E_1 = \begin{bmatrix} b_2 & -b_1 \\ 0 & 0 \end{bmatrix}, \quad \hat{q} = \begin{bmatrix} b_2q_1 - b_1q_2 \\ -r \end{bmatrix}$$

and

$$y = (CB)^{-1}C[X' - AX - q]. \tag{5}$$

Proof: presented in [18].

In this paper, first we implement this proposed index reduction method to linear semi-explicit DAEs. Then we employ the VIM using Chebyshev approximation to solve the obtained problem. Furthermore, we use some examples to demonstrate the efficiency and effectiveness of the proposed method.

3. HE'S VARIATIONAL ITERATION METHOD

In this section, we briefly review the main points of the powerful method, known as the He's variational iteration method. This method is a modification of a general Lagrange multiplier method proposed by Inokuti [20]. In the VIM, the differential equation

$$L[u(t)] + N[u(t)] = g(t), \tag{6}$$

is considered, where L and N are linear and nonlinear operators, respectively, and $g(t)$ is an inhomogenous term . Using the method, the correction functional

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda[L(u_n(s)) + N[\tilde{u}_n(s)] - g(s)]ds \tag{7}$$

is considered, where λ is a general Lagrange multiplier, u_n is the n-th approximate solution and $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. In this method, first we determine the Lagrange multiplier λ that can be identified via variational theory, i.e., the multiplier should chosen such that the correction functional is stationary, i.e., $\delta\tilde{u}_{n+1}(u_n(t), t) = 0$. Then the successive approximation u_n, $n \geq 0$ of the solution u will be obtained by using any selective initial function u_0 and the calculated Lagrange multiplier λ. Consequently, $u = \lim_{n\to\infty} u_n$. It means that, by the correction functional (7) several approximations will be obtained and therefore, the exact solution emerges as the limit of the resulting successive approximations.

To perform the VIM, in general, for an arbitrary natural number ν, $g(t)$ express in Taylor series,

$$g(t) \approx \sum_{i=0}^{\nu} g_i(t). \tag{8}$$

In this paper, we suggest that $g(t)$ be expressed in Chebyshev series,

$$g(t) \approx \sum_{i=0}^{\nu} a_i T_i(t), \tag{9}$$

where $T_i(t)$ is the first kind of orthogonal Chebyshev polynomial,

$T_0(t) = 1,$

$T_1(t) = t,$

$T_2(t) = 2t^2 - 1,$

and in general,

$T_{k+1} = 2tT_k - T_{k-1}, \qquad k \geq 1.$

In the next section, this method is successfully applied for solving differential-algebraic equations.

4. TEST PROBLEMS

In this section, to show the ability and efficiency of the proposed method, some examples are presented. In all the examples, to simplify the computations, for an arbitrary natural number ν, every coefficient function $g(t)$ is expressed in Chebyshev series (9). The results are compared with the case in which Taylor expansion is used with the same number of terms. The algorithms are performed by Maple 12 with 20 digits precision.

Example 1: Consider index-2 problem:

$$X' = AX + By + q, \tag{10a}$$

$$0 = CX + r, \tag{10b}$$

where $0 \leq t \leq 1$ and

$A = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1+2t \end{bmatrix}$, $q = \begin{bmatrix} -\sin(t) \\ 0 \end{bmatrix}$, $C^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $r(t) = -(e^{-t} + \sin(t))$, with

$x_1(0) = 1$, $x_2(0) = 0$ and the exact solutions $x_1(t) = e^{-t}$, $x_2(t) = \sin(t)$ and

$y(t) = \dfrac{\cos(t)}{1+2t}$.

From Theorem 1, problem (10) can be converted to the index-1 DAE:

$$\begin{cases} x_1 + x_2 = e^{-t} + \sin(t) & \text{(a)} \\ x_1' + x_1 - x_2 + \sin(t) = 0 & \text{(b)} \end{cases} \tag{11}$$

with $x_1(0) = 1$, and $x_2(0) = 0$. To solve the new problem, we transform the algebraic equation (11a) in the iterative form with respect to x_2 and by the He's variational iteration method and using (7), we construct the correction functional in x_1-direction for the differential equation (11b). Therefore, we obtain the following

$$\text{syste}\begin{cases} x_2^{(n+1)}(t) = e^{-t} + \sin(t) - x_1^{(n)}(t) & \text{(a)} \\ x_1^{(n+1)}(t) = x_1^{(n)}(t) + \int_0^t \lambda(s)[x'^{(n)}(s) + x_1^{(n)}(s) - \tilde{x}_2^{(n)}(s) + \sin(s)]ds & \text{(b)} \end{cases} \tag{12}$$

where $\tilde{x}_2^{(n)}$ is considered as a restricted variation ,i.e., $\delta\tilde{x}_2^{(n)}=0$. By taking the variation from both sides of the correction functional (12b), we have

$$\delta\tilde{x}_1^{(n+1)}(t)=\delta\tilde{x}_1^{(n)}+\delta\int_0^t \lambda(s)[x_1'^{(n)}(s)+x_1^{(n)}(s)-\tilde{x}_2^{(n)}(s)+\sin(s)]ds,$$

or

$$\delta\tilde{x}_1^{(n+1)}(t)=\delta\tilde{x}_1^{(n)}+\lambda(s)\delta x_1^{(n)}(s)\big|_{s=t}+\int_0^t[-\lambda'(s)+\lambda(s)]\delta x_1^{(n)}(s)ds.$$

By imposing $\delta\tilde{x}_1^{(n+1)}=0$, we obtain the stationary conditions

$$\begin{cases} 1+\lambda(s)\,|_{s=t}=0 \\ -\lambda'(s)-\lambda(s)=0. \end{cases} \tag{13}$$

Therefore

$$\lambda(s)=-e^{s-t}. \tag{14}$$

By substituting the optimal value (14) into functional (12b), we obtain the following iteration formula:

$$\begin{cases} x_2^{(n+1)}=e^{-t}+\sin(t)-x_1^{(n)}(t) & \text{(a)} \\ x_1^{(n+1)}=x_1^{(n)}-\int_0^t e^{s-t}[x_1'^{(n)}(s)+x_1^{(n)}(s)-\tilde{x}_2^{(n)}(s)+\sin(s)]ds, & \text{(b)} \end{cases} \tag{15}$$

with $n=0,1,2,\ldots,$ $x_1^{(0)}=x_1(0)=1$ and $x_2^{(0)}=x_2(0)=0.$

Now, we expand the coefficient functions e^{-t} and $\sin(t)$ at $t=0$ and e^{s-t} at $t=s$ by Taylor series expansion with $v=10$ as follow:

$$e^{-t}\cong 1-t+\tfrac{1}{2}t^2-\tfrac{1}{6}t^3+\tfrac{1}{24}t^4-\tfrac{1}{120}t^5+\tfrac{1}{720}t^6-\tfrac{1}{5040}t^7+\tfrac{1}{40320}t^8-\tfrac{1}{362880}t^9, \tag{16a}$$

$$\sin(t)\cong t-\tfrac{1}{6}t^3+\tfrac{1}{120}t^5-\tfrac{1}{5040}t^7+\tfrac{1}{362880}t^9, \tag{16b}$$

$$\begin{aligned} e^{s-t}\cong{} & 1-t+s+\tfrac{1}{2}(t-s)^2-\tfrac{1}{6}(t-s)^3+\tfrac{1}{24}(t-s)^4-\tfrac{1}{120}(t-s)^5 \\ & +\tfrac{1}{720}(t-s)^9-\tfrac{1}{5040}(t-s)^7+\tfrac{1}{40320}(t-s)^8-\tfrac{1}{362880}(t-s)^9. \end{aligned} \tag{16c}$$

So after 16 iterations, (15) and (16) yield:

$$\begin{cases} x_2^{(1)}(t)=\tfrac{1}{2}t^2-\tfrac{1}{3}t^3+\tfrac{1}{24}t^4+\cdots \\ x_1^{(1)}(t)=1-t+\tfrac{1}{6}t^3-\tfrac{1}{12}t^4+\cdots \end{cases}$$

$$\vdots$$

$$\begin{cases} x_2^{(16)}(t)=t-\tfrac{1}{6}t^3+\tfrac{1}{120}t^5-\tfrac{1}{5040}t^7+\tfrac{1}{362880}t^9-\tfrac{1}{3628800}t^{10}+\tfrac{1}{19958400}t^{11}-+\cdots \\ x_1^{(16)}(t)=1-t+\tfrac{1}{2}t^2-\tfrac{1}{6}t^3+\tfrac{1}{24}t^4-\tfrac{1}{120}t^5+-\cdots+\tfrac{1}{3628800}t^{10}-\tfrac{1}{19958400}t^{11}+-\cdots \end{cases}$$

In the alternative method, we expanded the coefficient functions e^{-t}, $\sin(t)$ and e^{s-t} by Chebyshev series with the same number of terms. Using (15) and (16), after 16 iterations, we obtain

$$\begin{cases} x_2^{(1)}(t) = -1.79(-14) + 4.36(-12)t + - \cdots + 3.67(-8)t^{10} \\ x_1^{(1)}(t) = 1.00 - 1.00t + - \cdots + 7.28\ (-21)t^{21} \end{cases}$$

$$\vdots$$

$$\begin{cases} x_2^{(16)}(t) = -1.79(-14) + 1.00\,t - + \cdots - 3.30(-15)t^{19} + - \cdots \\ x_1^{(16)}(t) = 1. - 1.00\,t + - \cdots - 2.14(-15)t^{19} + - \cdots \end{cases}$$

For the sake of comparison, we have illustrated the absolute errors for $x_1^{(16)}$ and $x_2^{(16)}$ using both Taylor and Chebyshev cases in figures 1 and 2. These figures, obviously, show the superiority of using Chebyshev approximation instead of Taylor approximation in this example. It should be noted that in order to increase the rate of convergence, we have used $x_2^{(n+1)}$ instead of $x_2^{(n)}$ in (15b) for computing $x_1^{(n+1)}$.

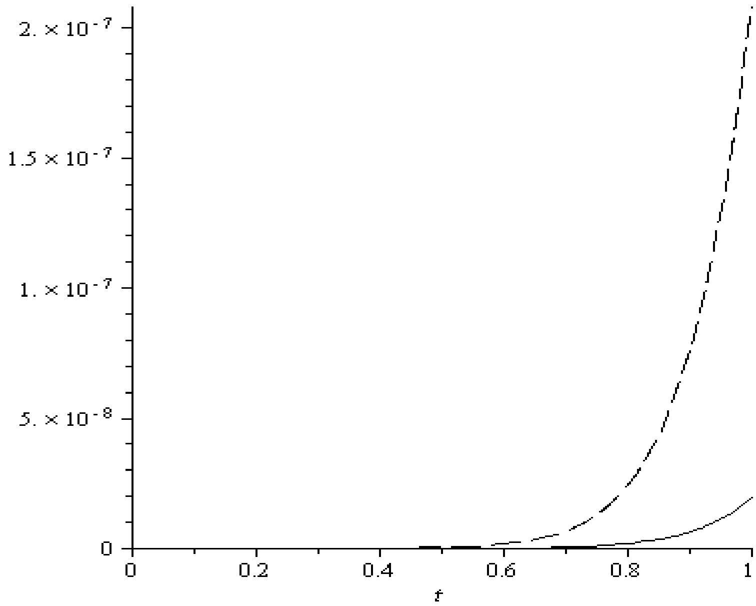

Fig. 1. Absolute errors of using Taylor polynomials for computation of $x_1^{(n+1)}$ __ and $x_2^{(n+1)}$... in Ex. 1.

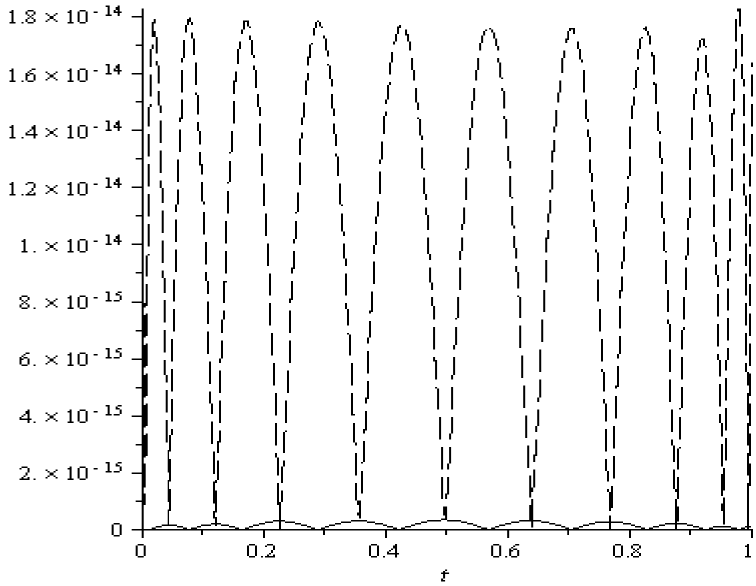

Fig. 2. Absolute errors of using Chebyshev polynomials for computation of $x_1^{(n+1)}$ __ and $x_2^{(n+1)}$.... in Ex. 1.

Example 2: Consider index-2 problem:

$$X' = AX + By + q, \tag{17a}$$

$$0 = CX + r, \tag{17b}$$

where $0 \le t \le 1$ and

$$A = \begin{bmatrix} 2 & t \\ 0 & \frac{1}{t+1} \end{bmatrix}, \quad C^T = B = \begin{bmatrix} 1 \\ t \end{bmatrix}, \quad q = \begin{bmatrix} e^t(1 - t - t\sin(t) - \frac{1}{1+t}) \\ e^t(\sin(t) + \cos(t) - \frac{\sin(t)+t}{1+t}) \end{bmatrix},$$

$$r(t) = -t\,e^t(1 + \sin(t))$$

with $x_1(0) = x_2(0) = 0$ and the exact solutions $x_1(t) = t\,e^t$, $x_2(t) = e^t\sin(t)$ and $y(t) = \dfrac{e^t}{1+t}$. By Theorem 1, the index -2 DAEs (17) transforms to the following index-1 DAEs:

$$\begin{cases} x_1 = -t x_2 + g_1(t) & \text{(a)} \\ x' = t x_1' - 2t x_1 + g_2(t) x_2 + g_3(t), & \text{(b)} \end{cases} \tag{18}$$

with $x_1(0) = x_2(0) = 0$, when $g_1(t) = t\,e^t(1 + \sin(t),\ g_2(t) = \dfrac{1 - t^2 - t^3}{1+t}$ and

$$g_3(t) = e^t(\cos(t) - t + t^2 + \frac{t + t^2 + t^3}{1+t}\sin(t)).$$

Similar to example 1, we transform the algebraic equation (18a) in the iterative form with respect to x_1 and construct the suitable correction functional in x_2-direction by using (7) for the differential equation (18a). Therefore, system (18) is expressed as

$$\begin{cases} x_1^{(n+1)} = -t\,x_2^{(n)}(t) + g_1(t) & \text{(a)} \\ x_2^{(n+1)} = x_2^{(n)}(t) + \int_0^t \lambda(s)\left[x_2'(s) - s\,\tilde{x}_1'^{(n)}(s) + 2s\,\tilde{x}_1^{(n)}(s) - g_2(s)\tilde{x}_2^{(n)}(s) - g_3(s)\right]ds, & \text{(b)} \end{cases} \tag{19}$$

where $\tilde{x}_1^{(n)}$ and $\tilde{x}_2^{(n)}$ denote restricted variations, i.e. , $\delta\tilde{x}_1^{(n)} = \delta\tilde{x}_2^{(n)} = 0$.To find the optimal value of λ in (19b), we have

$$\delta x_2^{(n+1)}(t) = \delta x_2^{(n)}(t) + \delta\int_0^t \lambda(s)\left[x_2'^{(n)}(s) - s\,\tilde{x}_1'^{(n)}(s) + 2s\,\tilde{x}_1^{(n)}(s) - g_2(s)\tilde{x}_2^{(n)}(s) - g_3(s)\right]ds.$$

Therefore

$$\delta x_2^{(n+1)}(t) = \delta x_2^{(n)}(t) + \lambda(s)\delta x_2^{(n)}(s)\big|_{s=t} - \int_0^t \lambda'(s)\delta x_2^{(n)}(s)ds.$$

Imposing $\delta\tilde{x}_2^{(n+1)} = 0$ yields the following stationary conditions:

$$\begin{cases} 1 + \lambda(s)\,|_{s=t} = 0 \\ -\lambda'(s) = 0. \end{cases} \tag{20}$$

Therefore the optimal value of Lagrange multiplier is

$$\lambda(s) = -1. \tag{21}$$

By substituting this value into correction functional (19b), the following iteration formula is obtained:

$$\begin{cases} x_1^{(n+1)} = -t\,x_2^{(n)}(t) + g_1(t) & \text{(a)} \\ x_2^{(n+1)} = x_2^{(n)}(t) - \int_0^t \left[x_2'^{(n)}(s) - s\,\tilde{x}_1'^{(n)}(s) + 2s\,\tilde{x}_1^{(n)}(s) - g_2(s)\tilde{x}_2^{(n)}(s) - g_3(s)\right]ds & \text{(b)} \end{cases} \tag{22}$$

with $n = 0,1,2,\ldots$, $x_1^{(0)} = x_1(0) = 0$ and $x_2^{(0)} = x_2(0) = 0$. We expand the functions $g_1(t)$, $g_2(t)$ and $g_3(t)$ by Taylor series expansion at $t = 0$ with $v = 14$.So after 18 iterations, (22) yields

$$\begin{cases} x_1^{(1)}(t) = t + 2t^2 + \frac{3}{2}t^3 + \frac{1}{2}t^4 + \cdots \\ x_2^{(1)}(t) = t + \frac{1}{2}t^2 + t^3 + \frac{5}{12}t^4 + \cdots \end{cases}$$

$$\vdots$$

$$\begin{cases} x_1^{(18)}(t) = t + t^2 + \frac{1}{2}t^3 + \frac{1}{6}t^4 + \cdots + \frac{1}{681080400}t^{15} + \frac{172992877}{8382528000}t^{16} + \ldots \\ x_2^{(18)}(t) = t + t^2 + \frac{1}{3}t^3 - \frac{1}{30}t^5 + \cdots - \frac{1}{681080400}t^{14} - \frac{172992877}{8382528000}t^{15} - \ldots \end{cases}$$

In the alternative method, we expanded the coefficient functions e^{-t}, sin(t) and e^{s-t} by Chebyshev series with the same number of terms. Using (17) and (18), after 18 iterations, we obtain

$$\begin{cases} x_1^{(1)}(t) = 3.74(-18) + 1.00\,t + \cdots - 2.38(-8)t^{14} \\ x_2^{(1)}(t) = 9.99(-1)\,t + 5.00(-1)t^2 + \cdots + 2.97(-9)t^{16} \end{cases}$$

$$\vdots$$

$$\begin{cases} x_1^{(18)}(t) = 3.74(-18) + 1.00\,t - 6.00(-4)\,t^{19} \cdots \\ x_2^{(18)}(t) = 9.99(-1)\,t + 1.00\,t^2 + \cdots - 1.44(-3)t^{19} + \ldots \end{cases}$$

For the sake of comparison, we have illustrated the absolute errors for $x_1^{(18)}$ and $x_2^{(18)}$ using both Taylor and Chebyshev cases in table 1. This Table, obviously, shows the superiority of using Chebyshev approximation instead of Taylor approximation in this example. In this example, we have used $x_1^{(n+1)}$ instead of $x_1^{(n)}$ in (22b) to compute $x_2^{(n+1)}$.

Table 1

	Absolute errors of using Taylor polynomials for computation of $\tilde{x}_1$ and $\tilde{x}_2$ in Ex. 2.		Absolute errors of using Chebyshev polynomials for computation of $\tilde{x}_1$ and $\tilde{x}_2$ in Ex. 2.	
t	$\lvert\tilde{x}_1(t)-x_1(t)\rvert$	$\lvert\tilde{x}_2(t)-x_2(t)\rvert$	$\lvert\tilde{x}_1(t)-x_1(t)\rvert$	$\lvert\tilde{x}_2(t)-x_2(t)\rvert$
0.1	2.51(−18)	2.50(−17)	2.38(−15)	2.38(−14)
0.2	1.91(−13)	9.57(−13)	1.35(−14)	6.79(−14)
0.3	1.41(−10)	4.73(−10)	2.91(−14)	9.70(−14)
0.4	1.55(−8)	3.88(−8)	4.32(−14)	1.08(−14)
0.5	5.91(−7)	1.18(−6)	2.79(−14)	1.57(−13)

The comparison between the results mentioned in Figures 1 and 2 and Table 1 show the power of the proposed method of this paper, for these examples.

5. CONCLUSION

The variational iteration method (VIM) has been successful for solving many application problems. However, difficulties may arise in dealing with determining the components u_m, (7). To overcome these difficulties the modified VIM is proposed using Chebyshev polynomials and is applied for solving differential algebraic equations in this paper. The results are compared with the VIM using Taylor series. Numerical results show that using Chebyshev approximation is much more efficient than using Taylor approximation which is more popular. The proposed method can be easily generalized for more functional equations.

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