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On uniqueness of a spacewise-dependent heat source in a time-fractional heat diffusion process

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Abstract

In this paper, a multi-dimensional inverse source problem for the time-fractional diffusion equation is investigated. Uniqueness results have been proved under some conditions on the problem. The fractional differentiation is considered to be of Riesz-Caputo type.

Keywords: time-fractional equation, uniqueness result, heat source, inverse problem, parabolic heat equation.
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1 Introduction

In recent years, fractional differential equation have attracted wide attentions. Various models using fractional partial differential equations have been successfully applied to describe problems in biology, physics, chemistry and biochemistry, and finance. These new fractional-order models are more adequate than the integer-order models, because the fractional order derivatives and integrals enable the description of the memory and hereditary properties of different substance. Time-fractional diffusion equation is deduced by replacing the standard time derivative with a time fractional derivative and can be used to describe the superdiffusion and subdiffusion phenomena. The direct problems, i.e., initial value problem and initial boundary value problems for time-fractional diffusion equation have been studied extensively in recent years, for instances, on maximum principle, on some uniqueness and existence results, on numerical solutions by finite element methods and finite difference methods, on exact solutions [7]. The early papers on inverse problems were provided by Murio in [1, 2] for solving sideways fractional heat equations by mollification methods. After that, some works have been published. In [3], Cheng et al. considered an inverse problem for determining the order of fractional derivative and diffusion coefficient in fractional diffusion equation and gave a uniqueness result. In [4], Liu and Yamamoto solved a backward problem for the time-fractional diffusion equation by a quasi- reversibility regularization method. Zheng and Wei in [5, 6] solved the Cauchy problems for time fractional diffusion equation on a strip domain by a Fourier regularization and a modified equation method. In [7] the one dimentional initial-boundary value problem for time fractional diffusion equation has been dealt with in terms of left-sided

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Caputo fractional derivative. Following the ideas in [8], in this paper we are going to prove a uniqueness result for the inverse multi-dimentional problem

$$\begin{cases} {}_0^{RC}D_T^{\alpha}u + Lu = f(x) & \text{in} \quad \Omega \times (0,T), \quad 0<\alpha<1, \\ u = 0 & \text{on} \quad \Gamma \times (0,T), \\ u(x,0) = u_0(x) & \text{for} \quad x \in \Omega, \end{cases} \tag{1}$$

with additional information

$$u(x,T) = \psi_T(x), \tag{2}$$

where ${}_0^{RC}D_T^{\alpha}u$ is the Riesz-Caputo fractional derivative of u taken in terms of the time variable.

2 Main resulte

Definition 2.1.

1) The left and right Riemann-Liouville fractional integrals of order α are defined respectively by

$${}_aI_x^{\alpha}y(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}y(t)dt \quad \text{and} \quad {}_xI_b^{\alpha}y(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}y(t)dt.$$

2) The Riesz fractional integral ${}_a^RI_b^{\alpha}y$ is given by

$${}_a^RI_b^{\alpha} = \frac{1}{2}({}_aI_x^{\alpha}y(x) + {}_xI_b^{\alpha}y(x)).$$

3) The left and right Riamann-Liouville fractional derivatives of order α are defined respectively by

$${}_aD_x^{\alpha}y(x) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-t)^{-\alpha}y(t)dt \quad \text{and} \quad {}_xD_b^{\alpha}y(x) = \frac{-1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_x^b (t-x)^{-\alpha}y(t)dt.$$

4) The Riesz fractional derivative ${}_a^RD_b^{\alpha}y$ is given by

$${}_a^RD_b^{\alpha}y(x) = \frac{1}{2}({}_aD_x^{\alpha}y(x) - {}_xD_b^{\alpha}y(x)).$$

5) The left and right Caputo fractional derivatives of order α are defined respectively by

$${}_a^CD_x^{\alpha}y(x) = \frac{1}{\Gamma(1-\alpha)}\int_a^x (x-t)^{-\alpha}\frac{d}{dx}y(t)dt \quad \text{and} \quad {}_x^CD_b^{\alpha}y(x) = \frac{-1}{\Gamma(1-\alpha)}\int_x^b (t-x)^{-\alpha}\frac{d}{dx}y(t)dt.$$

6) The Riesz-Caputo fractional derivative ${}_a^{RC}D_b^{\alpha}y$ is given by

$${}_a^{RC}D_b^{\alpha}y(x) = \frac{1}{2}({}_a^CD_x^{\alpha}y(x) - {}_x^CD_b^{\alpha}y(x)).$$

Lemma 2.2. *Let ${}_aI_x^{\alpha}y(x)$, ${}_a^RI_b^{\alpha}y$, ${}_aD_x^{\alpha}y(x)$, ${}_a^RD_b^{\alpha}y(x)$, ${}_a^CD_x^{\alpha}y(x)$, ${}_a^{RC}D_b^{\alpha}y(x)$ be as above. Then we have*

$$\int_0^T {}_0^{RC}D_T^{\alpha}u(s)\cdot {}_0^{RC}D_T^{2\alpha}u(s)ds =$$

$$\frac{1}{2}{}_0^RI_T^{1-\alpha}\left({}_0^{RC}D_T^{\alpha}u(s)\right)^2\Big|_{s=0}^{T} + \frac{1}{4\Gamma(1-\alpha)}\int_0^T\left(\frac{{}_0^{RC}D_T^{\alpha}u(T)}{(T-s)^{\alpha}} - \frac{{}_0^{RC}D_T^{\alpha}u(0)}{s^{\alpha}}\right){}_0^{RC}D_T^{\alpha}u(s)ds.$$

Theorem 2.3. *Consider a linear differential operator*

$$Lu(x,t) = \nabla \cdot (-A(x)\nabla u(x,t)) + b^t(x)\nabla u(x,t) + c(x)u(x,t),$$

with bounded (dis-continuous) coefficients obeying

$$\forall u : \quad (Lu, u) \geq 0,$$

and Lu does not change sign. Let u_0, $\psi_T \in L^2(\Omega)$. Then there exists at most one spacewise-dependent heat source $f \in L^2(\Omega)$ such that (1) together with condition (2) hold.

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