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Theoretical Analysis of Rotating Hollow Cylinder's Vibrations

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Abstract

Recently, energy storage systems are one of necessity in smart microgrid systems. Electro-mechanical batteries have important advantages compared with chemical batteries. More efficiency and life time, lower weight and volume can be mentioned. The vibration of an elastic, thick, hollow and finite length cylinder when subjected to the constant angular velocity is studied in this paper. Based on the technique of variables separation and utilizing the finite circular cylinders solution, a general solution is developed to analyze the vibration of rotating cylinders. Convergence of the method is determined to calculate the natural frequencies of various geometrical configurations. It is shown that the results obtained from the present semi-analytical method are in good agreement with those obtained using the previously semi-analytical methods for zero angular velocity.

Keywords: Flywheel; Electromechanical battery; Vibration; Orthogonal Functions; Variable Separation; Body force

1. Introduction

Finite length hollow cylinders are necessary in many industries such as aerospace. These industries use cylinders for satellite electro-mechanical batteries and satellite attitude control. So it is indispensable to investigate their vibrations in different conditions. They are analytical, semi-analytical and finite element methods that presented in papers for free and forced vibration of cylinders with different end boundary conditions. The earliest investigation concerning the vibration of cylinders was performed by Pochhamer [1] and Chree [2]. Their solution was developed for an infinite long solid cylinder. The vibrations and Stability of Rotating cylinders discussed by Pidduck[3], from the general equations of elasticity, following Pochhammer's method. Gazis [4] and Armenakas [5] studied the vibration of long free hollow cylinders using linear three dimensional (3D) theory of elasticity. Hutchinson [6, 7, 8] developed a semi-

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analytical method to solve the vibrations of finite length rods and solid cylinders on the basis of linear 3D elasticity. Singal and Williams [9] investigated the vibrations of thick hollow cylinders using the energy method based on the 3D theory of elasticity. Wang and Williams [10] using the finite element method to obtain the natural frequencies or mode shapes. Zhou et al. [11] studied 3D vibrations of the solid and hollow cylinders using the Ritz method and Chebyshev Polynomials. This Modified method is used to obtain more accurate and better convergence of the results. Ebenezer et al. [12, 13] presented a method based on the use of two infinite series solutions to the governing equations to determine the response of solid cylinders to axisymmetric distributed excitations on their surfaces. The two series consist of terms that are orthogonal and form complete sets of functions in the axial and radial directions, respectively. Toudeshky, Mofakhami *et al.* [14 and 15] also presented a general semi-analytical solution using the technique of variables separation on the basis of linear 3D theory of elasticity for non-body force cylinders. They also satisfied some of the boundary conditions using orthogonalization technique while the others to be exact. In this paper using the technique of variables separation on the basis of linear 3D theory of elasticity, the semi analytical method that used in [14], is developed which covers different body forces of finite length cylinders such as rotating hollow cylinders. In this method some of the boundary conditions are approximately satisfied using orthogonalization technique and the others are satisfied complete. Comparing with the previously solutions, high accuracy are achieved using the present method.

2. Vibration Analysis

An elastic, hollow and finite cylinder that rotates with constant angular velocity Ω is considered and shown in Fig. 1. An orthogonal cylindrical coordinate system is considered as shown in this figure.

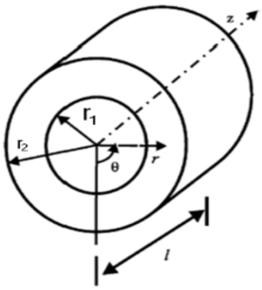

Fig. 1. A half rotating hollow cylinder

The displacement equations governing the motion of an isotropic media are Navier equations that presented below

$$(\lambda+\mu)\nabla(\nabla\mathcal{U})+\mu\nabla^2\mathcal{U}+\mathcal{F}=\rho\frac{\partial^2\mathcal{U}}{\partial t^2} \quad (1)$$

where the corresponding components of the displacement vector $\mathcal{U}$ at a point are, u_r, u_θ and u_z in the r, θ and z directions, respectively and ρ is the density, λ and μ are the lame constants, ∇^2 is the 3D Laplac ian operator and $\mathcal{F}$ is the body force. The most general solution of Eq. 1 may be obtained using Helmholt z decomposition as follows:

$$\mathcal{U}=\nabla\varphi+\nabla\times\mathcal{H} \quad (2)$$

With using the Newton's second law:

$$\mathcal{F} = \rho \boldsymbol{a} = \rho \frac{\partial^2 \mathcal{V}}{\partial t^2} \tag{3}$$

Vector $\mathcal{V}$ is the displacement at a point produced by the body force. Using the Helmholtz decomposition we can write

$$\mathcal{V} = \nabla \xi + \nabla \times \mathcal{B}$$
(4)

where in (2), (3) and (4), φ and ξ are scalar and $\mathcal{H}$ and $\mathcal{B}$ are vector potential functions. $\nabla \times \mathcal{H}$ and $\nabla \times \mathcal{B}$ are arbitrary functions that may be taken as zero. Substituting $\mathcal{U}$ and $\mathcal{V}$ from Eq. (2) into Eq. (1) results the following wave equation:

$$\nabla^2 \varphi = \frac{1}{c_1^2} \frac{\partial^2 (\varphi - \xi)}{\partial t^2}, \quad \nabla^2 \mathcal{H} = \frac{1}{c_2^2} \frac{\partial^2 (\mathcal{H} - \mathcal{B})}{\partial t^2} \tag{5}$$

where

$$c_1 = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_2 = \sqrt{\frac{\mu}{\rho}} \tag{6}$$

The constants c_1 and c_2 are the propagation velocity of dilatational and distortional waves in an infinite medium, respectively. For solving the Navier equation the following general solutions in cylindrical coordinates for Helmholtz decomposition is the technique of variable separation that can be obtained as:

$$\varphi(r,\theta,z,t) = [R_1(\alpha_1 r) T_1(\delta_1 z) + \overline{R_1}(\bar{\alpha}_1 r) \overline{T_1}(\bar{\delta}_1 z)] E_1(\nu\theta) e^{i\omega t}$$

$$\varphi(r,\theta,z,t) = [R_1(\alpha_1 r) T_1(\delta_1 z) + \overline{R_1}(\bar{\alpha}_1 r) \overline{T_1}(\bar{\delta}_1 z)] E_1(\nu\theta) e^{i\omega t}$$

$$H_r(r,\theta,z,t) = [R_2(\alpha_{23} r) T_2(\delta_{23} z) + \overline{R_2}(\bar{\alpha}_{23} r) \overline{T_2}(\bar{\delta}_{23} z)] E_2(\nu\theta) e^{i\omega t}$$

$$H_\theta(r,\theta,z,t) = \left[R_3(\alpha_{23} r) T_3(\delta_{23} z) + \overline{R_3}(\bar{\alpha}_{23} r) \overline{T_3}(\bar{\delta}_{23} z)\right] E_3(\nu\theta) e^{i\omega t}$$

$$H_z(r,\theta,z,t) = \left[R_4(\alpha_4 r) T_4(\delta_4 z) + \overline{R_4}(\bar{\alpha}_4 r) \overline{T_4}(\bar{\delta}_4 z)\right] E_4(\nu\theta) e^{i\omega t} \tag{7}$$

And also with this technique the displacements of body force ξ, B_r, B_θ and B_z can write the coefficients of φ, H_r, H_θ, and H_z. These coefficients must be obtained according to the body forces. So

$$\xi(r,\theta,z,t) = M_1[R_1(\alpha_1 r) T_1(\delta_1 z) + \overline{R_1}(\bar{\alpha}_1 r) \overline{T_1}(\bar{\delta}_1 z)] E_1(\nu\theta) e^{i\omega t}$$

$$B_r(r,\theta,z,t) = M[R_2(\alpha_{23} r) T_2(\delta_{23} z) + \overline{R_2}(\bar{\alpha}_{23} r) \overline{T_2}(\bar{\delta}_{23} z)] E_2(\nu\theta) e^{i\omega t}$$

$$B_\theta(r,\theta,z,t) = M_3[R_3(\alpha_{23} r) T_3(\delta_{23} z) + \overline{R_3}(\bar{\alpha}_{23} r) \overline{T_3}(\bar{\delta}_{23} z)] E_3(\nu\theta) e^{i\omega t}$$

$$B_z(r,\theta,z,t) = M_4[R_4(\alpha_4 r)T_4(\delta_4 z) + \overline{R_4}(\bar{\alpha}_4 r)\overline{T_4}(\bar{\delta}_4 z)]E_4(\nu\theta)e^{i\omega t} \tag{8}$$

$M_i, i = 1,2,3,4$ are the coefficients that must be obtained according to the body forces.
where:

$$R_1(\alpha_1 r) = F_1 J_\nu(\alpha_1 r) + G_1 Y_\nu(\alpha_1 r)$$

$$R_2(\alpha_{23} r) = F_2 J_{\nu+1}(\alpha_{23} r) + G_2 Y_{\nu+1}(\alpha_{23} r) + \frac{\nu}{\alpha_{23} r} \times \left[(F_3 - F_2) J_\nu(\alpha_{23} r) + (G_3 - G_2) Y_\nu(\alpha_{23} r)\right]$$

$$R_3(\alpha_{23} r) = F_3 J_{\nu+1}(\alpha_{23} r) + G_3 Y_{\nu+1}(\alpha_{23} r) + \frac{\nu}{\alpha_{23} r}\left[(F_2 - F_3) J_\nu(\alpha_{23} r) + (G_2 - G_3) Y_\nu(\alpha_{23} r)\right]$$

$$R_4(\alpha_4 r) = F_4 J_\nu(\alpha_4 r) + G_4 Y_\nu(\alpha_4 r)$$

$$T_1(\delta_1 z) = \begin{Bmatrix} \cos(\delta_1 z) \\ \sin(\delta_1 z) \end{Bmatrix}, \quad T_2(\delta_{23} z) = \begin{Bmatrix} \sin(\delta_{23} z) \\ \cos(\delta_{23} z) \end{Bmatrix}$$

$$T_3(\delta_{23} z) = \begin{Bmatrix} -\sin(\delta_{23} z) \\ \cos(\delta_{23} z) \end{Bmatrix}, \quad T_4(\delta_4 z) = \begin{Bmatrix} \cos(\delta_4 z) \\ \sin(\delta_4 z) \end{Bmatrix}$$

$$E_1(\upsilon\theta) = E_3(\upsilon\theta) = \begin{Bmatrix} \cos(\upsilon\theta) \\ \sin(\upsilon\theta) \end{Bmatrix}$$

$$E_2(\upsilon\theta) = E_4(\upsilon\theta) = \begin{Bmatrix} \sin(\upsilon\theta) \\ \cos(\upsilon\theta) \end{Bmatrix}$$

$$\delta_1^2 + \alpha_1^2 = \bar{\delta}_1^2 + \bar{\alpha}_1^2 = \left(\frac{\omega^2(1 - M_1)}{c_1^2}\right)$$

$$\delta_{23}^2 + \alpha_{23}^2 = \bar{\delta}_{23}^2 + \bar{\alpha}_{23}^2 = \left(\frac{\omega^2(2 - M_3 - M_2)}{c_2^2}\right)$$

$$\delta_4^2 + \alpha_4^2 = \bar{\delta}_4^2 + \bar{\alpha}_4^2 = \left(\frac{\omega^2(1 - M_4)}{c_2^2}\right) \tag{9}$$

J and Y are the Bessel functions and ν is real argument, F_k and G_k $(k = 1,2,3,4)$ are the constants. The functions of $\bar{R}_k(\bar{\alpha} r)$ and $\bar{T}_k(\bar{\delta} z)$ $(k = 1,2,3,4)$ can be calculated using $\bar{\alpha}$ and $\bar{\delta}$ in Eq. (7) instead of α and δ, respectively.

In comparison with this method that presented by Mofakhami et al. [14, 15] for cylinders, the presented solution is general and can used to analysis the wave propagation in infinite or finite circular cylinders that exited by the body forces. In this paper the body force that excited in the hollow cylinder is the centrifugal force and the strain that produce because of constant angular velocity. This is applicable in electro-mechanical batteries in energy storage system (flywheel). In addition the present method can be used to evaluate the wave propagation in cylinders with different end boundary conditions, which will be investigated in the future.

If an elastic circular cylinder with inner and outer radii of r_1 and r_2 and finite length of $2l$ is considered, the generalized solution form (7) and (8) is reduced to a simple form with 12 independent terms by identifying the variables as follows:

$$M_4 = M_2 + M_3 - 1, \quad \upsilon = n$$
$$\delta_2 = \delta_{23} = \delta_4, \quad \alpha = \alpha_1 = \alpha_2 = \alpha_{23} = \alpha_4$$
$$\bar{\delta} = \bar{\delta}_1 = \bar{\delta}_2 = \bar{\delta}_{23} = \bar{\delta}_4, \quad \bar{\alpha}_2 = \bar{\alpha}_{23} = \bar{\alpha}_4 \tag{10}$$

In these equations n is circumferential wave number. The displacement components can be obtained by substituting the Eq. (10) into eqs. (7) and (8) that presented in Appendix A. Then with combining these displacements leaded to the following displacement components can write:

$$u_r + v_r = \Bigg(\Bigg[\left(\frac{n}{r} J_n(\alpha r) - \alpha J_{n+1}(\alpha r) \right) A_1 + \left(\frac{n}{r} Y_n(\alpha r) - \alpha Y_{n+1}(\alpha r) \right) B_1 \Bigg] \begin{Bmatrix} \cos(\delta_1 z) \\ \sin(\delta_1 z) \end{Bmatrix} +$$
$$\left[\delta_2 \frac{n}{r} J_{n+1}(\alpha r) A_2 + \delta_2 Y_{n+1}(\alpha r) B_2 + \left\{ \frac{n}{r} J_n(\alpha r) A_3 + \frac{n}{r} Y_n(\alpha r) B_3 \right\} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \right] \begin{Bmatrix} \cos(\delta_2 z) \\ \sin(\delta_2 z) \end{Bmatrix} +$$
$$\Bigg[\left(\frac{n}{r} J_n(\bar{\alpha}_1 r) - \bar{\alpha}_1 J_{n+1}(\bar{\alpha}_1 r) \right) A_4 + \left(\frac{n}{r} Y_n(\bar{\alpha}_1 r) - \bar{\alpha}_1 Y_{n+1}(\bar{\alpha}_1 r) \right) B_4 + \bar{\delta} J_{n+1}(\bar{\alpha}_2 r) A_5 + \bar{\delta} Y_{n+1}(\bar{\alpha}_2 r) B_5 +$$
$$\left\{ \frac{n}{r} J_n(\bar{\alpha}_2 r) A_6 + \frac{n}{r} \bar{Y}_n(\bar{\alpha}_2 r) B_6 \right\} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \Bigg] \begin{Bmatrix} \cos(\bar{\delta} z) \\ \sin(\bar{\delta} z) \end{Bmatrix} \Bigg) \begin{Bmatrix} \cos n\theta \\ \sin n\theta \end{Bmatrix} e^{i\omega t}$$

$$u_\theta + v_\theta = \Bigg(\Bigg[\left(-\frac{n}{r} J_n(\alpha r) A_1 - \frac{n}{r} Y_n(\alpha r) B_1 \right) \Bigg] \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \begin{Bmatrix} \cos(\delta_1 z) \\ \sin(\delta_1 z) \end{Bmatrix} + \Bigg[\delta_2 \left(J_{n+1}(\alpha r) A_2 + \delta_2 Y_{n+1}(\alpha r) \right) B_2 \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} -$$
$$\left(\frac{n}{r} J_n(\alpha r) - \alpha J_{n+1}(\alpha r) \right) A_3 - \left(\frac{n}{r} Y_n(\alpha r) - \alpha Y_{n+1}(\alpha r) \right) B_3 \Bigg] \begin{Bmatrix} \cos(\delta_2 z) \\ \sin(\delta_2 z) \end{Bmatrix} +$$
$$\Bigg[\left(-\frac{n}{r} J_n(\bar{\alpha}_1 r) A_4 - \frac{n}{r} Y_n(\bar{\alpha}_1 r) B_4 + \bar{\delta} J_{n+1}(\bar{\alpha}_2 r) A_5 + \bar{\delta} Y_{n+1}(\bar{\alpha}_2 r) B_5 \right) \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} -$$
$$\left(\frac{n}{r} J_n(\bar{\alpha}_2 r) - \bar{\alpha}_2 J_{n+1}(\bar{\alpha}_2 r) \right) A_6 - \left(\frac{n}{r} Y_n(\bar{\alpha}_2 r) - \bar{\alpha}_2 Y_{n+1}(\bar{\alpha}_2 r) \right) B_6 \Bigg] \begin{Bmatrix} \cos(\bar{\delta} z) \\ \sin(\bar{\delta} z) \end{Bmatrix} \Bigg) \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix} e^{i\omega t}$$

$$u_z + v_z = -\Bigg(\left[\delta_1 J_n(\alpha r) A_1 - \delta_1 Y_n(\alpha r) B_1 \right] \begin{Bmatrix} \sin(\delta_1 z) \\ \cos(\delta_1 z) \end{Bmatrix} + \left[\alpha J_n(\alpha r) A_2 + \alpha Y_n(\alpha r) B_2 \right] \begin{Bmatrix} \sin(\delta_2 z) \\ \cos(\delta_2 z) \end{Bmatrix} +$$
$$\left[\bar{\delta} J_n(\bar{\alpha}_1 r) A_4 + \bar{\delta} Y_n(\bar{\alpha}_1 r) B_4 + \bar{\alpha}_2 J_n(\bar{\alpha}_2 r) A_5 + \bar{\alpha}_2 Y_n(\bar{\alpha}_2 r) B_5 \right] \times$$
$$\begin{Bmatrix} \sin(\bar{\delta} z) \\ \cos(\bar{\delta} z) \end{Bmatrix} \Bigg) \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \begin{Bmatrix} \sin n\theta \\ \cos n\theta \end{Bmatrix} e^{i\omega t} \tag{11}$$

where

$$A_1 = F_1(1+M_1), \quad A_2 = F_3(1+M_3), \quad A_3 = F_4(1+M_4) + \frac{\delta_2}{\alpha}(F_2(1+M_2) - F_3(1+M_3)) \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$
$$B_1 = G_1(1+M_1), \quad B_2 = G_3(1+M_3), \quad B_3 = G_4(1+M_4) + \frac{\delta_2}{\alpha}(G_2(1+M_2) - G_3(1+M_3)) \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$A_4 = \bar{F}_1(1+M_1),\quad A_5 = \bar{F}_3(1+M_3),\quad A_6 = \overline{F_4}(1+M_4)+\frac{\bar{\delta}_2}{\bar{\alpha}_2}(\bar{F}_2(1+M_2) - \bar{F}_3(1+M_3))\begin{Bmatrix}1\\-1\end{Bmatrix}$$

$$B_4 = \bar{G}_1(1+M_1),\quad B_5 = \bar{G}_3(1+M_3),\quad B_6 = \overline{G_4}(1+M_4)+\frac{\bar{\delta}_2}{\bar{\alpha}_2}(\bar{G}_2(1+M_2)-\bar{G}_3(1+M_3))\begin{Bmatrix}1\\-1\end{Bmatrix} \tag{12}$$

There are two forms of symmetric and anti-symmetric solutions. The relevant stress components can be obtained in a similar form as the displacement components that given in Appendix B. In the following, solution of the hollow rotating cylinder that can use in the electro-mechanical battery is performed.

3. Free-end Hollow Rotating Cylinder

The rotating cylinder considered that rotates with constant angular velocity Ω around the z direction. Because of the rotational around the z direction the centrifugation force is produced. Adopting Southwell's definitions of stress and stress-strain relations, there is an exact solution of the problem of steady motion even when the strain is finite. So the Navier equation for this cylinder is given in Eq. (13).

$$\mu\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}\right)+$$

$$(\lambda+\mu)\times\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)+\rho r\Omega^2+\rho\left(-2\Omega\frac{\partial u_\theta}{\partial t}\right)+\rho r\Omega^2\left(\frac{\partial u_r}{\partial r}+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)$$

$$=\frac{\partial^2 u_r}{\partial t^2}\,\mu\left(\nabla^2 u_\theta - \frac{u_\theta}{r^2}+\frac{2}{r^2}\frac{\partial u_r}{\partial \theta}\right)+(\lambda+\mu)\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)+\rho\left(2\Omega\frac{\partial u_r}{\partial t}\right)$$

$$=\frac{\partial^2 u_\theta}{\partial t^2}\,\mu\nabla^2 u_z + (\lambda+\mu)\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial u_z}{\partial z}\right)=\frac{\partial^2 u_z}{\partial t^2} \tag{13}$$

With the relation between body forces, Eqs. (7) and (8) the body forces can obtained with the new equations that M_i, $i = 1,2,3,4$ are determined and they are the functions of another obvious parameters.

The boundary conditions of the free- end hollow rotating cylinder are defined as follows:

$$\sigma_{rr} = \sigma_{r\theta} = \sigma_{rz} = 0 \text{ at } r = r_1, \mathrm{r}_2 \quad \sigma_{zz} = \sigma_{rz} = \sigma_{\theta z} = 0 \text{ at } z = -1, 1 \tag{14}$$

Exact satisfaction of the end boundary conditions for $r_1 \le r \le r_2$ forces more than on relationship between the 12 independent coefficients presented in Eq. (11) and Appendix B. So, some of the boundary conditions should be approximately satisfied using the orthogonalization technique. That is arbitrary to choose which boundary conditions are satisfied exactly or approximately. For example the following boundary conditions are satisfied exactly:

$$\sigma_{rz} = 0 \text{ at } r = r_1, \mathrm{r}_2 \quad \sigma_{rz} = \sigma_{\theta z} = 0 \text{ at } at\ z = -1, 1 \tag{15}$$

And the other boundary conditions are satisfied using orthogonalization technique:

$$\sigma_{rr} = \sigma_{r\theta} = 0 \text{ at } r = r_1, \mathrm{r}_2 \quad \sigma_{zz} = 0 \text{ at } z = -1, 1 \tag{16}$$

Satisfaction of the boundary conditions for the first form of the solution “symmetric mode” leads to the following expressions:

$$A_{2j} = -A_{1j}\,\frac{2\delta_{1j}\alpha_j}{\alpha_j^{\,2}-\delta_{2j}^2}\,\frac{\sin(\delta_{1j}l)}{sin(\delta_{2j}l)}$$
$$B_{2j} = -B_{1j}\,\frac{2\delta_{1j}\alpha_j}{\alpha_j^{\,2}-\delta_{2j}^2}\,\frac{\sin(\delta_{1j}l)}{sin(\delta_{2j}l)}$$
$$A_{3j} = -A_{1j}\,\frac{2\delta_{1j}\delta_{2j}}{\alpha_j^{\,2}-\delta_{2j}^2}\,\frac{\sin(\delta_{1j}l)}{sin(\delta_{2j}l)}$$
$$B_{3j} = -B_{1j}\,\frac{2\delta_{1j}\delta_{2j}}{\alpha_j^{\,2}-\delta_{2j}^2}\,\frac{\sin(\delta_{1j}l)}{sin(\delta_{2j}l)}$$
$$A_{5i} = K_{1i}A_{4i} + K_{2i}B_{4i} + K_{3i}A_{6i} + K_{4i}B_{6i}$$
$$B_{5i} = \Lambda_{1i}A_{4i} + \Lambda_{2i}B_{4i} + \Lambda_{3i}A_{6i} + \Lambda_{4i}B_{6i}$$
$$\bar{\delta}_i = \frac{(i-1)\pi}{l}\,,\quad i = 1,2,\cdots \tag{17}$$

where the coefficients K_{ki} and Λ_{ki} (k=1, 2, 3, 4) are the functions of $\bar{\alpha}_{2i}$, $\bar{\alpha}_{ij}$ and $\bar{\delta}_i$ given in Appendix C. The orthogonality on the second boundary conditions in Eq. (16), the following condition is considered:

$$J_n'\left(\alpha_j b\right)A_{1j} = -Y_n'\left(\alpha_j b\right)B_{1j} \tag{18}$$

where prime denotes differentiation with respect to the relevant argument α_j $(j = 1,2,3,\cdots)$ is the root of $p_n'\left(\alpha_j a\right)$ and $p_n\left(\alpha_j r\right)$ is the orthogonal function. So, the boundary conditions (14) are satisfied using orthogonalization as follows:

$$\int_{r_1}^{r_2}\sigma_{zz}(r,\theta,l)\,p_n\left(\alpha_j r\right)r\,dr = 0$$
$$\int_0^1\sigma_{rr}(r,\theta,z)\begin{Bmatrix}\cos\left(\bar{\delta}_i z\right)\\ \sin\left(\bar{\delta}_i z\right)\end{Bmatrix}dz = 0\,,\quad r = r_1, r_2$$
$$\int_0^1\sigma_{r\theta}(r,\theta,z)\begin{Bmatrix}\cos\left(\bar{\delta}_i z\right)\\ \sin\left(\bar{\delta}_i z\right)\end{Bmatrix}\mathrm{dz} = 0\,,\quad r = r_1, r_2 \tag{19}$$

The displacement and stress field are written in the series form with the indices i and j which are truncated with N_1 and N_2 terms respectively. Using Appendix B and Eq. (19) the linear algebraic system of equation $\left(N_1 + 4N_2\right)\left(N_1 + 4N_2\right)$ is obtained:

$$\left[\boldsymbol{Y}_{st}\right]_{5\times5}\left[\mathbf{X}_{\mathrm{t}}\right]_{5\times1} = \left[0\right]_{5\times1} \tag{20}$$

The components of matrix and vectors in equation [20] are

$$\boldsymbol{Y}_{st} = \left[\boldsymbol{Y}_{st,ij}\right]_{\tau*\eta}\,,\quad \tau = \begin{Bmatrix}N_1 & s=1\\ N_2 & s\neq1\end{Bmatrix}$$
$$\eta = \begin{Bmatrix}N_1 & t=1\\ N_2 & t\neq1\end{Bmatrix}$$

$$\mathbf{X}_1 = \left\{A_{1j}\right\}_{N_1\times 1},\ \ \mathbf{X}_2 = \left\{A_{4j}\right\}_{N_2\times 1}$$
$$\mathbf{X}_3 = \left\{B_{4j}\right\}_{N_2\times 1},\ \ \mathbf{X}_4 = \left\{A_{6j}\right\}_{N_2\times 1},\ \boldsymbol{X}_5 = \left\{B_{6j}\right\}_{N_2\times 1} \tag{21}$$

where the components of the matrix $\boldsymbol{Y}_{st}$ are given in Appendix D. setting the determinant value of $\left[\boldsymbol{Y}_{\mathrm{st}}\right]_{5\times 5}$ equal to zero, the natural frequencies, the eigenvalues of the rotating cylinder can be obtained. Therefore the body forces are proportional with the displacement we can transfer them and combine them with rigidity matrix in free vibration.

4. Conclusion

A semi-analytical solution basis on the 3-D linear elastic theory is given for the vibration of finite circular cylinders with different body forces. In this method with using the technique of variables separation the minimum coefficients is required to analyze the vibration of finite circular cylinders. To evaluate the precision of the present method the natural frequencies are calculated for different geometries and compared with those reported in the previous studies for non-body force cylinders. The results are in good agreement with the reported results in the literatures. Then this method is evaluated for a rotating cylinder that used in electro-mechanical batteries. The natural frequencies are calculated for different angular velocities. The advantages of the proposed approach are its generality, accuracy and good convergence of the solution with a relatively small sized coefficient matrix.

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