جلسه دوم

The Laplace Transform

wlg

The Laplace Transform

تبدیل لاپلاس یک تابع پیوسته خطی که تابع در حوزه زمان را به حوزه فرکانس منتقل میکند

$$L[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

Eq A

به محدوده ای از صفحه s که انتگرال فوق در آن همگرا میباشد ناحیه همگرایی و یا ROC مینامند.

معکوس تبدیل لاپلاس

$$L^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{ts} ds$$

Eq B

*notes

The Laplace Transform

تبدیل لاپلاس تابع پله واحد

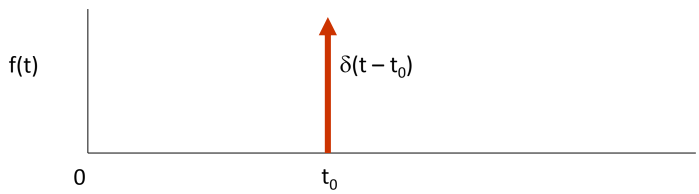
$$L[u(t)] = \int_0^{\infty} 1 e^{-st} dt = \frac{-1}{s} e^{-st} \Big|_0^{\infty}$$

$$L[u(t)] = \frac{1}{s}$$

*notes

The Laplace Transform

Pictorially, the unit impulse appears as follows:



Mathematically:

$\delta(t - t_0) = 0 \quad t \neq 0$

$$\int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)\,dt = 1 \qquad \varepsilon > 0$$

*note

The Laplace Transform

The Laplace transform of a unit impulse:

An important property of the unit impulse is a sifting or sampling property. The following is an important.

$$\int_{t_1}^{t_2} f(t)\delta(t - t_0)\,dt = \begin{cases} f(t_0) & t_1 < t_0 < t_2 \\ 0 & t_0 < t_1, t_0 > t_2 \end{cases}$$

The Laplace Transform

The Laplace transform of a unit impulse:

In particular, if we let f(t) = δ(t) and take the Laplace

$$L[\delta(t)] = \int_0^\infty \delta(t) e^{-st} dt = e^{-0s} = 1$$

The Laplace Transform

Building transform pairs:

$$L[e^{-at}u(t)] = \int_0^\infty e^{-at} e^{-st} dt = \int_0^{\infty e} e^{-(s+a)t} dt$$

$$L[e^{-at}u(t)] = \frac{-e^{-st}}{(s+a)} \Big|_0^\infty = \frac{1}{s+a}$$

A transform pair

$$e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$$

The Laplace Transform

Building transform pairs:

$$L[tu(t)] = \int_0^\infty te^{-st}dt$$

$$\int_0^\infty udv = uv\Big|_0^\infty - \int_0^\infty vdu$$

u = t
dv = e^{-st}dt

$$tu(t) \Leftrightarrow \frac{1}{s^2}$$

A transform pair

The Laplace Transform

Building transform pairs:

$$L[\cos(wt)] = \int_0^\infty \frac{(e^{jwt} + e^{-jwt})}{2} e^{-st}dt$$

$$= \frac{1}{2}\left[\frac{1}{s - jw} - \frac{1}{s + jw}\right]$$

$$= \frac{s}{s^2 + w^2}$$

$$\cos(wt)u(t) \Leftrightarrow \frac{s}{s^2 + w^2}$$

A transform pair

The Laplace Transform

Time Shift

$$L[f(t-a)u(t-a)] = \int_a^\infty f(t-a)e^{-st}$$

$$Let\ x = t-a, then\ dx = dt\ and\ t = x+a$$

$$As\ t \to a,\ x \to 0\ and\ as\ t \to \infty, x \to \infty.\ So,$$

$$\int_0^\infty f(x)e^{-s(x+a)}dx = e^{-as}\int_0^\infty f(x)e^{-sx}dx$$

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$

The Laplace Transform

Frequency Shift

$$L[e^{-at}f(t)] = \int_0^\infty [e^{-at}f(t)]e^{-st}dt$$

$$= \int_0^\infty f(t)e^{-(s+a)t}dt = F(s+a)$$

$$L[e^{-at}f(t)] = F(s+a)$$

The Laplace Transform

Example: Using Frequency Shift

Find the L[e^{-at}cos(wt)]

In this case, f(t) = cos(wt) so,

$$F(s) = \frac{s}{s^2 + w^2}$$

$$and \;\; F(s+a) = \frac{(s+a)}{(s+a)^2 + w^2}$$

$$L[e^{-at}\cos(wt)] = \frac{(s+a)}{(s+a)^2 + (w)^2}$$

The Laplace Transform

Time Integration:

The property is:

$$L\left[\int_0^\infty f(t)dt\right] = \int_0^\infty \left[\int_0^t f(x)dx\right] e^{-st}dt$$

$$Integrate \;\; by \;\; parts:$$

$$Let \;\; u = \int_0^t f(x)dx, \quad du = f(t)dt$$

$$and$$

$$dv = e^{-st}dt, \quad v = -\frac{1}{s}e^{-st}$$

The Laplace Transform

Time Integration:

Making these substitutions and carrying out
The integration shows that

$$L\left[\int_0^\infty f(t)dt\right] = \frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$
$$= \frac{1}{s}F(s)$$

The Laplace Transform

Time Differentiation:

If the L[f(t)] = F(s), we want to show:

$$L[\frac{df(t)}{dt}] = sF(s) - f(0)$$

Integrate by parts:

$$u = e^{-st},\ du = -se^{-st}dt\ \ and$$
$$dv = \frac{df(t)}{dt}dt = df(t),\ so\ \ v = f(t)$$

*note

The Laplace Transform

Time Differentiation:

Making the previous substitutions gives,

$$L\left[\frac{df}{dt}\right] = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)\left[-se^{-st}\right]dt$$

$$= 0 - f(0) + s\int_0^\infty f(t)e^{-st}dt$$

So we have shown:

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

The Laplace Transform

Time Differentiation:

We can extend the previous to show;

$$L\left[\frac{df(t)^2}{dt^2}\right] = s^2F(s) - sf(0) - f'(0)$$

$$L\left[\frac{df(t)^3}{dt^3}\right] = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

general case

$$L\left[\frac{df(t)^n}{dt^n}\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)$$

$$- \ldots - f^{(n-1)}(0)$$

The Laplace Transform

Transform Pairs:

f(t)	F(s)
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-st}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$

The Laplace Transform

Transform Pairs:

f(t)	F(s)
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(wt)$	$\frac{w}{s^2+w^2}$
$\cos(wt)$	$\frac{s}{s^2+w^2}$

The Laplace Transform

Transform Pairs:

f(t)	F(s)
$e^{-at}\sin(wt)$	$\dfrac{w}{(s+a)^2+w^2}$
$e^{-at}\cos(wt)$	$\dfrac{s+a}{(s+a)^2+w^2}$
$\sin(wt+\theta)$	$\dfrac{s\sin\theta + w\cos\theta}{s^2+w^2}$
$\cos(wt+\theta)$	$\dfrac{s\cos\theta - w\sin\theta}{s^2+w^2}$

The Laplace Transform

Common Transform Properties:

f(t)	F(s)
$f(t-t_0)u(t-t_0), t_0 \ge 0$	$e^{-t_0 s}F(s)$
$f(t)u(t-t_0), t \ge 0$	$e^{-t_0 s}L[f(t+t_0)$
$e^{-at}f(t)$	$F(s+a)$
$\dfrac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - s^0 f^{n-1}f(0)$
$tf(t)$	$-\dfrac{dF(s)}{ds}$
$\int_0^t f(\lambda)d\lambda$	$\dfrac{1}{s}F(s)$

The Laplace Transform

دستور متلب برای تبدیل لاپلاس و معکوس آن

Example

Use Matlab to find the transform of te^{-4t}

The following is written in italic to indicate Matlab code

```
syms t,s
laplace(t*exp(-4*t),t,s)
ans =
      1/(s+4)^2
```

The Laplace Transform

دستور متلب برای تبدیل لاپلاس معکوس

Example

Use Matlab to find the inverse transform of

$$F(s) = \frac{s(s+6)}{(s+3)(s^2+6s+18)} \qquad prob.12.19$$

```
syms s t

ilaplace(s*(s+6)/((s+3)*(s^2+6*s+18)))

ans =
      -exp(-3*t)+2*exp(-3*t)*cos(3*t)
```

The Laplace Transform

Theorem: قضیه مقدار اولیه در حوزه زمان

If the function f(t) and its first derivative are Laplace transformable and f(t) Has the Laplace transform F(s), and the $\lim_{s \to \infty} sF(s)$ exists, then

$$\lim_{s \to \infty} sF(s) = \lim_{t \to 0} f(t) = f(0)$$

Initial Value Theorem

The utility of this theorem lies in not having to take the inverse of F(s) in order to find out the initial condition in the time domain. This is particularly useful in circuits and systems.

The Laplace Transform

Example: Initial Value Theorem:

Given;

$$F(s) = \frac{(s+2)}{(s+1)^2 + 5^2}$$

Find f(0)

$$f(0) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s\frac{(s+2)}{(s+1)^2 + 5^2} = \lim_{s \to \infty}\left[\frac{s^2 + 2s}{s^2 + 2s + 1 + 25}\right]$$

$$= \lim_{s \to \infty} \frac{s^2/s^2 + 2s/s^2}{s^2/s^2 + 2s/s^2 + (26/s^2)} = 1$$

The Laplace Transform

Theorem: قضیه مقدار نهایی در حوزه زمان

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s), then

$$\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t) = f(\infty)$$

Final Value Theorem

Again, the utility of this theorem lies in not having to take the inverse of F(s) in order to find out the final value of f(t) in the time domain. This is particularly useful in circuits and systems.

The Laplace Transform

Example: Final Value Theorem:

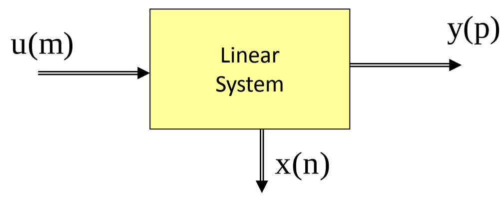
Given:

$$F(s) = \frac{(s+2)^2 - 3^2}{[(s+2)^2 + 3^2]} \qquad note\ F^{-1}(s) = te^{-2t}\cos 3t$$

Find $f(\infty)$.

$$f(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} s\frac{(s+2)^2 - 3^2}{[(s+2)^2 + 3^2]} = 0$$

State variable technique



$$u(m) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_i(t) \\ u_m(t) \end{bmatrix} \qquad x(n) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_i(t) \\ x_n(t) \end{bmatrix} \qquad y(p) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_i(t) \\ y_p(t) \end{bmatrix}$$

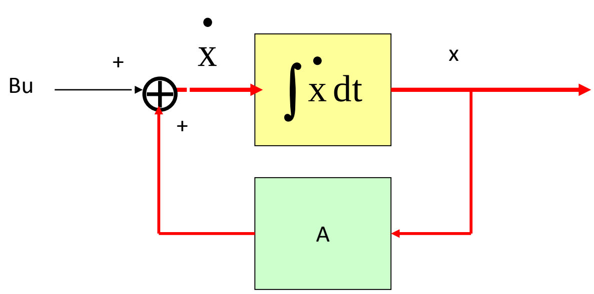
Input vector   State vector   Output vector

$$\dot{x} = \frac{dx}{dt} = A(t)x(t) + B(t)u(t)$$

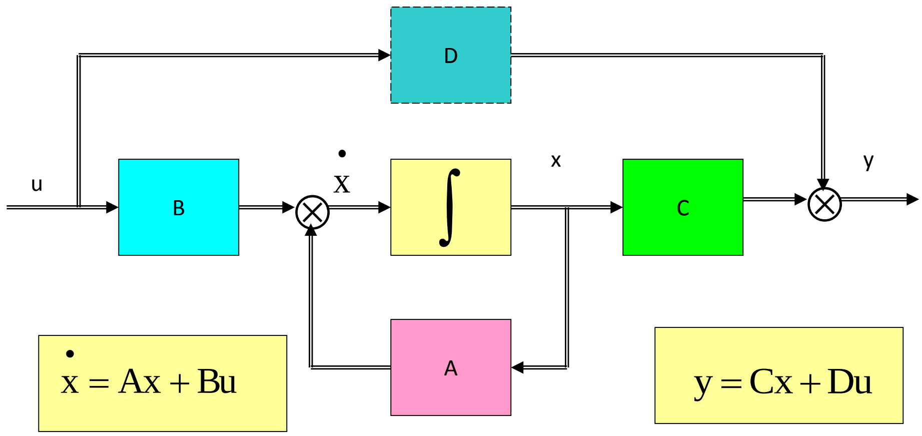
$$y(t) = C(t)x(t) + D(t)u(t)$$

Multivariable linear system (MIMO)

Representation of $\dot{x} = Ax + Bu$



State representation of a linear system



A= System Matrix(n,n)
B= Input Matrix (n,m)
x= State Vector (n,1)
u= Input Vector (m,1)

C= Output Matrix (r,n)
D= Direct Transmission Matrix (r,m)
y= Output Vector (r,1)

Transition matrix

$$\dot{x} = Ax + Bu$$

Assuming that the system is continuous and linear
that A and B are time-invariant and
Using Laplace transform

$$sX(s) - x(0) = AX(s) + BU(s)$$

$$(sI - A)X(s) = x(0) + BU(s)$$

$$X(s) = (sI - A)^{-1}[x(0) + BU(s)]$$

Taking the inverse Laplace transform of resolvent matrix

Transition matrix $$\Phi(t) = L^{-1}[(sI - A)^{-1}]$$

Transition matrix

The state vector will take the following form (convolution)

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t-\tau)Bu(\tau)d\tau$$

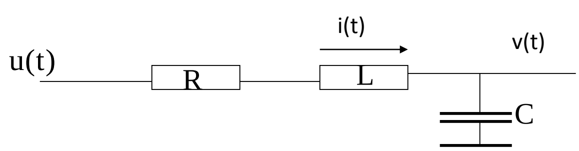
Or more generally

$$x(t) = \Phi(t-t_0)x(0) + \int_{t_0}^t \Phi(t-\tau)Bu(\tau)d\tau$$

The output vector will take the following form
Assuming that C and D are time-invariant

$$y(t) = Cx(t) + Du(t)$$

State variable technique



Basic circuit equation

$$u = Ri + L\frac{di}{dt} + v$$

$$i = C\frac{dv}{dt}$$

Can be arranged

$$\frac{di}{dt} = u = \frac{-R}{L}i - \frac{1}{L}v + \frac{1}{L}u$$

$$\frac{dv}{dt} = \frac{i}{C}$$

In matrix form

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} \frac{R_2}{R_1 + R_1} \\ \frac{R_1}{R_1 + R_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

State variable technique

$$\begin{bmatrix} \frac{di}{dt} \\ \frac{dv}{dt} \end{bmatrix} = \begin{bmatrix} \frac{-R}{L} & \frac{-1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i \\ v \end{bmatrix} + \begin{bmatrix} \frac{R_2}{L(R_1+R_1)} \\ \frac{R_1}{L(R_1+R_1)} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = C\begin{bmatrix} i \\ v \end{bmatrix} = [0 \quad 1]\begin{bmatrix} i \\ v \end{bmatrix}$$

Solution: resolvent matrix

$$[sI - A]^{-1} = \begin{bmatrix} s + \frac{R}{L} & \frac{1}{L} \\ \frac{-1}{C} & s \end{bmatrix}^{-1}$$

$$d(s) = s^2 + \frac{sR}{L} + \frac{1}{LC}$$

$$[sI - A]^{-1} = \frac{1}{s^2 + \frac{sR}{L} + \frac{1}{LC}} \begin{bmatrix} s & \frac{-1}{L} \\ \frac{1}{C} & s + \frac{R}{L} \end{bmatrix} = \begin{bmatrix} \frac{1}{sd(s)} & \frac{-1}{Ld(s)} \\ \frac{1}{Cd(s)} & \frac{s}{d(s)} + \frac{R}{Ld(s)} \end{bmatrix}$$

State variable technique

$$d(s) = s^2 + \frac{sR}{L} + \frac{1}{LC} = (s+a)(s+b)$$

$$\frac{1}{d(s)} = \frac{1}{(s+a)(s+b)} = \frac{1}{b-a}[\frac{1}{s+a} - \frac{1}{s+b}]$$

$$f(t) = L^{-1}(\frac{1}{d(s)}) = \frac{1}{b-a}[e^{-at} - e^{-bt}]$$

$$\frac{df}{dt} = \frac{1}{b-a}[-ae^{-at} + be^{-bt}]$$

$$\int f(t)dt = \frac{1}{b-a}[-\frac{e^{-at}}{a} + \frac{e^{-bt}}{b}]$$

تبدیل لاپلاس در متلب (مثال)

a. Calculate the Laplace Transform using Matlab

Calculating the Laplace **F**(s) transform of a function f(t) is quite simple in Matlab. First you need to specify that the variable t and s are symbolic ones. This is done with the command

$$f(t) = -1.25 + 3.5te^{-2t} + 1.25e^{-2t}$$

```
>> syms t s
>> f=-1.25+3.5*t*exp(-2*t)+1.25*exp(-2*t);
>> F=laplace(f,t,s)

F =

-5/4/s+7/2/(s+2)^2+5/4/(s+2)

>> simplify(F)

ans =

(s-5)/s/(s+2)^2

>> pretty(ans)
                         s - 5
                       ----------
                                2
                        s (s + 2)
```

$$F(s) = \frac{(s-5)}{s(s+2)^2}$$

```
>> syms t s
>> F=(s-5)/(s*(s+2)^2);
>> ilaplace(F)
ans =
-5/4+(7/2*t+5/4)*exp(-2*t)
>> simplify(ans)
ans =
-5/4+7/2*t*exp(-2*t)+5/4*exp(-2*t)
>> pretty(ans)
        - 5/4 + 7/2 t exp(-2 t) + 5/4 exp(-2 t)
```

Which corresponds to f(t)

$$f(t) = -1.25 + 3.5te^{-2t} + 1.25e^{-2t}$$

سیستمهای خطی در حوزه فرکانس

- سیستمهای علی: سیستم هایی علی هستند که پاسخ ضربه آنها شرط زیر را برآورده کند:

$$h(t)=0 \quad for\ t<0$$

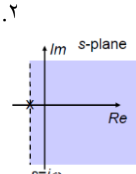
- ناحیه همگرایی سیستم های علی همیشه در سمت راست محور jw قرار دارند

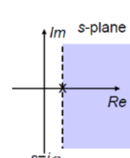
سیستم های پایدار: برای پایداری تعاریف مختلفی ارائه شده است. برای یک سیستم خطی که در حوزه زمان توصیف میشود پایداری به معنای محدود بودن سطح زیر منحنی پاسخ ضربه است یعنی:

$$\int |h(t)|dt < \infty.$$

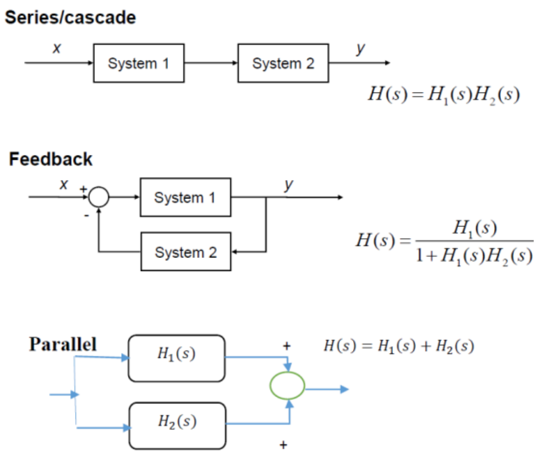
برای سیستم های توصیف شده در حوزه فرکانس به یکی از دو معنی زیر میتواند باشد:

۱. سیستم علی باشد و محور موهومی جزو ناحیه همگرایی باشد.
۲. تمام ریشه های مخرج تابع انتقال (قطبهای سیستم) برای سیستم های علی در سمت چپ محور موهومی باشند.

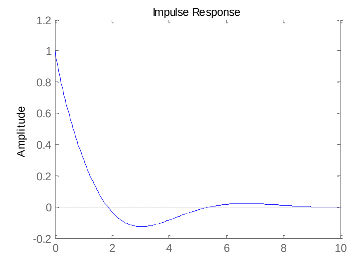


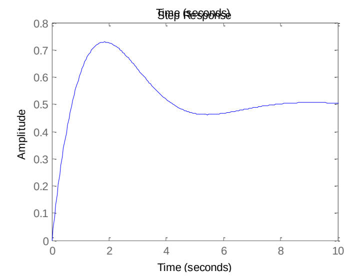


سیستمهای سری، موازی و فیدبک

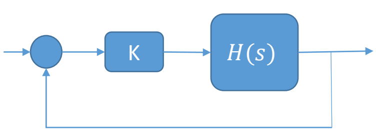



```
clear all
% system and subsystems g1(s)=s/(s(s+1)) and g2(s)=1/(s^2+2s+2)
s=tf('s');
g1=s/(s*(s+1));
g2=1/(s^2+2*s+1);
%%%%%%%%%%
g_seri=series(g1,g2);
%%%%%%%%%%
g_feed=feedback(g1,g2);
%%%%%%%%%%%%%
g_paral=parallel(g1,g2);
%%%%%%%%%%%%%
figure(1)
impulse(g_feed)
figure(2)
step(g_feed)
```





قطبها و صفرهای یک تابع انتقال



- ریشه های صورت یک تابع انتقال را «صفر» و ریشه های مخرج را «قطب» مینامند.
- برای پایداری یک سیستم حلقه بسته لازم است محل ریشه های قطبهای حلقه بسته در سمت چپ محور jw باشند.

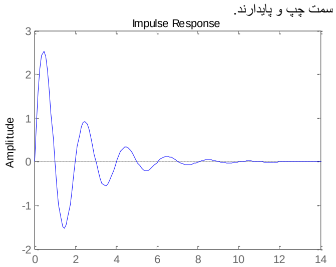
$$H(s)=\frac{1}{s(s+1)(s+5)} \text{ and } k=10$$

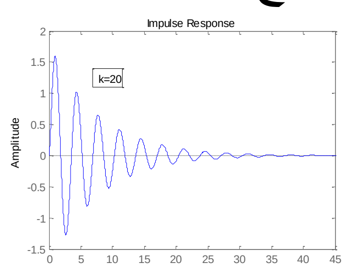
$$\frac{Y(s)}{R(s)}=\frac{k}{s^2+s+k}$$

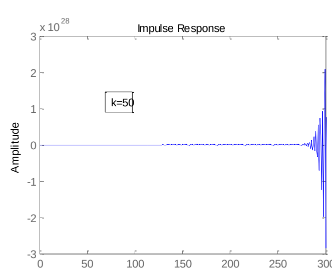
محل قطبهای حلقه بسته بازاء k=10 با دستور متلب

```
roots([1 1 10])
ans =
 -0.5000 + 3.1225i
 -0.5000 - 3.1225i
```

همگی سمت چپ و پایدارند.







$$H(s)=\frac{1}{s(s+1)(s+5)} \text{ and k=20 and k=50}$$

$$\frac{Y(s)}{R(s)}=\frac{k}{s^3+6s^2+5s+k}$$

محل قطبهای حلقه بسته با k=20 و k=50

```
>> roots([1 6 5 20])
ans =
 -5.7362 + 0.0000i
 -0.1319 + 1.8626i
 -0.1319 - 1.8626i
>> roots([1 6 5 50])
ans =
 -6.4314 + 0.0000i
  0.2157 + 2.7799i
  0.2157 - 2.7799i

>>
```