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Reducing index, and pseudospectral method for high index differential-algebraic equations

M. Ghovatmand[1,2], M.M. Hosseini[1,*], M. Jafari[3]

[1] *Faculty of Mathematics, Yazd University, Yazd, Iran.*
[2] *Department of Mathematics, Shahrood University of Technology, Shahrood, Iran.*
[3] *I.A.U., Najafabad Branch, Department of Mathematics, Najafabad, Iran.*

Abstract. In this paper, an efficient index reduction technique for high index linear differential algebraic equations is suggested. Also, their numerical solution is considered by pseudospectral method. For Hessenberg high index system, the index reduction technique is defined and its use is demonstrated. We give a condition under which the general linear form of the problem can be transformed to the index reduced form with index-1 by a simple formulation. Furthermore, with providing some examples, the aforementioned cases are dealt with numerically.

Keywords: Differential-algebraic equations; Index reduction techniques; Pseudospectral method.
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1 Introduction

Many physical problems are most easily initially modelled as a system of differential-algebraic equations (DAEs) [7]. Some numerical methods have been developed, using both BDF [7,10] and implicit Runge-Kutta methods [1]. These methods are only directly suitable for low index problems and often require that the problem to have special structure. Although some important applications can be solved by these methods; there is a need for more general approaches. Some more general approaches were proposed in [2–4, 11, 12].

In this paper, we consider a linear (or linearized) model problem,

$$\begin{aligned} X^{(m)} &= \textstyle\sum_{j=1}^{m} A_j X^{(j-1)} + By + q, \quad (a) \\ 0 &= CX + r, \quad (b) \end{aligned} \tag{1.1}$$

*Correspondence to: Faculty of Mathematics, Yazd University, Yazd, Iran
Email: hosse_m@yazduni.ac.ir
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where A_j, B and C are smooth functions of t, $t_0 \leq t \leq t_f$, $A_j(t) \in R^{n\times n}$, $j = 1,\dots,m$, $B(t) \in R^{n\times 1}$, $C(t) \in R^{1\times n}$,$n \geq 2$, and CB is nonsingular for each t (hence the DAE has index-$m+1$). The inhomogeneities are $q(t) \in R^n$ and $r(t) \in R$. It is well-known that the DAE (1.1) can be difficult to solve when it has a higher index (index greater than one, [2]). In this case an alternative treatment is the use of index reduction methods (see, e.g., [7, 10, 13]), until a well-posed problem (index-1 DAE or ordinary differential equations) is obtained. Early in [5] and [9], the index-(m+1) DAE (1.1) has been reduced to index-m DAE. In this paper, we will transform the index-(m+1) DAE (1.1) to index-1 DAE by introducing a simple formulation.

2 A simple formulation for index reduction

First, consider problem (1.1) with arbitrary m,

$$\begin{aligned} X^{(m)} &= \textstyle\sum_{j=1}^{m} A_j X^{(j-1)} + By + q, \quad (a)\\ 0 &= CX + r, \qquad\qquad\qquad (b) \end{aligned} \tag{2.1}$$

this problem is called the Hessenberg system where $A = (a_{ij})_{n\times n}$, $B = (b_i)_{n\times 1}$,$C = (c_i)_{1\times n}$, $q = (q_i)_{n\times 1}$,$n \geq 2$, and $CB(t) \neq 0$ for all $t \in [0, t_f]$, i.e.,

$$CB(t) = \sum_{i=1}^{n} (c_i b_i)(t) \neq 0, \qquad \forall t \in [0, t_f]. \tag{2.2}$$

By (2.1) and (2.2), we have,

$$y = (CB)^{-1} C \left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q \right], \tag{2.3}$$

and substituting (2.3) into (2.1) implies,

$$X^{(m)} = \sum_{j=1}^{m} A_j X^{(j-1)} + B(CB)^{-1} C \left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q \right] + q,$$

$$CX + r = 0.$$

So, the problem (2.1) can be written as:

$$\begin{aligned} \left(I - B(CB)^{-1}C\right) \left[X^{(m)} - \textstyle\sum_{j=1}^{m} A_j X^{(j-1)} - q\right] = 0, \quad (a)\\ CX + r = 0, \qquad\qquad\qquad\qquad (b) \end{aligned} \tag{2.4}$$

here, the overdetermined system (2.4) will transform to a full rank DAE system with n equations and n unknowns which has index-1.

Theorem 2.1. The index-$(m+1)$ DAE system (2.1), with $n=2$ is equivalent to the following index-m DAE system:

$$E_m X^{(m)} + E_{m-1} X^{(m-1)} + \cdots + E_1 X' + E_0 X = \hat{q}. \tag{2.5}$$

such that,

$$E_0 = \begin{bmatrix} b_1 a_{21}^{(1)} - b_2 a_{11}^{(1)} & b_1 a_{22}^{(1)} - b_2 a_{12}^{(1)} \\ c_1 & c_2 \end{bmatrix}, \ E_1 = \begin{bmatrix} b_2 a_{11}^{(2)} - b_1 a_{21}^{(2)} & b_1 a_{22}^{(2)} - b_2 a_{12}^{(2)} \\ 0 & 0 \end{bmatrix}, \ldots,$$

$$E_{m-1} = \begin{bmatrix} b_2 a_{11}^{(m)} - b_1 a_{21}^{(m)} & b_1 a_{22}^{(m)} - b_2 a_{12}^{(m)} \\ 0 & 0 \end{bmatrix}, \ E_m = \begin{bmatrix} b_2 & -b_1 \\ 0 & 0 \end{bmatrix} \hat{q} = \begin{bmatrix} b_2 q_1 - b_1 q_2, -r \end{bmatrix},$$

$$y = (CB)^{-1} C \left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q \right] \mathit{and} \ \ A_j = \begin{bmatrix} a_{11}^{(j)} & a_{12}^{(j)} \\ a_{21}^{(j)} & a_{22}^{(j)} \end{bmatrix}, \ j = 1, 2, \cdots, m.$$

Proof. As it is seen, the DAE system (2.1) is transformed to overdetermined system (2.4) by using (2.3). Now by considering $n=2$, we have

$$\left(I - B(CB)^{-1}C\right) = \frac{1}{c_1 b_1 + c_2 b_2} \begin{bmatrix} c_2 b_2 & -b_1 c_2 \\ -b_2 c_1 & c_1 b_1 \end{bmatrix},$$

thus, (2.4) has a form as below,

$$\left(I - B(CB)^{-1}C\right) \left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q \right], \tag{2.6}$$

$$= \begin{cases} c_2 [\left(b_1 a_{21}^{(1)} - b_2 a_{11}^{(1)}\right) x_1 + \left(b_1 a_{22}^{(1)} - b_2 a_{12}^{(1)}\right) x_2 + \left(b_1 a_{21}^{(2)} - b_2 a_{11}^{(2)}\right) x_1^{(1)} \\ + \left(b_1 a_{22}^{(2)} - b_2 a_{12}^{(2)}\right) x_2^{(1)} + \cdots + \left(b_1 a_{21}^{(m)} - b_2 a_{11}^{(m)}\right) x_1^{(m-1)} \\ + \left(b_1 a_{22}^{(2)} - b_2 a_{12}^{(1)}\right) x_2^{(m-1)} + b_2 x_1^{(m)} - b_1 x_1^{(m)} - b_2 q_1 + b_1 q_2] = 0 \\ -c_1 [\left(b_1 a_{21}^{(1)} - b_2 a_{11}^{(1)}\right) x_1 + \left(b_1 a_{22}^{(1)} - b_2 a_{12}^{(1)}\right) x_2 + \left(b_1 a_{21}^{(2)} - b_2 a_{11}^{(2)}\right) x_1^{(1)} \\ + \left(b_1 a_{22}^{(2)} - b_2 a_{12}^{(2)}\right) x_2^{(1)} + \cdots + \left(b_1 a_{21}^{(m)} - b_2 a_{11}^{(m)}\right) x_1^{(m-1)} \\ + \left(b_1 a_{22}^{(2)} - b_2 a_{12}^{(1)}\right) x_2^{(m-1)} + b_2 x_1^{(m)} - b_1 x_1^{(m)} - b_2 q_1 + b_1 q_2] = 0 \end{cases}$$

relation (2.2) implies that, $c_1(t) \neq 0$ or $c_2(t) \neq 0, \forall t \in [0, t_f]$. So, we can arbitrarily eliminate one of the equations of the system (2.6) and the overdetermined system (2.4) can transform to the following full rank DAE system:

$$\begin{cases} \left(b_1a_{21}^{(1)} - b_2a_{11}^{(1)}\right)x_1 + \left(b_1a_{22}^{(1)} - b_2a_{12}^{(1)}\right)x_2 + \left(b_1a_{21}^{(2)} - b_2a_{11}^{(2)}\right)x_1^{(1)} \\ + \left(b_1a_{22}^{(2)} - b_2a_{12}^{(2)}\right)x_2^{(1)} + \cdots + \left(b_1a_{21}^{(m)} - b_2a_{11}^{(m)}\right)x_1^{(m-1)} \\ + \left(b_1a_{22}^{(2)} - b_2a_{12}^{(1)}\right)x_2^{(m-1)} + b_2x_1^{(m)} - b_1x_1^{(m)} = b_2q_1 - b_1q_2, \\ c_1x_1 + c_2x_2 = -r, \end{cases} \tag{2.7}$$

which has the form of system (2.5). By using algorithm (4.1) mentioned in [10], it is easy to see that the system (2.5) has index-m. □

In case $n > 2$, for transforming the overdetermined system (2.4) to a full rank system with index m, there is a need for one additional condition on the problem (2.1). To proceed further, we define matrix, $M_{n\times n}$, as below:

$$M = \sum_{i=1}^{n} c_i b_i \left(I - B(CB)^{-1}C\right),$$

and the lth-row and (l,s)th-element of matrix Mdenote by $M[l]$ and $M[l,s]$ respectively, where $1 \le l, s \le n$.

Theorem 2.2. Consider the problem (2.1), when$n > 2$, if

$$\exists k, 1 \le k \le n, c_k(t) \ne 0, \forall t \in [0, t_f] \tag{2.8}$$

then the kth-row of matrix $\left(I - B(CB)^{-1}C\right)$ is linearly dependent with respect to other rows.

Proof. presented in [6]. □

Now, if $\overline{M}$ is obtained by eliminating kth-row of M (k is defined in (2.8)), then the overdetermined system (2.4) can be transformed to the following DAE system with n equation and n unknowns:

$$\begin{cases} \overline{M}\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right] = 0, \\ CX + r = 0, \end{cases} \tag{2.9}$$

Here, we will show that system (2.9) is full rank and has index-1.

Theorem 2.3. If $F = \begin{bmatrix} \overline{M} \\ C \end{bmatrix}_{n\times n}$ and k is denoted as in (2.8) then,

$$|\det(F(t))| = |c_k(t)| \left|\sum_{i=1}^{n} (c_i b_i)(t)\right|^{(n-1)} \quad \forall t \in [0, t_f]. \tag{2.10}$$

Proof. presented in [6]. $\square$

Now, since $\det(F(t)) \neq 0$, for all $t \in [0, t_f]$, the following corollaries will obtain.

Corollary 2.4. *Rank*$(M) = n - 1$.

Corollary 2.5. The DAE system (2.9) is full rank.

Now, having Theorems 2.1 and 2.2, the system (2.4) can be transformed to the following full rank DAE system, with n equations and n unknowns,

$$\begin{cases} \overline{M}\left[X^{(m)} - \sum_{j=1}^{m} A_j X^{(j-1)} - q\right] = 0, \\ CX + r = 0, \end{cases}$$

i.e.,

$$\begin{bmatrix} \overline{M} \\ 0 \end{bmatrix} X^{(m)} + \begin{bmatrix} -\overline{M}A_m \\ 0 \end{bmatrix} X^{(m-1)} + \cdots + \begin{bmatrix} -\overline{M}A_2 \\ 0 \end{bmatrix} X' + \begin{bmatrix} -\overline{M}A_1 \\ C \end{bmatrix} X = \begin{bmatrix} \overline{M}q \\ -r \end{bmatrix},$$

or

$$E_m X^{(m)} + E_{m-1} X^{(m-1)} + \cdots + E_1 X' + E_0 X = \hat{q}. \tag{2.11}$$

Lemma 2.6. Index of system (2.11) is equal to m.

Proof. To prove this, we define,

$$\begin{cases} U_0 = X, \\ U_1 = X' \\ \vdots \\ U_{m-1} = X^{(m-1)} \end{cases} \tag{2.12}$$

so, the system (2.11) can be written as:

$$\begin{bmatrix} I & & & \\ & & 0 & \\ & 0 & I & \\ & & & E_m \end{bmatrix} \begin{bmatrix} U_0' \\ \vdots \\ U_{m-2}' \\ U_{m-1}' \end{bmatrix} + \begin{bmatrix} 0 & -I & & \\ & & \ddots & 0 \\ & 0 & & -I \\ E_0 & E_1 & \cdots & E_{m-1} \end{bmatrix} \begin{bmatrix} U_0 \\ \vdots \\ U_{m-2} \\ U_{m-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hat{q} \end{bmatrix}. \tag{2.13}$$

The non singularity of $F = \begin{bmatrix} \overline{M} \\ C \end{bmatrix}$ implies (by using algorithm (4.1) mentioned in [10]) that the above system, and identically, the system (2.11) has index-m. $\square$

So, the linear (or linearized) problem (1.1) (which has index-$m+1$), with holding (2.8), can be transformed to the implicit DAE system (2.4) (which has index-m) by simple proposed formulation. Here, we give a formulation which the general linear form of the problem can be easily transformed to the index reduced form with index 1 by a simple formulation.

Theorem 2.7. The problem (2.11) with index-m is equivalent to the following index-1 problem,

$$F_m X^{(m)} + F_{m-1} X^{(m-1)} + \cdots + F_1 X' + F_0 X = \hat{q}^{(m-1)}. \tag{2.14}$$

where,

$$F_m = \begin{bmatrix} \overline{M} \\ 0 \end{bmatrix}, F_i = \begin{bmatrix} -\overline{M} A_{i+1} \\ \begin{pmatrix} m-1 \\ m-1-i \end{pmatrix} c^{(m-1-i)}(t) \end{bmatrix}, i = m-1, m-2, \ldots, 1, 0, \tag{2.15}$$

that $c^{(k)}(t)$, $\hat{q}^{(k)}(t)$ denote differential of order kth and $\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!}$, $k = 0, 1, \cdots n$, are coefficient of Newton's binomial.

Proof. It can be proved by induction on m. For $m = 2$, we have,

$$E_2 X'' + E_1 X' + E_0 X = \hat{q}.$$

By considering (2.13), the above system can be written as,

$$\begin{bmatrix} I_n & 0 \\ 0 & E_2 \end{bmatrix} \begin{bmatrix} U_0' \\ U_1' \end{bmatrix} + \begin{bmatrix} 0 & -I_n \\ E_0 & E_1 \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{q} \end{bmatrix},$$

or,

$$\begin{bmatrix} I_n & 0 \\ 0 & \begin{bmatrix} \overline{M} \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} U_0' \\ U_1' \end{bmatrix} + \begin{bmatrix} 0 & -I_n \\ \begin{bmatrix} -\overline{M} A_1 \\ C \end{bmatrix} & \begin{bmatrix} -\overline{M} A_2 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{q} \end{bmatrix},$$

then by using algorithm (4.1) mentioned in [2], we have,

$$\begin{bmatrix} I & 0 \\ \begin{bmatrix} 0 \\ C \end{bmatrix} & \begin{bmatrix} \overline{M} \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} U_0' \\ U_1' \end{bmatrix} + \begin{bmatrix} 0 & -I \\ \begin{bmatrix} -\overline{M} A_1 \\ C' \end{bmatrix} & \begin{bmatrix} -\overline{M} A_2 \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{q}' \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} I & 0 \\ \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \overline{M} \\ 0 \end{bmatrix} \end{bmatrix} \begin{bmatrix} U_0' \\ U_1' \end{bmatrix} + \begin{bmatrix} 0 & -I \\ \begin{bmatrix} -\overline{M} A_1 \\ C' \end{bmatrix} & \begin{bmatrix} -\overline{M} A_2 \\ C \end{bmatrix} \end{bmatrix} \begin{bmatrix} U_0 \\ U_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \hat{q}' \end{bmatrix},$$

or,

$$F_2 X'' + F_1 X' + F_0 X = \hat{q}'.$$

Now, we assume that the relation (2.14) is correct for $m-1$. By (2.11) we have,

$$E_m X^{(m)} + E_{m-1} X^{(m-1)} + \cdots + E_1 X' + E_0 X = \hat{q}.$$

or,

$$\begin{bmatrix} \overline{M} \\ 0 \end{bmatrix} X^{(m)} + \begin{bmatrix} -\overline{M}A_m \\ 0 \end{bmatrix} X^{(m-1)} + \cdots + \begin{bmatrix} -\overline{M}A_2 \\ 0 \end{bmatrix} X' + \begin{bmatrix} -\overline{M}A_1 \\ C \end{bmatrix} X = \begin{bmatrix} \overline{M}q \\ -r \end{bmatrix}.$$

so, by assumption of induction following system,

$$\begin{bmatrix} -\overline{M}A_m \\ 0 \end{bmatrix} X^{(m-1)} + \cdots + \begin{bmatrix} -\overline{M}A_2 \\ 0 \end{bmatrix} X' + \begin{bmatrix} -\overline{M}A_1 \\ C \end{bmatrix} X = \hat{q}(t),$$

is equivalent to:

$$\begin{bmatrix} -\overline{M}A_m \\ 0 \end{bmatrix} X^{(m-1)} + \begin{bmatrix} -\overline{M}A_m \\ \begin{pmatrix} m-2 \\ 0 \end{pmatrix} c^{(0)}(t) \end{bmatrix} X^{(m-2)}$$

$$+ \cdots + \begin{bmatrix} -\overline{M}A_1 \\ \begin{pmatrix} m-2 \\ m-3 \end{pmatrix} c^{(m-2)}(t) \end{bmatrix} X' + \begin{bmatrix} -\overline{M}A_1 \\ \begin{pmatrix} m-2 \\ m-2 \end{pmatrix} c^{(m-2)}(t) \end{bmatrix} X = \hat{q}(t)^{(m-2)}.$$

again by using algorithm (4.1) mentioned in [10], the system (2.11) is equivalent to

$$\begin{bmatrix} \overline{M} \\ 0 \end{bmatrix} X^{(m)} + \begin{bmatrix} -\overline{M}A_m \\ \begin{pmatrix} m-1 \\ 0 \end{pmatrix} c^{(0)}(t) \end{bmatrix} X^{(m-1)} + \cdots + \begin{bmatrix} -\overline{M}A_2 \\ \begin{pmatrix} m-1 \\ m-2 \end{pmatrix} c^{(m-2)}(t) \end{bmatrix} X'$$

$$+ \begin{bmatrix} -\overline{M}A_1 \\ \begin{pmatrix} m-1 \\ m-1 \end{pmatrix} c^{(m-1)}(t) \end{bmatrix} X = \hat{q}(t)^{(m-1)}$$

or, $F_m X^{(m)} + F_{m-1} X^{(m-1)} + \cdots + F_1 X' + F_0 X = \hat{q}^{(m-1)}$. □

In Section 3, the advantages of these proposed formulation and index reduction are shown.

Corollary 2.8. The index-$(m+1)$ DAE system (2.1), with $n = 2$ is equivalent to the following index -1 DAE system.

$$\begin{bmatrix} b_2 & -b_1 \\ 0 & 0 \end{bmatrix} X^{(m)} + \begin{bmatrix} b_2 a_{11}^{(m)} - b_1 a_{21}^{(m)} & b_1 a_{22}^{(m)} - b_2 a_{12}^{(m)} \\ \begin{pmatrix} m-1 \\ 0 \end{pmatrix} c_1(t) & \begin{pmatrix} m-1 \\ 0 \end{pmatrix} c_2(t) \end{bmatrix} X^{(m-1)}$$

$$+ \cdots + \begin{bmatrix} b_2 a_{11}^{(2)} - b_1 a_{21}^{(2)} & b_1 a_{22}^{(2)} - b_2 a_{12}^{(2)} \\ \begin{pmatrix} m-1 \\ m-2 \end{pmatrix} c_1^{(m-2)}(t) & \begin{pmatrix} m-1 \\ m-2 \end{pmatrix} c_2^{(m-2)}(t) \end{bmatrix} X'$$

$$+ \begin{bmatrix} b_2 a_{11}^{(1)} - b_1 a_{21}^{(1)} & b_1 a_{22}^{(1)} - b_2 a_{12}^{(1)} \\ \begin{pmatrix} m-1 \\ m-1 \end{pmatrix} c_1^{(m-1)}(t) & \begin{pmatrix} m-1 \\ m-1 \end{pmatrix} c_2^{(m-1)}(t) \end{bmatrix} X = \hat{q}(t)^{(m-1)}$$

3 Pseudospectral method and linear DAE systems

It is known that the eigenfunctions of certain singular Sturm–Liouville problems allow the approximation of functions in $C^{\infty}[a,b]$ where truncation error approaches zero faster than any negative power of the number of basic functions used in the approximation, as that number (order of truncation N) tends to infinity. This phenomenon is usually referred to as "spectral accuracy". The accuracy of derivatives obtained by direct, term-by-term differentiation of such truncated expansion naturally deteriorates, but for low-order derivatives and sufficiently high-order truncations this deterioration is negligible, compared to the restrictions in accuracy introduced by typical difference approximations (for more details, refer to [6, 8]). Throughout, we are using first kind orthogonal Chebyshev polynomials $\{T_k\}_{k=0}^{+\infty}$ which are eigenfunctions of singular Sturm-Liouville problem:

$$\left(\sqrt{1-x^2}T'(x)\right)' + \frac{k^2}{\sqrt{1-x^2}}T_k(x) = 0.$$

Here, the implementation of pseudospectral method for DAE system (2.9) is represented. It must be noted that, the discussion in this section can simply extended to the general forms (1.1) and (2.11). Now, consider the DAE system,

$$\begin{cases} f_{11}(t)x' + f_{12}(t)x_2' + f_{13}(t)x_1 + f_{14}(t)x_2 = q_1(t) \\ f_{23}(t)x_1 + f_{24}(t)x_2 = q_2(t) \end{cases} \quad t \in [-1,1], \quad (a)$$
$$\begin{cases} x_1(-1) = \alpha, \\ x_2(-1) = \beta, \end{cases} \quad (b) \tag{3.1}$$

where f_{ij}, q_1 and q_2 are sufficiently smooth functions of t, $-1 \le t \le 1$ and α and β are constants. For an arbitrary natural number$N$, we suppose that the approximate solution of DAE (3.1), is:

$$\begin{cases} x_1(t) \approx \sum_{i=o}^{N} a_i T_i(t), \\ \\ x_2(t) \approx \sum_{i=o}^{N} a_{N+1+i} T_i(t), \end{cases} \tag{3.2}$$

where $\underline{a} = (a_0, \ldots, a_{2N+1})^t \in R^{2N+2}$ and $\{T_k\}_{k=0}^{N}$ is the sequence of Chebyshev polynomials of the first kind. Here, the main purpose is to find $\underline{a} = (a_0, \ldots, a_{2N+1})^t$. If we put:

$$V(x) = \sum_{k=o}^{N} a_k T_k(x),$$

matrices $A^{(0)}, A^{(1)}AA, BB$, CC and DD can be defined as follows [6]:

$$V \Leftrightarrow A^{(0)}, A_{ij}^{(0)} = \begin{cases} 1, i = j, \\ 0, i \ne j, \end{cases}$$

$$V' \Leftrightarrow A^{(1)}, A_{ij}^{(1)} = \begin{cases} (1/c_i)\ 2j, i + j\text{odd},\ j > i, \\ 0, \text{otherwise}, \end{cases}$$

with $0 \leq i, j \leq N$, $c_i = \begin{cases} 2, i = 0 \\ 1,\ i > 0 \end{cases}$ and

$$\begin{cases} AA = f_{11}(t)\, A^{(1)} + f_{13}(t)\, A^{(0)}, \\ BB = f_{12}(t)\, A^{(1)} + f_{14}(t)\, A^{(0)}, \\ CC = f_{23}(t)\, A^{(0)}, \\ DD = f_{24}(t)\, A^{(0)}, \end{cases} \tag{3.3}$$

such that by the part (a) of (3.1) converts to:

$$\begin{cases} \sum_{i=0}^{2N+1} a_i \phi_i(t) \approx q_1(t), \\ \\ \sum_{i=0}^{2N+1} a_i \psi_i(t) \approx q_2(t), \end{cases} \tag{3.4}$$

and also by the part (b) of (3.1) converts to:

$$\begin{cases} \sum_{i=0}^{N} a_i T_i(-1) = \sum_{i=0}^{N} a_i (-1)^i = \alpha, \\ \\ \sum_{i=0}^{N} a_{N+1+i} T_i(-1) = \sum_{i=0}^{N} a_{N+1+i} (-1)^i = \beta. \end{cases} \tag{3.5}$$

in which,

$$\phi_i(t) = \begin{cases} \sum_{i=o}^{N} (AA)_{ki}\, T_k(t), 0 \leq i \leq N, \\ \\ \sum_{i=o}^{N} (BB)_{ki}\, T_k(t), N+1 \leq i \leq 2N+1, \end{cases} \qquad (a)$$

$$\psi_i(t) = \begin{cases} \sum_{i=o}^{N} (CC)_{ki}\, T_k(t), 0 \leq i \leq N, \\ \\ \sum_{i=o}^{N} (DD)_{ki}\, T_k(t), N+1 \leq i \leq 2N+1, \end{cases} \qquad (b) \tag{3.6}$$

Relation (3.4) forms a system with two equations and $2N+2$ unknowns, to construct the remaining $2N$ equations we substitute Chebyshev-Guass-Radau points, i.e.,

$$t_j = \cos\left(\frac{2\pi j}{(2N-1)}\right), j = 0, \ldots, N-1 \tag{3.7}$$

in (3.4) and put:

$$\begin{cases} \sum_{i=0}^{2N+1} a_i \phi_i(t_j) \approx q_1(t_j), \\ \\ \sum_{i=0}^{2N+1} a_i \psi_i(t_j) \approx q_2(t_j), \end{cases} \quad j = 0, 1, \ldots, N-1,$$

to obtain $2N$ equations. In addition, according to initial conditions (3.1), the Chebyshev-Guass or Chebyshev-Guass-Lobatto points can be chosen in (3.7).

4 Numerical examples

This section deals with some numerical tests on simple, but interesting, problems. All the examples are solved, directly, using (2.1), and reduced index, using (2.14), by pseudospectral method. Results show the advantages of index reduction technique, mentioned in section 2. In this section $v \geq 1$ is a parameter. The presented algorithms in this article are performed using Maple 12.

Example 4.1. Consider for $0 \leq t \leq 1$,

$$\begin{cases} x_1'' = -vx_{1'}' + \left(v - \frac{1}{2-t}\right) x_1 + (2-t)\, vy + q_1(t), \\ x_2'' = x_2' + \frac{v}{2-t} x_1' + \frac{v-1}{2-t} x_1 - x_2 + (v-1)\, y + q_2(t) \\ 0 = (t+2)\, x_1 + (t_2 - 4)\, x_2 + r\,(t)\,, \end{cases} , \tag{4.1}$$

with $x_1\,(0) = x_2\,(0) = 1$. The inhomogeneities $q(t)$ and $r(t)$ are chosen to be

$$q = \begin{pmatrix} \left(\frac{t-3+vt-2v}{t-2}\right) e^t \\ \left(\frac{t-2+v}{t-2}\right) e^t \end{pmatrix}, r\,(t) = -\left(t^2 + t - 2\right) e^t,$$

so that the exact solution is $x_1 = x_2 = e^t$, $y = -e^t/(t-2)$ and $v = 100$. The problem has index-3, by Theorem 2.2, Theorem 2.7 and Corollary 2.8, it can be converted to the index-1 DAE system as below,

$$\begin{aligned} &\begin{bmatrix} 1-v & (2-t)\, v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1'' \\ x_2'' \end{bmatrix} + \begin{bmatrix} 1-2v^2 & (t-2)\, v \\ t+2 & t^2-4 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \\ &+ \begin{bmatrix} (v-1)\left(v - \frac{1}{2-t}\right) - v\,(v-1) & (2-t)\, v \\ 1 & 2t \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} (2-t)\, v \frac{t-2+v}{t-2} - (v-1)\, \frac{t-3+vt-2v}{t-2} \\ 2t+1 \end{bmatrix}. \end{aligned} \tag{4.2}$$

The above example has been solved with and without reduced index techniques. To compare both techniques we have presented the absolute errors for $n = 12$ in figures 1-4.

Example 4.2. Consider for $0 \leq t \leq 1$,

$$\begin{cases} x_1'' = vx_{1'}' - x_1 + \left(\frac{1-v}{t-2}\right) x_2 + (2-t)\, vy + q_1(t), \\ x_2'' = vx_2' + \frac{v}{2-t} x_1 - x_2 + (v-1)\, y + q_2(t) \\ x_3'' = x_1' - vx_3' - x_2 - vy + q_3(t) \\ 0 = \cos(t)\, x_1 - \sin(t)\, x_2 + tx_3 + r\,(t)\,, \end{cases} , \tag{4.3}$$

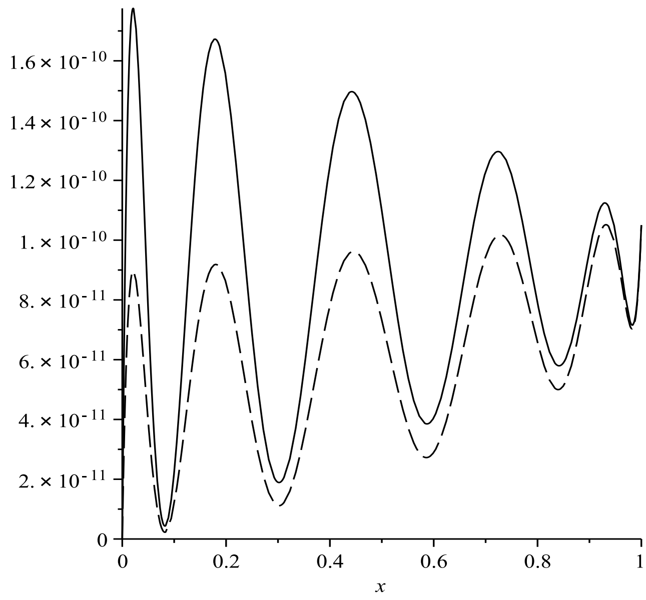

Figure 1: Absolute errors of x_1,—–, and x_2 , −− ,for example 1, without index reduction.

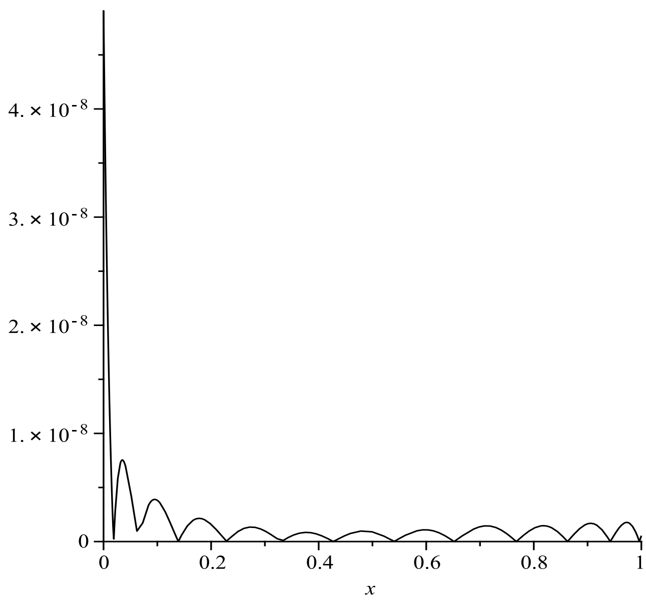

Figure 2: Absolute error of y for example 1, without index reduction.

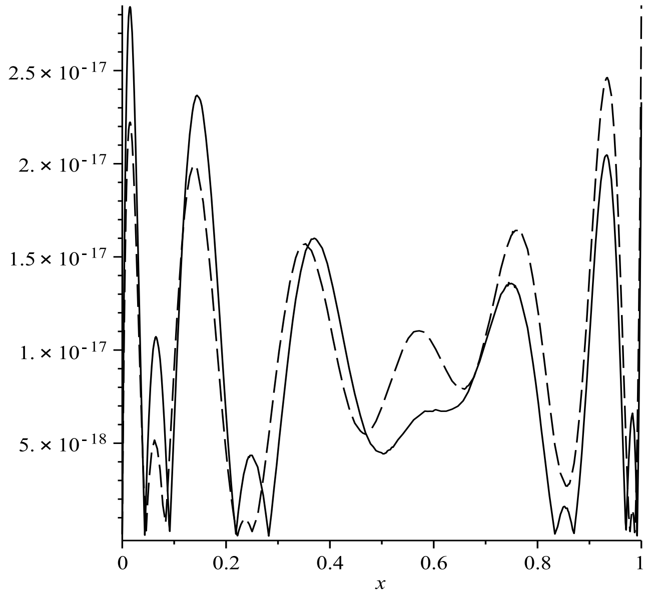

Figure 3: Absolute errors of x_1,—, and x_2 , −− ,for example 2.

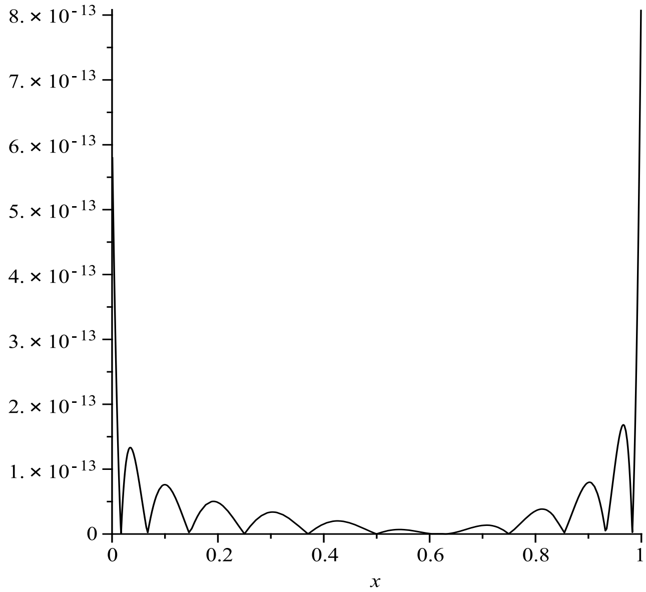

Figure 4: Absolute error of y for example 2.

with $x_1(0) = 0$ and $x_2(0) = x_3(0) = 1$. The inhomogeneities $q(t)$ and $r(t)$ are compatible with the exact solutions $x_1 = \sin(t)$, $x_2 = \cos(t)$, $x_3 = e^t$ and $y = \frac{t}{(t-2)}$. The problem has index-3 which can be converted to the index-1 DAE system as below,

$$E_2X'' + E_1X' + E_0X = \hat{q}, \tag{4.4}$$

where,

$$E_2 = \begin{bmatrix} (1-v)\cos(t) & v\,((2-t)\cos(t)-t) & (1-v)t \\ v\cos(t) & -v\sin(t) & v\,(2-t)\cos(t)+(1-v)\sin(t) \\ 0 & 0 & 0 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} (v-1)(v\cos(t)+t) & v^2((t-2)\cos(t)+t) & vt(1-v) \\ v\cos(t)(t-v-2)+(v-1)\sin(t) & v^2\sin(t) & v^2(2-t)\cos(t)+v(1-v)\sin(t) \\ \cos(t) & -\sin(t) & t \end{bmatrix},$$

$$E_0 = \begin{bmatrix} \cos(t)\left(1-v-v^2\right)+\frac{v^2t}{(2-t)} & \frac{\cos(t)}{(2-t)}\left((v+1)^2+vt\,(4-t)\right)+t\,(1-2v) & 0 \\ v\cos(t)-\frac{v^2\sin(t)}{(t-2)} & \frac{1}{(t-2)}\left(v\cos(t)\left(t^2-4t-v+5\right)\right)+(1-2v)\sin(t) & 0 \\ -\sin(t) & -\cos(t) & 1 \end{bmatrix},$$

and

$$\hat{q} = \begin{bmatrix} (1-v)\cos(t)\left(vt+\left(\frac{v-1}{t-2}-v\right)\cos(t)\right)+((2-t)\cos(t)-vt)\left(\left(\frac{v}{t-2}v\right)\sin(t)+\frac{(1-v)t}{(t-2)}\right) \\ +(1-v)t\left(\frac{vt}{t-2}+e^t(1+v)\right) \\ v\cos(t)\left(vt+\left(\frac{v-1}{t-2}-v\right)\cos(t)\right)-v\sin(t)\left(\left(\frac{v}{t-2}+v\right)\sin(t)+\frac{(1-v)t}{(t-2)}\right) \\ +((2-t)v\cos(t)+(1-v)\sin(t))\left(\frac{vt}{t-2}+e^t(v+1)\right) \\ (t+1)e^t \end{bmatrix}$$

In Table 1, we record results of running pseudospectral method with, i.e. (4.4), and without, i.e. (4.3), index reduction, when $v = 1000$. we use "e_x" and "e_y" to denote the maximum absolute error in $X = (x_1, x_2, x_3)$ and y, respectively. These values are approximately obtained through their graphs. The advantage of using index reduction method (proposed in Section 2) is clearly demonstrated for this example.

Table 1
Maximum absolute error for Example 2, $v = 1000$

N	Index-3			Index-1		
	e_x	e_y	Comp. times (s)	e_x	e_y	Comp. times (s)
6	3.5(-3)	8.5(-2)	9.35	5.7(-6)	8.0(-5)	6.40
10	9.5(-6)	8.5(-4)	17.52	6.5(-12)	1.3(-10)	14.49
14	3.0(-8)	7.0(-6)	26.75	1.5(-18)	3.8(-17)	26.49
18	1.8(-10)	5.0(-8)	41.05	9.5(-26)	1.25(-24)	38.97

5 Conclusion

In this paper, an efficient index reduction technique for high index linear differential-algebraic equations are introduced. Also, numerical approach of them is considered by pseudospectral method. Some numerical examples have been solved to illustrate the efficiency and accuracy of the proposed methods.

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