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Coefficient Bounds for Some Families of Bi-Univalent Functions with Missing Coefficients †

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† Dedicated to the memory of Professor Bogdan Tadeusz Bojarski (1931–2018).
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Abstract: A branch of complex analysis with a rich history is geometric function theory, which first appeared in the early 20th century. The function theory deals with a variety of analytical tools to study the geometric features of complex-valued functions. The main purpose of this paper is to estimate more accurate bounds for the coefficient $|a_n|$ of the functions that belong to a class of bi-univalent functions with missing coefficients that are defined by using the subordination. The significance of our present results consists of improvements to some previous results concerning different recent subclasses of bi-univalent functions, and the aim of this paper is to improve the results of previous outcomes. In addition, important examples of some classes of such functions are provided, which can help to understand the issues related to these functions.

Keywords: analytic and univalent function; bi-univalent function; coefficient estimates; subordination

MSC: 30C45; 30C50; 30C80

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1. Introduction

The study of univalent functions is traditional, and it is categorized under geometric function theory (GFT) since numerous noteworthy characteristics of univalent functions can be found in the basic geometrical properties. In 1851 [1], the Reimann mapping theorem led to the development of GFT. Nevertheless, it helps to discover new results in a wide range of topics, including contemporary mathematical physics and more established branches of physics, like fluid dynamics, nonlinear integrable systems theory, and the theory of partial differential equations. One of the most fascinating areas of geometric function theory is the theory of univalent functions, which is a well-known classical topic of complex analytic functions. Around the 20th century, many geometric aspects of analytical functions were introduced and studied, like starlikeness, convexity, close-to-convexity, typically real functions, etc.

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane $\mathbb{C}$, and let $\mathcal{A}$ be the class of functions f analytic in $\mathbb{D}$ that has the following representation:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ z \in \mathbb{D}. \tag{1}$$

Denote $\mathcal{S}$ as the subclass of all functions of $\mathcal{A}$ that are univalent in $\mathbb{D}$. The study of the characteristics of normalized univalent functions that fall under the class and are defined in the open unit disk $\mathbb{D}$ is the main focus of the geometry theory of functions.

Furthermore, let $\mathcal{B}$ represent the category of all analytic functions ϖ in $\mathbb{D}$ that fulfil the criteria $\varpi(0) = 0$ and $|\varpi(z)| < 1$ for all $z \in \mathbb{D}$. If the image of the open unit disk by a univalent function has some geometrical characteristics, it may be of interest to find an analytic characterization of such functions. The best example of a domain with desirable features is a convex domain and a starlike one with regard to a point. Many subclasses of those analytic univalent functions that map onto these above-mentioned domains were introduced and thoroughly studied, such as the well-known classes $\mathcal{K}$ and $\mathcal{S}^*$ of convex and starlike functions, respectively.

In geometric function theory, determining the bounds for the coefficients $|a_n|$ is a crucial task since it reveals details about the geometric characteristics of these functions. For instance, the growth and distortion bounds, as well as the covering theorems, are given by the bound for the second coefficient $|a_2|$ of functions $f \in \mathcal{S}$.

Every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{D}) \quad \text{and} \quad f\Big(f^{-1}(w)\Big) = w \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right),$$

with the expansion of the power series

$$f^{-1}(w) = w + \sum_{k=2}^{\infty} b_k w^k = w - a_2 w^2 + \Big(2a_2^2 - a_3\Big)w^3 - \Big(5a_2^3 - 5a_2 a_3 + a_4\Big)w^4 + \ldots.$$

A function $f \in \mathcal{A}$ is said to be *bi-univalent* in $\mathbb{D}$ if f is univalent in $\mathbb{D}$ and f^{-1} has a univalent analytic extension in $\mathbb{D}$. For brevity, we will denote this analytic extension by $g := f^{-1}$. The studies of the class of bi-univalent functions in $\mathbb{D}$ was initiated by Levin [2], who proved that

$$|a_2| < 1.51.$$

Following these studies, Branan and Clunie [3] improved Levin's result by the subsequent variant

$$|a_2| \leq \sqrt{2}.$$

Furthermore, Netanyahu [4] showed that for the bi-univalent functions,

$$\max |a_2| = \frac{3}{4}.$$

The fact that the following functions are bi-univalent must be mentioned:

$$f_1(z) = \frac{z}{1-z}, \quad f_2(z) = \log\left(\frac{1}{1-z}\right).$$

And, these correspond to the inverse functions of

$$f_1^{-1}(w) = \frac{w}{1+w}, \quad f_2^{-1}(w) = \frac{e^w - 1}{e^w}.$$

Let Σ denote the family of bi-univalent functions in $\mathbb{D}$. The study of Srivastava et al. [5] provides a brief historical review of the roles in the family Σ along with a few examples. Regarding [5], the class Σ of bi-univalent functions has numerous subfamilies, each of which has a different set of analytic features, and many authors have attempted to explore these families, for example, [6–13]. In a few of these articles, the authors studied some subclasses of bi-univalent functions connected with the Faber and Laguerre polynomials, determined estimates for coefficients and Hankel determinants for different subclasses of bi-univalent functions associated with Hohlov operator and Horadam polynomials, and gave some estimates for the Fekete–Szegő functional. Other related issues can be found in [14–16], while, in general, it is still difficult to determine the extremal functions for bi-univalent functions.

The Faber polynomials expansion method was first described by Faber [17], and he used this method to study the coefficient boundaries of $|a_m|$ for $m \geq 3$. In the mathematical sciences, notably in the field of geometric function theory, these Faber polynomials are crucial. In this regard, in order to obtain the optimal bounds of $|a_n|$ for the coefficients of bi-univalent functions, some researchers used the Faber polynomial expansions [18–23].

Let f and F be two analytic functions in $\mathbb{D}$; the function f is considered *subordinate* to F, denoted by $f(\zeta) \prec F(\zeta)$, if there exists a function $\omega : \mathbb{D} \to \mathbb{D}$ analytic in $\mathbb{D}$ with $\omega(0) = 0$, such that $f = F \circ \omega$. The above function ω is considered a *subordination function* (see [24], p. 125). If $f(\zeta) \prec F(\zeta)$, then $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$, and with the additional assumption that F is univalent in $\mathbb{D}$, the subordination $f(\zeta) \prec F(\zeta)$ is equivalent to $f(0) = F(0)$ and $f(\mathbb{D}) \subset F(\mathbb{D})$ (see [25], p. 15).

The conceptual underpinnings of the current research problem and important research-related issues are shown in this section. A review of comparable studies sheds some light on the advantages and shortcomings of the earlier investigations.

Let h be an analytic function with positive real part in $\mathbb{D}$ and the power series expansion

$$h(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots,\ z \in \mathbb{D}, \quad \text{with} \quad B_1 \neq 0.$$

With the help of the aforementioned type of function, we define a subclass of $\mathcal{A}$ that is a generalization of Definition 1 from [20], assuming the weaker assumption $\lambda \geq 0$ as follows:

Definition 1. *A function $f \in \Sigma$ is said to be in the class $\mathcal{N}_\Sigma(\lambda, \delta, h)$ for $\lambda \geq 0$ and $\delta \geq 0$ if*

$$\mathrm{I}_{\lambda,\delta}[f](z) := (1-\lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) \prec h(z), \quad \textit{and}$$
$$\mathrm{I}_{\lambda,\delta}[g](w) := (1-\lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) \prec h(w), \quad g = f^{-1}.$$

Here, we present an example that helps prove that this class is nonempty and contains functions other than the identity one.

Remark 1. *(i) We emphasize that the class $\mathcal{N}_\Sigma(\lambda, \delta, h)$ is not empty for appropriate choices of the parameters. Thus, letting*

$$h_*(z) = 1 + 0.35z + 0.1z^2,$$

like we may see in Figure 1a made using the MAPLE™ computer software, we have

$$\operatorname{Re} h_*(z) > 0, z \in \mathbb{D},\ B_1 = 0.35 = h_*'(0) \neq 0, B_2 = 0.1,$$

and

$$B_n = 0 \quad \text{for} \quad n \geq 3.$$

It is easy to show that

$$\operatorname{Re}\frac{z h_*'(z)}{h_*(z) - 1} = \operatorname{Re}\frac{2z + 3.5}{z + 3.5} > 0.6 > 0,\ z \in \mathbb{D}.$$

Hence, h_* is a starlike (univalent) function in $\mathbb{D}$ with respect to the point $z_0 = 1$.

The function

$$f_*(z) = \frac{z}{1 + 0.2z} \in \mathcal{S}$$

and its inverse

$$g_*(w) = f_*^{-1}(w) = \frac{w}{1 - 0.2w}$$

are analytic in $\mathbb{D}$; Hence, $f_* \in \Sigma$.

Also, for $\lambda = 0.2$ and $\delta = 0.1$, a simple computation shows that

$$\mathrm{I}_{0.2,0.1}[f_*](z) = \frac{1}{0.2z+1} - \frac{0.04z}{(0.2z+1)^2} + 0.1z\left(-\frac{0.4}{(0.2z+1)^2} + \frac{0.08z}{(0.2z+1)^3}\right) \quad \textit{and}$$
$$\mathrm{I}_{0.2,0.1}[g_*](w) = \frac{1}{-0.2w+1} + \frac{0.04w}{(-0.2w+1)^2} + 0.1w\left(\frac{0.4}{(-0.2w+1)^2} + \frac{0.08w}{(-0.2w+1)^3}\right).$$

Since h_* is univalent in $\mathbb{D}$, using the inclusions

$$\mathrm{I}_{0.2,0.1}[f_*](\mathbb{D}) \subset h(\mathbb{D}) \quad \textit{and} \quad \mathrm{I}_{0.2,0.1}[g_*](\mathbb{D}) \subset h(\mathbb{D})$$

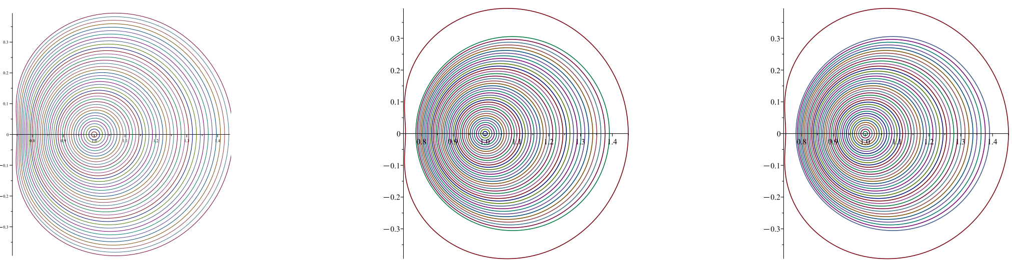
that follow from Figure 1b and Figure 1c, respectively, also made using MAPLE™, we conclude that

$$\mathrm{I}_{0.2,0.1}[f_*](z) \prec h_*(z) \quad \textit{and} \quad \mathrm{I}_{0.2,0.1}[g_*](w) \prec h_*(w).$$

Therefore, $f_* \in \mathcal{N}_\Sigma(0.2, 0.1, h_*)$. Therefore, there exists values of the parameters λ, δ, and functions h, such that

$$\mathcal{N}_\Sigma(\lambda, \delta, h) \setminus \{\mathrm{Id}\} \neq \emptyset,$$

where Id *denotes the identity function. To not lengthen the paper unnecessarily, we omit the MAPLE™ codes for the figures we used throughout the article.*



(**a**) The image $h(\mathbb{D})$ (**b**) The inclusion $\mathrm{I}_{0.2,0.1}[f_*](\mathbb{D}) \subset h_*(\mathbb{D})$ (**c**) The inclusion $\mathrm{I}_{0.2,0.1}[g_*](\mathbb{D}) \subset h_*(\mathbb{D})$

Figure 1. Figures for Remark 1.

(ii) If, in the above example, the values of $|\lambda|$ and $|\delta|$ decrease to 0, then the behavior of the functions $\mathrm{I}_{\lambda,\delta}[f_*]$ and $\mathrm{I}_{\lambda,\delta}[g_*]$ becomes very similar to that of the functions $\dfrac{f_*(z)}{z}$ and $\dfrac{g_*(z)}{z}$. In some examples we made using MAPLE™ software, we saw that the above set inclusions hold. Hence, these new functions belong to the classes of Definition 1. These indicate a consequence of the general fact that

$$\lim_{(\lambda,\delta)\to(0,0)} \mathrm{I}_{\lambda,\delta}[f_*](z) = \frac{f_*(z)}{z} \quad \textit{and} \quad \lim_{(\lambda,\delta)\to(0,0)} \mathrm{I}_{\lambda,\delta}[g_*](z) = \frac{f_*(z)}{z},\ z \in \mathbb{D};$$

that is,

$$\mathcal{N}_\Sigma(0,0,h) = \mathcal{N}_\Sigma\left(0,0,\frac{f(z)}{z}\right) \textit{ for all } f \in \Sigma.$$

(iii) If, in similar examples, the values of $|\lambda|$ and $|\delta|$ increase, then there are some cases when the subordinations of Definition 1 do or do not hold*, as follows (to not lengthen the paper, we omit the corresponding graphical representations):*

(a) $\widetilde{f}(z) = \dfrac{z}{1+0.1z} \in \mathcal{N}_{\Sigma}(1.1, 0.3, h_*)$;
(b) $f_*(z) = \dfrac{z}{1+0.2z} \notin \mathcal{N}_{\Sigma}(1.1, 0.3, h_*)$, if $h_*(z) = 1 + 0.35z + 0.1z^2$.

In a similar way, the authors of [26] defined the following family of analytic functions:

$$\mathscr{S}(\nu,\rho;h) = \left\{ f \in \mathcal{A} : 1 + \frac{1}{\rho}\left(\frac{zf'(z) + \nu z^2 f''(z)}{(1-\nu)f(z) + \nu z f'(z)} - 1 \right) \prec h(z),\ 0 \le \nu \le 1,\ \rho \in \mathbb{C} \setminus \{0\} \right\}$$

and obtained a bound for the general coefficients of the bi-univalent functions of this class by using the Faber polynomials subject to a series of assumptions.

In our paper, we replace the assumptions for the function h from [26] with some weaker ones as stated above (i.e., omitting the conditions that $h(\mathbb{D})$ is symmetric with respect of the real axis and $B_1 > 0$).

Here, we present an example that helps to better understand the above explanation for the function h and proves that this family is nonempty, containing other functions than the identity one.

Remark 2. *In the below example, we consider a case when $h(\mathbb{D})$ is not symmetric with respect of the real axis and $B_1 \neq 0$, as we assumed in Definition 1. We show that for some values of the parameters, the class $\mathscr{S}(\nu,\rho;h)$ is not empty. Taking*

$$\widehat{h}(z) = 1 + 0.35(1+i)z + 0.1z^2,$$

since $\widehat{h}(\overline{z}) \neq \overline{\widehat{h}(z)}$ for all $z \in \mathbb{D}$, it follows that the domain $\widehat{h}(\mathbb{D})$ is not symmetric with respect of the real axis and

$$B_1 = 0.35(1+i) = \widehat{h}'(0) \neq 0.$$

Like we may see in Figure 2a, we have $\operatorname{Re}\widehat{h}(z) > 0$, $z \in \mathbb{D}$, and Figure 2b, also made with MAPLE™ software, shows that

$$\operatorname{Re} \mathrm{J}(z) := \operatorname{Re} \frac{z\widehat{h}'(z)}{\widehat{h}(z) - 1} = \operatorname{Re} \frac{3.5(1+i) + 2z}{3.5(1+i) + z} > 0.7 > 0,\ z \in \mathbb{D}.$$

Hence, $\widehat{h}$ is a starlike (univalent) function with respect to the point $z_0 = 1$. Denoting

$$\mathrm{L}_{\nu,\rho}[f](z) := 1 + \frac{1}{\rho}\left(\frac{zf'(z) + \nu z^2 f''(z)}{(1-\nu)f(z) + \nu z f'(z)} - 1 \right),$$

with the same notation as in Remark 1, we have (see Figure 2c)

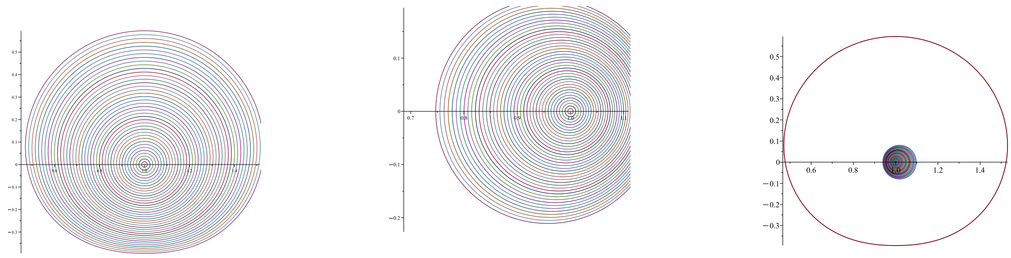
$$\mathrm{L}_{0.5,4}[f_*](\mathbb{D}) \subset \widehat{h}(\mathbb{D}).$$

Using the fact that $\widetilde{h}$ is univalent in $\mathbb{D}$, the above inclusion shows that

$$\mathrm{L}_{4,0.5}[f_*](z) \prec \widehat{h}(z),\ i.e.\ f_* \in \mathscr{S}\left(0.5, 4; \widetilde{h}\right).$$

In conclusion, for the above choices of functions and the corresponding parameters, we have

$$\mathscr{S}(\nu,\rho;h) \setminus \{\mathrm{Id}\} \neq \varnothing.$$



(**a**) The image $\widehat{h}(\mathbb{D})$ (**b**) The inclusion $\mathrm{J}(\mathbb{D}) \subset \{w \in \mathbb{C} : \operatorname{Re} w > 0\}$ (**c**) The inclusion $\mathrm{L}_{0.5,4}[f_*](\mathbb{D}) \subset \widehat{h}(\mathbb{D})$

Figure 2. Figures for Remark 2.

In [18], the researchers proved the following result for analytic functions of the family $\mathscr{S}(\nu, \rho; h)$:

Theorem ([18] Theorem 4)**.** *Let* $f(z) = z + \sum_{k=n}^{\infty} a_k z^k$ $(n \geq 2)$ *and its inverse map* $g = f^{-1}$ *be in* $\mathscr{S}(\nu, \rho; h)$ *with* $|B_2| \leq B_1$*. Then,*

(i)

$$|a_n| \leq \min\left\{ \frac{|\rho| B_1}{(n-1)[1+\nu(n-1)]}; \sqrt{\frac{2|\rho| B_1}{n(2n-2)[1+\nu(2n-2)]}} \right\},$$

(ii)

$$\left| n a_n^2 - a_{2n-1} \right| \leq \frac{|\rho| B_1}{(2n-2)[1+\nu(2n-2)]}.$$

The goal of the current study is to estimate upper bounds for the coefficients $|a_n|$ for those functions that belong to the set of bi-univalent functions with missing coefficients and defined by the $\mathcal{N}_{\Sigma}(\lambda, \delta, h)$. This paper aims to improve some of the results from [18,27]. Additionally, connections to some previously obtained results are made.

The below lemmas are required to prove our results.

Lemma 1 ([28,29])**.** *Let* $f \in \mathcal{S}$ *be given by* (1)*. Then, the coefficients of its inverse map* $g = f^{-1}$ *are given in terms of the Faber polynomials of* f *with*

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \ldots, a_n) w^n,$$

where

$$\begin{aligned} K_{n-1}^{-n} = & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \left[a_5 + (-n+2) a_3^2 \right] \\ & + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4] + \sum_{j \geq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j $(7 \leq j \leq n)$ *is a homogeneous polynomial in the variables* $a_2, a_3, \ldots, a_n$ *and the expressions such as (for example)* $(-m)!$ *are to be interpreted symbolically by*

$$(-m)! \equiv \Gamma(1-m) := (-m)(-m-1)(-m-2)\ldots, \quad m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},\ \mathbb{N} := \{1, 2, \ldots\}.$$

We see that the initial three terms of K_{n-1}^{-n} are given by

$$K_1^{-2} = -2a_2, \quad K_2^{-3} = 3\left(2a_2^2 - a_3\right), \quad \text{and} \quad K_3^{-4} = -4\left(5a_2^3 - 5a_2a_3 + a_4\right).$$

Typically, for every real number p, the expansion of K_n^p is given below (see [28] for details; see also [29], p. 349):

$$K_n^p = pa_{n+1} + \frac{p(p-1)}{2}D_n^2 + \frac{p!}{(p-3)!3!}D_n^3 + \ldots + \frac{p!}{(p-n)!n!}D_n^n.$$

Lemma 2 ([30]). *Let* $f(z) = z + \sum\limits_{k=n}^{\infty} a_k z^k$, $n \geq 2$ *be a univalent function in* $\mathbb{D}$ *and*

$$f^{-1}(w) = w + \sum_{k=n}^{\infty} b_k w^k \quad \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

Then,

$$b_{2n-1} = na_n^2 - a_{2n-1}, \quad \text{and} \quad b_k = -a_k \quad \text{for} \quad n \leq k \leq 2n-2.$$

Lemma 3 ([31] Exercise 9, p. 172). *Assume that* $\varpi(z) = \sum\limits_{j=1}^{\infty} p_j z^j \in \mathcal{B}$. *Then,*

$$|p_n| \leq 1,\ n \geq 2.$$

This lemma represents a special case of the result in [31] [Exercise 9, p. 172] obtained from this exercise for $p_0 = 0$.

2. Main Results

First, we prove the next lemma.

Lemma 4. *Let* $u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \in \mathcal{B}$ *and* s *be a complex number. Then, for all* $n \in \mathbb{N}$, *the following inequality holds:*

$$\left|u_{2n} - su_n^2\right| \leq 1 + (|s| - 1)\left|u_n^2\right| \leq \max\{1; |s|\}.$$

Moreover, the functions $u(z) = z$ *and* $u(z) = z^2$ *prove that the above inequality is sharp for* $|s| \geq 1$ *and for* $|s| < 1$, *respectively.*

Proof. For $u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \in \mathcal{B}$ and a fixed $n \in \mathbb{N}$, let

$$\varepsilon_k := e^{2k\pi i/n}, \quad k \in \{1, 2, \ldots, n\}$$

be the nth order complex roots of the unit. If we define the function $v : \mathbb{D} \to \mathbb{C}$ by

$$v(z) := \frac{1}{n}\sum_{k=1}^{n} u(\varepsilon_k z),\ z \in \mathbb{D}, \tag{2}$$

using the well-known relation

$$\sum_{k=1}^{n} \varepsilon_k^m = \begin{cases} 0, & \text{if} \quad m \in \mathbb{N} \quad \text{is not a multiple of } n, \\ n, & \text{if} \quad m \in \mathbb{N} \quad \text{is a multiple of } n, \end{cases}$$

it follows that

$$v(z) = u_n z^n + u_{2n} z^{2n} + \ldots,\ z \in \mathbb{D}. \tag{3}$$

Since u is an analytic function in $\mathbb{D}$, from Definition (2), it follows that v ia also analytic in $\mathbb{D}$ and $v(0) = 0$. Moreover, since $u \in \mathcal{B}$, we have

$$|v(z)| \leq \frac{1}{n}\sum_{k=1}^{n}\left|u\left(e^{-2ik\pi/n}z\right)\right| < \frac{n}{n} = 1,\ z \in \mathbb{D}.$$

Therefore, $v \in \mathcal{B}$.

Because the function $\chi(z) := z^n$ is a surjective endomorphism of the unit disk $\mathbb{D}$, setting $\zeta := z^n$ in (3) and using the fact that $v \in \mathcal{B}$, we deduce that the function $\psi : \mathbb{D} \to \mathbb{C}$ by

$$\psi(\zeta) := u_n\zeta + u_{2n}\zeta^2 + u_{3n}\zeta^3 + \ldots,\ \zeta \in \mathbb{D}$$

belongs to the class $\mathcal{B}$. Now, using [32] (page 10, inequality (7)) for the function $\psi \in \mathcal{B}$, we obtain the desired outcome with the aforementioned power series expansion. □

We now prove the following main theorem using the aforementioned lemmas and a new method.

Theorem 1. *Let the function* $f(z) = z + \sum_{k=n_0}^{\infty} a_k z^k \in \mathcal{N}_\Sigma(\lambda, \delta, h)$, $n_0 \geq 2$. *Then,*

$$|a_{n_0}| \leq \min\left\{\frac{|B_1|}{1 + (n_0 - 1)(\lambda + n_0\delta)}; \sqrt{\frac{2|B_1|\max\left\{1; \left|\frac{B_2}{B_1}\right|\right\}}{n_0(1 + (2n_0 - 2)[\lambda + (2n_0 - 1)\delta])}}\right\}, \tag{4}$$

and

$$\left|n_0 a_{n_0}^2 - a_{2n_0-1}\right| \leq \frac{|B_1|\max\left\{1; \left|\frac{B_2}{B_1}\right|\right\}}{1 + (2n_0 - 2)[\lambda + (2n_0 - 1)\delta]}. \tag{5}$$

Proof. If $f(z) = z + \sum_{k=n_0}^{\infty} a_k z^k \in \mathcal{N}_\Sigma(\lambda, \delta, h)$, then there are two functions as defined by the quasi-subordination $u, v \in \mathcal{B}$ of the form

$$u(z) = \sum_{k=1}^{\infty} u_k z^n \quad \text{and} \quad v(z) = \sum_{k=1}^{\infty} v_k z^k,$$

satisfying

$$(1 - \lambda)\frac{f(z)}{z} + \lambda f'(z) + \delta z f''(z) = 1 + \sum_{k=n_0}^{\infty}[1 + (k - 1)(\lambda + k\delta)]a_k z^{k-1} = h(u(z)) \tag{6}$$

and

$$(1 - \lambda)\frac{g(w)}{w} + \lambda g'(w) + \delta w g''(w) = 1 + \sum_{k=2}^{\infty}[1 + (k - 1)(\lambda + k\delta)]b_k w^{k-1} = h(v(w)), \tag{7}$$

respectively, where, according to Lemma 1,

$$b_k = \frac{1}{k}K_{n-1}^{-k}(a_2, a_3, \ldots, a_k),\ k \geq 2. \tag{8}$$

We have

$$h(u(z)) = 1 + B_1\left(u_1 z + u_2 z^2 + \ldots\right) + B_2\left(u_1 z + u_2 z^2 + \ldots\right)^2 + \ldots, \tag{9}$$

and, according to (6) and (9), the corresponding coefficients of the power expansions are equal. Hence, we equate these coefficients step by step.

First, from (6), we have $a_k = 0$ for $2 \leq k \leq n_0 - 1$. Thus, the term containing "z" in (9) is equal to zero, that is, $B_1 u_1 = 0$. Using the fact that $B_1 \neq 0$, it follows $u_1 = 0$. Therefore, (9) becomes

$$h(u(z)) = 1 + B_1\left(u_2 z^2 + \dots\right) + B_2\left(u_2 z^2 + \dots\right)^2 + \dots. \tag{10}$$

Secondly, since, in (6), the term containing "z^2" is zero, it follows that, for the corresponding term of (10), we have $B_1 u_2 = 0$. Since $B_1 \neq 0$, it follows that $u_2 = 0$. Hence, (10) becomes

$$h(u(z)) = 1 + B_1\left(u_3 z^3 + \dots\right) + B_2\left(u_3 z^3 + \dots\right)^2 + \dots.$$

We repeat the same method $n_0 - 2$ times and take into account that from the "$n_0 - 3$" step we obtain

$$h(u(z)) = 1 + B_1\left(u_{n_0-2} z^{n_0-2} + \dots\right) + B_2\left(u_{n_0-2} z^{n_0-2} + \dots\right)^2 + \dots. \tag{11}$$

Since the coefficient of term containing "z^{n_0-2}" in (6) is zero, we obtain that the relevant coefficient in (11) is $B_1 u_{n_0-2} = 0$. Thus, the assumption $B_1 \neq 0$ implies $u_{n_0-2} = 0$. Hence, (11) becomes

$$h(u(z)) = 1 + B_1\left(u_{n_0-1} z^{n_0-1} + \dots\right) + B_2\left(u_{n_0-1} z^{n_0-1} + \dots\right)^2 + \dots. \tag{12}$$

Now, by comparing the terms in "z^{n_0-1}" in (6) and (12), we obtain that

$$B_1 u_{n_0-1} = [1 + (n_0 - 1)(\lambda + n_0\delta)]a_{n_0},$$

that is,

$$a_{n_0} = \frac{B_1 u_{n_0-1}}{1 + (n_0 - 1)(\lambda + n_0\delta)}. \tag{13}$$

On the other hand, since $a_k = 0$ for $2 \leq k \leq n_0 - 1$, from (8), we obtain $b_k = 0$ for $2 \leq k \leq n_0 - 1$, and from (8), we have

$$b_{n_0} = \frac{1}{n_0} K_{n_0-1}^{-n_0}(0, 0, \dots, 0, a_{n_0}) = -a_{n_0}.$$

Furthermore, similar to the method described above, from the relation (7), we obtain that the term containing "w^{n_0-1}" is given by

$$B_1 v_{n_0-1} = -[1 + (n_0 - 1)(\lambda + n_0\delta)]a_{n_0},$$

that is,

$$a_{n_0} = -\frac{B_1 v_{n_0-1}}{1 + (n_0 - 1)(\lambda + n_0\delta)}. \tag{14}$$

From (13) and (14), using Lemma 3 and considering the previous reasons, we obtain

$$|a_{n_0}| = |b_{n_0}| \leq \frac{|B_1|}{1 + (n_0 - 1)(\lambda + n_0\delta)}. \tag{15}$$

Also, equating the terms that contain "z^{2n_0-2}" from (6) for $k = 2n_0 - 1$ and those of (12), we obtain

$$\Big(1 + (2n_0 - 2)[\lambda + (2n_0 - 1)\delta]\Big)a_{2n_0-1} = B_1 u_{2n_0-2} + B_2 u_{n_0-1}^2 = B_1\left(u_{2n_0-2} + \frac{B_2}{B_1}u_{n_0-1}^2\right).$$

Thus, based on the previous equality and according to the Lemma 4, it follows that

$$\Big(1+(2n_0-2)[\lambda+(2n_0-1)\delta]\Big)|a_{2n_0-1}|\leq|B_1|\max\left\{1;\left|\frac{B_2}{B_1}\right|\right\}.$$

Hence,

$$|a_{2n_0-1}|\leq\frac{|B_1|\max\left\{1;\left|\frac{B_2}{B_1}\right|\right\}}{1+(2n_0-2)[\lambda+(2n_0-1)\delta]}. \tag{16}$$

From Definition 1, because $f\in\mathcal{N}_\Sigma(\lambda,\delta,h)$ implies $g\in\mathcal{N}_\Sigma(\lambda,\delta,h)$ and using the above method of proof, we have

$$(1-\lambda)\frac{g(w)}{w}+\lambda g'(w)+\delta w g''(w)=1+\sum_{k=n_0}^{\infty}[1+(k-1)(\lambda+k\delta)]b_k w^{k-1}=h(v(w)).$$

Hence, we obtain

$$|b_{2n_0-1}|\leq\frac{|B_1|\max\left\{1;\left|\frac{B_2}{B_1}\right|\right\}}{1+(2n_0-2)[\lambda+(2n_0-1)\delta]}. \tag{17}$$

Furthermore, in view of Lemma 2, using the relations (16) and (17), we deduce that

$$|a_{n_0}|\leq\sqrt{\frac{|a_{2n_0-1}|+|b_{2n_0-1}|}{n_0}}\leq\sqrt{\frac{2|B_1|\max\left\{1;\left|\frac{B_2}{B_1}\right|\right\}}{n_0\big(1+(2n_0-2)[\lambda+(2n_0-1)\delta]\big)}}, \tag{18}$$

and from (15) and (18), we obtain the inequality (4).

In addition, using (17) and Lemma 2, it follows that

$$|n_0a_{n_0}^2-a_{2n_0-1}|=|b_{2n_0-1}|\leq\frac{|B_1|\max\left\{1;\left|\frac{B_2}{B_1}\right|\right\}}{1+(2n_0-2)[\lambda+(2n_0-1)\delta]},$$

which completes our proof. □

Next, this study shows why this theorem improves and generalizes some previous ones by a suitable choice of parameters.

Remark 3. *By choosing λ, δ, and h properly, we obtain from Theorem 1 the bounds that are better, in some ranges of the parameters, than the estimates obtained before.*

1. If

$$h(z)=\frac{1+(1-2\alpha)z}{1-z},\ 0\leq\alpha<1,$$

then the bounds are better than those in [20, Theorem 2];

2. If

$$h(z)=\frac{1+(1-2\alpha)z}{1-z},\ 0\leq\alpha<1,$$

and $\delta=0$ or $\lambda=1$, then the bounds are better than those in [20] [Corollary 3] and [20] [Corollary 4], respectively;

3. If $\delta=0$, then the bounds are better than those in [33] [Theorems 3.1] in the case of subordination.

In the following part, we emphasize the significance of our present results that improve some previous results concerning different recent subclasses of bi-univalent functions.

Remark 4. *In the proof of Theorem 4 of [18], assume for convenience that* $\rho = 1$, $\vartheta = 0$ *with* $f(z) = z + \sum_{k=n}^{\infty} a_k z^k \in \mathscr{S}(0,1;h)$. *By the definition of the subordination, there exist two functions* $u, v \in \mathcal{B}$ *with*

$$u(z) = \sum_{k=1}^{\infty} u_k z^n \quad \text{and} \quad v(z) = \sum_{k=1}^{\infty} v_k z^k,$$

satisfying

$$\frac{zf'(z)}{f(z)} = h(u(z)) \quad and \quad \frac{wg'(w)}{g(w)} = h(v(w)),$$

respectively.

Since

$$\left.\frac{zf'(z)}{f(z)}\right|_{z=0} = 1,$$

it follows that

$$\frac{zf'(z)}{f(z)} = 1 + \beta_1 z + \ldots + \beta_n z^n + \ldots, \; z \in \mathbb{D}, \tag{19}$$

that is,

$$z + na_n z^n + \ldots = (z + a_n z^n + \ldots)\Big(1 + \beta_1 z + \beta_2 z^2 + \ldots\Big), \; z \in \mathbb{D}.$$

Equating the corresponding coefficients of the above relation, we obtain

$$\beta_1 = \beta_2 = \cdots = \beta_{n-2} = 0,$$
$$\beta_{n-1} = (n-1)a_n,$$

and from (19), it follows that

$$\frac{zf'(z)}{f(z)} = 1 + (n-1)a_n z^{n-1} + \ldots, \; z \in \mathbb{D}.$$

Let us consider again, for convenience, that $n = 3$. *Thus,*

$$f(z) = z + a_3 z^3 + a_4 z^4 + \ldots,$$

then

$$\frac{zf'(z)}{f(z)} = 1 + \beta_2 z^2 + \beta_3 z^3 + \ldots,$$

where

$$\beta_2 = 2a_3, \; \beta_3 = 3a_4, \; \beta_4 = 4a_5 - 2a_3^2.$$

Consequently, if f *has the above form, then it is impossible that* $\beta_2 = 2a_3$ *and* $\beta_4 = 4a_5$ *at the same time. We have* $\beta_4 = 4a_5$ *while* $n_0 = 5$, *but in this case,* $\beta_2 = 0$. *Therefore, the relation (2.11) of [18] and the Theorem 4 of [18] are not correct. Similarly, for the same reason, Theorem 2.6 of [27] is not correct.*

Example 1. *As an example of Theorem 1, if we consider the analytic function in* $\mathbb{D}$ *defined by*

$$f(z) := \frac{1}{\ell}\log\left(\frac{1}{1-\ell z}\right) = z + \frac{\ell z^2}{2} + \frac{\ell^2 z^3}{3} + \ldots, \; z \in \mathbb{D}, \quad with \quad 0 < |\ell| \le 1,$$

then $f \in \mathcal{A}$ *and its inverse is* $f^{-1}(w) = \dfrac{e^{\ell w} - 1}{\ell e^{\ell w}}$, *which have an analytic extension in* $\mathbb{D}$ *denoted as* $g(z) = \dfrac{e^{\ell z} - 1}{\ell e^{\ell z}}$.

Letting

$$h(z) := 1 + 0.35z + 0.1z^2 + 0.1z^3, \; z \in \mathbb{D}, \tag{20}$$

like we may see in Figure 3a made with MAPLE™ computer software, we have

$$\operatorname{Re} h(z) > 0,\ z \in \mathbb{D},\ B_1 = 0.35 = h'(0) \neq 0, B_2 = 0.1, B_3 = 0.1,$$

and

$$B_n = 0 \quad \text{for} \quad n \geq 4.$$

Also, we see that

$$\operatorname{Re} \frac{zh'(z)}{h(z)-1} = \operatorname{Re} \frac{3z^2 + 2z + 3.5}{z^2 + z + 3.5} > 0.1 > 0,\ z \in \mathbb{D}.$$

Hence, h is a starlike (univalent) function in $\mathbb{D}$ with respect to the point $z_0 = 1$. For some "very small" values of the parameter $|\ell|$ (i.e., close to zero), we have $f \in \mathcal{N}_\Sigma(1.1, 0.15, h)$ with h given by (20) since the ranges $f(\mathbb{D})$ and $g(\mathbb{D})$ with small neighborhoods of the point $w_0 = 1$ are included in $h(\mathbb{D})$. According to Theorem 1, the inequalities (4) and (5) reduce to

$$|\ell| \leq 0.5843487098\ldots \quad \textit{and} \quad |\ell| \leq 0.7156780854\ldots,$$

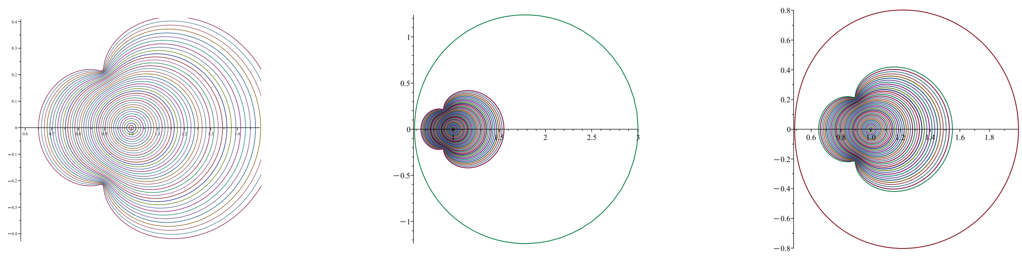
respectively. Hence,

$$0 < |\ell| \leq 0.5843487098\ldots. \tag{21}$$

(i) Unfortunately, the upper bound of (21) represents a necessary but not sufficient condition for $f \in \mathcal{N}_\Sigma(1.1, 0.15, h)$ with h given by (20). Let us consider $\lambda = 1.1$ and $\delta = 0.15$. Thus, for $\ell = 0.5843487098$ from Figure 3b,c, we see that

$$\mathrm{I}_{1.1,0.15}[f](\mathbb{D}) \not\subset h(\mathbb{D}) \quad \textit{and} \quad \mathrm{I}_{1.1,0.15}[g](\mathbb{D}) \not\subset h(\mathbb{D}),$$

but the reverse inclusions are true. Hence, for $\ell = 0.5843487098$, we have $f \notin \mathcal{N}_\Sigma(1.1, 0.15, h)$.



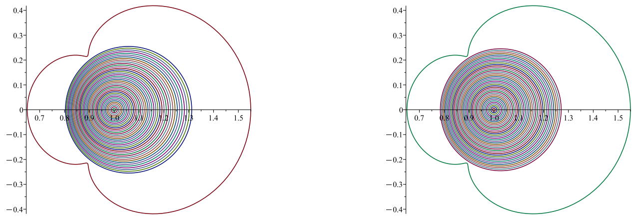
(**a**) The image $h(\mathbb{D})$ (**b**) The inclusion $h(\mathbb{D}) \subset \mathrm{I}_{1.1,0.15}[f](\mathbb{D})$ (**c**) The inclusion $h(\mathbb{D}) \subset \mathrm{I}_{1.1,0.15}[g](\mathbb{D})$

Figure 3. Figures for Example 1(i).

(ii) As we see in Figure 4a,b, for $\lambda = 1.1$, $\delta = 0.15$, and, for example, $\ell = 0.201$, we have the inclusions

$$\mathrm{I}_{1.1,0.15}[f](\mathbb{D}) \subset h(\mathbb{D}) \quad \textit{and} \quad \mathrm{I}_{1.1,0.15}[g](\mathbb{D}) \subset h(\mathbb{D}),$$

and from the fact that h is univalent in $\mathbb{D}$, it follows that both of the subordinations of Definition 1 hold, i.e., $f \in \mathcal{N}_\Sigma(1.1, 0.15, h)$.



(**a**) The inclusion $\mathrm{I}_{1.1,0.15}[f](\mathbb{D}) \subset h(\mathbb{D})$

(**b**) The inclusion $\mathrm{I}_{1.1,0.15}[g](\mathbb{D}) \subset h(\mathbb{D})$

Figure 4. Figures for Example 1(ii).

3. Conclusions

The present studies have been extensively made in order to make conclusions that support the justification for the current research, taking into account the aims, methodology, conclusions, and results of the investigations. The coefficient boundaries of analytic functions can be found with the use of the Faber polynomial expansion approach, which has been proven to be effective.

We have defined a new subclass of bi-univalent functions in this article, along with several useful examples. In the concluding part, we underline that by utilizing subordination, we were able to determine the bounds for the coefficient $|a_n|$ for the class of bi-univalent functions with missing coefficients, emphasizing the novelty of the methods used for the proofs and comments.

Moreover, by applying Lemma 4, the inequalities of Theorem 1 for these function classes represent an improvement of a few results for some ranges of the parameters.

We expect that this method can be applied to the classes of harmonic and meromorphic functions in some future works.

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