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Upper bounds of the logarithmic coefficients for some subclasses of analytic functions

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Dedicated to Prof. Davood Alimohammadi on his 63th birth anniversary.

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Abstract
Due to the major importance of the study of the logarithmic coefficients for univalent functions, in this paper we find the sharp upper bounds for some expressions associated with logarithmic coefficients of functions that belong to some well-known classes of analytic functions in the open unit disk of the complex plane.

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1 Introduction and preliminaries

Let $\mathcal{A}$ be a class of analytic functions in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}, \tag{1.1}$$

and let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{D}$. For $\alpha \in [0,1)$, we denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of all functions $f \in \mathcal{A}$ for which f is a *starlike function of order* α in $\mathbb{D}$, that is,

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}.$$

Note that $\mathcal{S}^*(0) =: \mathcal{S}^*$ represents the class of *starlike functions* in $\mathbb{D}$. A function $f \in \mathcal{A}$ is said to be *starlike of reciprocal order* $\alpha \in [0,1)$ (see [19, p. 2734]), denoted by $f \in \mathcal{S}^*_r(\alpha)$, if

$$\operatorname{Re} \frac{f(z)}{zf'(z)} > \alpha, \quad z \in \mathbb{D}.$$

It is well known that every starlike function of reciprocal order 0 is starlike. In particular, every starlike function of reciprocal order $\alpha \in [0,1)$ is starlike, and hence univalent in $\mathbb{D}$

(see [22]). Moreover, for a function $f \in \mathcal{A}$ and $0 < \alpha < 1$, the following equivalence holds:

$$f \in \mathcal{S}_r^*(\alpha) \quad \Longleftrightarrow \quad \left| \frac{zf'(z)}{f(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad z \in \mathbb{D}.$$

Various authors have studied many problems for the classes $\mathcal{S}_r^*(\alpha)$ and one may see, for example, the works [8, 19, 22].

If is well known that if F and G are two analytic functions in $\mathbb{D}$, then F is said to be subordinated to G, denoted by $F(z) \prec G(z)$, if there exists an analytic function w in $\mathbb{D}$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in \mathbb{D}$, such that $F(z) = G(w(z))$ for all $z \in \mathbb{D}$. It follows from the *Schwarz lemma* that $F(z) \prec G(z)$ implies $F(\mathbb{D}) \subset G(\mathbb{D})$, while if G is univalent in $\mathbb{D}$ then

$$F(z) \prec G(z) \quad \Longleftrightarrow \quad F(0) = G(0) \quad \text{and} \quad F(\mathbb{D}) \subset G(\mathbb{D}).$$

In [6] the authors introduced the class $\mathcal{S}_{4\ell}^*$ of functions $f \in \mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{5z}{6} + \frac{z^5}{6} =: Q_{4\ell}(z).$$

A function $f \in \mathcal{S}_{4\ell}^*$ maps the open unit disk $\mathbb{D}$ onto a domain that consists in a four-leaf-shaped region.

In [12] (see also [26]), by using the polynomial function $\Phi_\ell(z) := 1 + \sqrt{2}z + \frac{z^2}{2}$, the corresponding class $\mathcal{S}_\ell^*$ of functions $f \in \mathcal{A}$ such that

$$\frac{zf'(z)}{f(z)} \prec \Phi_\ell(z)$$

was widely investigated. Note that every function in $\mathcal{S}_\ell^*$ is univalent in $\mathbb{D}$ and maps the open unit disk onto the domain bounded by the limacon curve

$$\left(4u^2 + 4v^2 - 8u - 5\right)^2 + 8\left(4u^2 + 4v^2 - 12u - 3\right) = 0, \quad z = u + iv.$$

The logarithmic coefficients γ_n of a function $f \in \mathcal{S}$ are defined with the aid of the following power series expansion:

$$F_f(z) := \log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad z \in \mathbb{D}, \quad \text{where } \log 1 = 0. \tag{1.2}$$

These coefficients play an important role in different estimates in the theory of univalent functions, and note that we will use the notation γ_n instead of $\gamma_n(f)$; in this regard see [16, Chap. 2] and [17, 18].

The logarithmic coefficients γ_n of an arbitrary function $f \in \mathcal{S}$ (see [11, Theorem 4]) satisfy the inequality

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \le \frac{\pi^2}{6},$$

and the equality is obtained for the *Koebe function*. For $f \in \mathcal{S}^*$, the inequality $|\gamma_n| \le 1/n$ holds but it is not true for the whole class $\mathcal{S}$ (see [10, Theorem 8.4]). However, the problem of the best upper bounds for the logarithmic coefficients of univalent functions for $n \geq 3$ is presumably still open.

Because of the major importance of the study of the logarithmic coefficients, in recent years several authors have recently investigated the issues regarding the logarithmic coefficients and various related problems for some subclasses of analytic functions (for example, see [1–5, 9, 13–15, 20, 21, 24, 25]).

In [2] the authors obtained bounds for the logarithmic coefficients γ_n, $n \in \mathbb{N}$, of the general class

$$\mathcal{S}^*(\varphi) := \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\},$$

with the given bounds generalizing many of the earlier obtained results.

Theorem A ([2, Theorem 1(i)]) *Let $f \in \mathcal{S}^*(\varphi)$. If $\varphi(z) = 1 + B_1 z + \cdots + B_n z^n + \cdots$, $z \in \mathbb{D}$, with $B_1 \neq 0$, is convex (univalent), then the logarithmic coefficients of f satisfy the inequalities*:

$$|\gamma_n| \leq \frac{|B_1|}{2n}, \quad n \in \mathbb{N} := \{1, 2, 3, \dots\}, \tag{1.3}$$

and

$$\sum_{n=1}^{\infty} |\gamma_n|^2 \leq \frac{1}{4} \sum_{n=1}^{\infty} \frac{|B_n|^2}{n^2}. \tag{1.4}$$

All the inequalities in (1.3) and (1.4) are sharp, so that for any $n \in \mathbb{N}$ there is a function f_n given by $zf_n'(z)/f_n(z) = \varphi(z^n)$ and a function f given by $zf'(z)/f(z) = \varphi(z)$, respectively.

In [6] the authors found upper bounds for the logarithmic coefficients γ_n for $n = 1, 2, 3, 4$ as follows:

Theorem B ([6, Theorem 6]) *Let f be the series as in (1.1) and suppose $f \in \mathcal{S}^*_{4\ell}$. Then*

$$|\gamma_1| \leq \frac{5}{12}, \qquad |\gamma_2| \leq \frac{5}{24}, \qquad |\gamma_3| \leq \frac{5}{36}, \qquad |\gamma_4| \leq \frac{5}{48}.$$

These bounds are sharp.

The next lemma will be used to obtain our first main result.

Lemma 1.1 ([23, Theorem II(i)], [10, Theorem 6.3, p. 192]) *Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be analytic in $\mathbb{D}$, and suppose that $f(z) \prec g(z)$ where g is univalent in $\mathbb{D}$. Then*

$$\sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |b_k|^2, \quad n \in \mathbb{N}.$$

To prove our second result that gives sharp estimates for the logarithmic coefficients γ_n, $n \in \mathbb{N}$, for functions belonging to the class $\mathcal{S}^*_{4\ell}$, we will use a different method given by the following lemma.

Lemma 1.2 ([23, Theorem VI(i)]) *Let $f(z)=\sum_{k=1}^{\infty}a_k z^k$ and $F(z)=\sum_{k=1}^{\infty}A_k z^k$, $z\in\mathbb{D}$, be analytic functions in $\mathbb{D}$ such that $f(z)\prec F(z)$. If $A_1>0$ and there is a function*

$$F_1(z)=\frac{1}{2}A_1+A_2z+\cdots+A_nz^{n-1}+\sum_{k=n+1}^{\infty}B_kz^{k-1},\quad z\in\mathbb{D},\tag{1.5}$$

which is regular in $\mathbb{D}$ and satisfies $\operatorname{Re}F_1(z)>0$, $z\in\mathbb{D}$, then

$$|a_n|\le A_1,\quad n\in\mathbb{N}.$$

Equality is possible only if either $f(z)=F(\varepsilon z^n)$, $|\varepsilon|=1$, or if A_k are of the form

$$A_k=\sum_{j=1}^{n-1}\rho_j e^{i(k-1)\vartheta_j},\quad \rho_j\ge 0,\ k=1,2,\dots,n.$$

In [7] the author defined the class $S(A,B,p,\beta)$ of functions $f(z)=z^p+\sum_{k=1}^{\infty}a_kz^k$ analytic in $\mathbb{D}$ which satisfy

$$\frac{zf'(z)}{f(z)}\prec(p-\beta)\frac{1+Az}{1+Bz}+\beta=:\psi(z),\tag{1.6}$$

with $-1\le B<A\le 1$ and $0\le\beta<p$, $p\in\mathbb{N}$. Theorem 3 of [7] gives upper bounds for the coefficients a_n of functions belonging to this class $S(A,B,p,\beta)$.

We emphasize that, if we assume also that $-1\le A<B\le 1$ instead of $-1\le B<A\le 1$, the conclusion of Theorem 3 in [7] holds. We will use the following lemma to determine upper bounds for the coefficients of functions in the class $\mathcal{S}_r^*(\alpha)$.

Lemma 1.3 ([7, Theorem 3]) *If $f(z)=z^p+\sum_{k=1}^{\infty}a_kz^k\in S(A,B,p,\beta)$, then*

$$|a_n|\le\prod_{k=0}^{n-(p+1)}\frac{|(B-A)(p-\beta)+Bk|}{k+1},\quad n\ge p+1,$$

and these bounds are sharp for all $-1\le A<B\le 1$ and for each $n\ge p+1$.

The main purpose of this paper is to get sharp bounds for some relations associated with logarithmic coefficients of functions belonging to the well-known classes $\mathcal{S}_r^*(\alpha)$, $\mathcal{S}_{4\ell}^*$, and $\mathcal{S}_\ell^*$. Moreover, we find sharp bounds for the coefficients of functions in $\mathcal{S}_r^*(\alpha)$.

2 Main results

The first result of this section deals with the logarithmic coefficients of the class $\mathcal{S}_r^*(\alpha)$.

Theorem 2.1 *Let $f\in\mathcal{S}_r^*(\alpha)$.*

1. *Then, the logarithmic coefficients of f satisfy the following inequalities*:

$$|\gamma_n|\le\frac{1-\alpha}{n},\quad n\in\mathbb{N},\tag{2.1}$$

$$\sum_{n=1}^{\infty}|\gamma_n|^2\le\begin{cases}(\alpha-1)^2\sum_{n=1}^{\infty}\frac{(1-2\alpha)^{2(n-1)}}{n^2}, & \text{if } \alpha\in[0,1)\setminus\{1/2\},\\ \frac{1}{4}, & \text{if } \alpha=1/2,\end{cases}\tag{2.2}$$

and

$$\sum_{n=1}^{\infty} n^2|\gamma_n|^2 \le \begin{cases} (\alpha-1)^2\sum_{n=1}^{\infty}(1-2\alpha)^{2(n-1)}, & \text{if } \alpha \in [0,1)\setminus\{1/2\}, \\ \frac{1}{4}, & \text{if } \alpha = 1/2. \end{cases} \tag{2.3}$$

2. *The first inequality is sharp* (*the best possible*) *because for an arbitrary* $n_0 \in \mathbb{N}$, *for the function*

$$\kappa(z) = f_{n_0}(z) := \begin{cases} z[1+(1-2\alpha)z^{n_0}]^{\frac{2(\alpha-1)}{n_0(1-2\alpha)}}, & \text{if } \alpha \in [0,1)\setminus\{1/2\}, \\ ze^{-\frac{z^{n_0}}{n_0}}, & \text{if } \alpha = 1/2, \end{cases}$$

we obtain equality in (2.1).

3. *The second and third inequalities are sharp because for the function*

$$\mathrm{k}(z) = f_1(z) := \begin{cases} z[1+(1-2\alpha)z]^{\frac{2(\alpha-1)}{1-2\alpha}}, & \text{if } \alpha \in [0,1)\setminus\{1/2\}, \\ ze^{-z}, & \text{if } \alpha = 1/2, \end{cases} \tag{2.4}$$

we obtain equalities in (2.2) *and* (2.3).

Proof 1. To prove the first part of our result, suppose that $f \in \mathcal{S}_r^*(\alpha)$. Then

$$\frac{f(z)}{zf'(z)} \prec \frac{1+(1-2\alpha)z}{1-z},$$

and, according to the definition of subordination, this is equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1-z}{1+(1-2\alpha)z} =: \phi(z)$$
$$= \begin{cases} 1+\frac{2(\alpha-1)}{1-2\alpha}\sum_{k=1}^{\infty}(-1)^{k-1}(1-2\alpha)^k z^k, & \text{if } \alpha \in [0,1)\setminus\{1/2\}, \\ 1-z, & \text{if } \alpha = 1/2, \end{cases}$$

where $\phi(z) = 1 + B_1 z + \cdots$, $z \in \mathbb{D}$, with $B_1 = 2(\alpha-1) \neq 0$ for $\alpha \in [0,1)$. Now, we will show that ϕ is convex (univalent) in $\mathbb{D}$. Since $\phi'(0) = B_1 \neq 0$ and

$$\operatorname{Re}\left(1+\frac{z\phi''(z)}{\phi'(z)}\right) = \operatorname{Re}\frac{1-(1-2\alpha)z}{1+(1-2\alpha)z} > 0, \quad z \in \mathbb{D},$$

for all $\alpha \in [0,1)$, it follows that ϕ is convex (univalent) in $\mathbb{D}$. Since all conditions of Theorem A are satisfied, from (1.3) and (1.4) it follows that the first two inequalities of our theorem hold.

Now we will prove the last inequality of our theorem. Thus, given $f \in \mathcal{S}_r^*(\alpha)$ and using the power series expansion formula (1.2), we get

$$\sum_{n=1}^{\infty} 2n\gamma_n z^n = z\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\frac{f(z)}{z}\right) = \frac{zf'(z)}{f(z)} - 1 \prec \frac{1-z}{1+(1-2\alpha)z} - 1 = \phi(z) - 1 =: \psi(z).$$

We already proved that the function ϕ is univalent in $\mathbb{D}$, hence ψ will be such too, and it has the form

$$\psi(z) = \begin{cases} \frac{2(\alpha-1)}{1-2\alpha}\sum_{n=1}^{\infty}(-1)^{nk-1}(1-2\alpha)^n z^n, & \text{if } \alpha \in [0,1)\setminus\{1/2\}, \\ -z, & \text{if } \alpha = 1/2. \end{cases}$$

According to Lemma 1.1, the above subordination leads to

$$\sum_{n=1}^{p} 4n^2|\gamma_n|^2 \leq \begin{cases} 4\sum_{n=1}^{p}(\alpha-1)^2(1-2\alpha)^{2(n-1)}, & \text{if } \alpha \in [0,1)\setminus\{1/2\}, \\ 1, & \text{if } \alpha = 1/2, \end{cases}$$

and, letting $p \to +\infty$, the last assertion is proved.

2. For the proof of the sharpness of (2.1), if $\alpha \in [0,1)\setminus\{1/2\}$, we have $\kappa \in \mathcal{A}$, and since

$$\frac{z\kappa'(z)}{\kappa(z)} = \frac{1-z^{n_0}}{1+(1-2\alpha)z^{n_0}} \prec \frac{1-z}{1+(1-2\alpha)z},$$

it follows that $\kappa \in \mathcal{S}_r^*(\alpha)$. Also,

$$\begin{aligned} \frac{1}{2}\log\frac{\kappa(z)}{z} &= \frac{1}{2}\frac{2(\alpha-1)}{n_0(1-2\alpha)}\log\bigl[1+(1-2\alpha)z^{n_0}\bigr] \\ &= \frac{\alpha-1}{n_0}z^{n_0} + \sum_{k=2}^{\infty}\frac{(-1)^{k-1}(\alpha-1)(1-2\alpha)^{k-1}}{n_0 k}z^{n_0 k}, \quad z \in \mathbb{D}. \end{aligned}$$

Thus $\gamma_{n_0} = \frac{\alpha-1}{n_0}$ and hence $|\gamma_{n_0}| = \frac{1-\alpha}{n_0}$.

For $\alpha = \frac{1}{2}$, it follows that $\kappa \in \mathcal{A}$ and

$$\frac{z\kappa'(z)}{\kappa(z)} = 1 - z^{n_0} = \frac{1-z^{n_0}}{1+(1-2\cdot\frac{1}{2})z^{n_0}} \prec \frac{1-z}{1+(1-2\cdot\frac{1}{2})z},$$

which implies $\kappa \in \mathcal{S}_r^*(1/2)$. Since

$$\frac{1}{2}\log\frac{\kappa(z)}{z} = -\frac{z^{n_0}}{2n_0},$$

it follows that $\gamma_{n_0} = -\frac{1}{2n_0}$ and hence $|\gamma_{n_0}| = \frac{1}{2n_0} = \frac{1-\alpha}{n_0}$ for $\alpha = \frac{1}{2}$.

3. To prove the sharpness of (2.2) and (2.3), for $\alpha \in [0,1)\setminus\{1/2\}$, like in the above proof, by replacing $n_0 := 1$, we get $\mathrm{k} \in \mathcal{S}_r^*(\alpha)$, and

$$\frac{1}{2}\log\frac{\mathrm{k}(z)}{z} = \sum_{k=1}^{\infty}\frac{(-1)^{k-1}(\alpha-1)(1-2\alpha)^{k-1}}{k}z^k, \quad z \in \mathbb{D}.$$

Hence, $\gamma_n = \frac{(-1)^{n-1}(\alpha-1)(1-2\alpha)^{n-1}}{n}$, $n \in \mathbb{N}$ and thus

$$|\gamma_n|^2 = \frac{(\alpha-1)^2(1-2\alpha)^{2(n-1)}}{n^2}, \quad n \in \mathbb{N}. \tag{2.5}$$

Therefore we have

$$\sum_{n=1}^{\infty} |\gamma_n|^2 = (\alpha - 1)^2 \sum_{n=1}^{\infty} \frac{(1-2\alpha)^{2(n-1)}}{n^2}.$$

For $\alpha = \frac{1}{2}$, similarly to the above proof, we get $\mathrm{k} \in \mathcal{S}_r^*(1/2)$ and

$$\gamma_1 = \frac{1}{2} \quad \text{and} \quad \gamma_k = 0, \quad k \geq 2, \tag{2.6}$$

which implies

$$\sum_{n=1}^{\infty} |\gamma_n|^2 = \frac{1}{4}.$$

Similarly, if $\alpha \in [0,1) \setminus \{1/2\}$, from (2.5) we get

$$n^2|\gamma_n|^2 = (\alpha - 1)^2(1-2\alpha)^{2(n-1)}, \quad n \in \mathbb{N}.$$

Hence

$$\sum_{n=1}^{\infty} n^2|\gamma_n|^2 = (\alpha - 1)^2 \sum_{n=1}^{\infty}(1-2\alpha)^{2(n-1)},$$

and, for $\alpha = 1/2$, according to (2.6) it follows that

$$\sum_{n=1}^{\infty} n^2|\gamma_n|^2 = \frac{1}{4}.$$

Therefore the proof of the theorem is completed. □

Remark 2.1 The power series on the right-hand side of (2.3) converges only for $|1-2\alpha| < 1$, which is equivalent to $\alpha \in (0,1)$. If $\alpha = 0$, then

$$\sum_{n=1}^{\infty}(1-2\alpha)^{2(n-1)} = \sum_{n=1}^{\infty} 1 = +\infty,$$

hence this series is divergent. Consequently, for $\alpha = 0$ the third inequality is not useful from the point of view that for $\sum_{n=1}^{\infty} n^2|\gamma_n|^2$ the upper bound given by (2.3) is not finite.

The following result provides the best possible upper bounds for the logarithmic coefficients of the functions belonging to $\mathcal{S}_{4\ell}^*$ and it is an extension of Theorem B.

Theorem 2.2 *If $f \in \mathcal{S}_{4\ell}^*$, then*

$$|\gamma_n| \leq \frac{5}{12n}, \quad n \in \mathbb{N}. \tag{2.7}$$

This inequality is sharp for each $n \in \mathbb{N}$.

Proof If $f \in \mathcal{S}^*_{4\ell}$, then from the definition of $\mathcal{S}^*_{4\ell}$ it follows that

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\frac{f(z)}{z}\right) = \frac{zf'(z)}{f(z)} - 1 \prec \frac{5z}{6} + \frac{z^5}{6},$$

and using the logarithmic coefficients γ_n of f given by (1.2), we get

$$\sum_{n=1}^{\infty} 2n\gamma_n z^n \prec \frac{5z}{6} + \frac{z^5}{6} =: F(z). \tag{2.8}$$

Now, consider in Lemma 1.2 the sequence

$$A_1 = \frac{5}{6}, \qquad A_2 = A_3 = A_4 = 0, \qquad A_5 = \frac{1}{6}, \tag{2.9}$$

$A_n = 0$ for all $n \geq 6$, and $B_k = 0$ for all $k \geq 7$. Then the function F_1 given by (1.5) becomes

$$F_1(z) = \frac{5}{12} + \frac{z^4}{6}.$$

Since $A_1 = 5/6 > 0$, the analytic function F_1 in $\mathbb{D}$ satisfies $\operatorname{Re} F_1(z) > 1/4 > 0$, $z \in \mathbb{D}$. Hence all the assumptions of Lemma 1.2 are satisfied. Therefore, according to this lemma, the subordination (2.8) implies that

$$2n|\gamma_n| \leq A_1 = \frac{5}{6}, \quad n \in \mathbb{N},$$

which yields our inequality.

The sharpness of the inequality (2.7) could be proved much easier, and without using Lemma 1.2, as follows. For every $n \in \mathbb{N}$, define the function

$$f_n(z) := z\exp\left(\int_0^z \frac{Q_{4\ell}(t^n) - 1}{t}\,\mathrm{d}t\right) = z + \frac{5}{12n}z^{n+1} + \cdots, \quad z \in \mathbb{D}.$$

It is easy to check that $f_n \in \mathcal{A}$ and

$$\frac{zf_n'(z)}{f_n(z)} = 1 + \frac{5z^n}{6} + \frac{z^{5n}}{6} \prec 1 + \frac{5z}{6} + \frac{z^5}{6},$$

hence $f_n \in \mathcal{S}^*_{4\ell}$. Since

$$\log\frac{f_n(z)}{z} = 2\sum_{k=1}^{\infty}\gamma_k(f_n)z^k = \frac{5}{12n}z^n + \cdots, \quad z \in \mathbb{D},$$

it follows that the upper bound of the inequality (2.7) is sharp for each $n \in \mathbb{N}$ if $f = f_n$, that is, $|\gamma_n| = \frac{5}{12n}$, $n \in \mathbb{N}$. □

Sharp bounds of the logarithmic coefficients γ_n for the functions of the class $\mathcal{S}^*_{\ell}$ are obtained in the next result.

Theorem 2.3 *If $f \in \mathcal{S}^*_\ell$, then*

$$|\gamma_n| \leq \frac{1}{\sqrt{2}n}, \quad n \in \mathbb{N}. \tag{2.10}$$

This inequality is sharp for each $n \in \mathbb{N}$.

Proof Supposing that $f \in \mathcal{S}^*_\ell$, by the definition of $\mathcal{S}^*_\ell$ it follows that

$$z\frac{\mathrm{d}}{\mathrm{d}z}\left(\log\frac{f(z)}{z}\right) = \frac{zf'(z)}{f(z)} - 1 \prec \sqrt{2}z + \frac{z^2}{2},$$

which, with regard to the logarithmic coefficients γ_n of f given by (1.2), leads to

$$\sum_{n=1}^{\infty} 2n\gamma_n z^n \prec \sqrt{2}z + \frac{z^2}{2} =: F(z). \tag{2.11}$$

If we consider in Lemma 1.2 the sequence $A_1 = \sqrt{2}$, $A_2 = 1/2$, $A_n = 0$ for all $n \geq 3$, and $B_k = 0$ for all $k \geq 4$, then the function F_1 given by (1.5) becomes

$$F_1(z) = \frac{1}{\sqrt{2}} + \frac{z}{2}.$$

Since $A_1 = \sqrt{2} > 0$, the analytic function F_1 in $\mathbb{D}$ satisfies $\operatorname{Re} F_1(z) > (\sqrt{2}-1)/2 > 0$, $z \in \mathbb{D}$. Hence all the assumptions of Lemma 1.2 are satisfied. Therefore, from the subordination (2.11) we get

$$2n|\gamma_n| \leq A_1 = \sqrt{2}, \quad n \in \mathbb{N},$$

and the inequality (2.10) is proved.

Moreover, for every $n \in \mathbb{N}$, define the function

$$f_n(z) := z\exp\left(\int_0^z \frac{\Phi_\ell(t^n) - 1}{t}\,\mathrm{d}t\right) = z + \frac{1}{\sqrt{2}n}z^{n+1} + \cdots, \quad z \in \mathbb{D}.$$

Since $f_n \in \mathcal{A}$ and

$$\frac{zf_n'(z)}{f_n(z)} = 1 + \sqrt{2}z^n + \frac{z^{2n}}{2} \prec 1 + \sqrt{2}z + \frac{z^2}{2},$$

it follows that $f_n \in \mathcal{S}^*_\ell$. Also, a simple computation shows that

$$\log\frac{f_n(z)}{z} = 2\sum_{k=1}^{\infty} \gamma_k(f_n)z^k = \frac{1}{\sqrt{2}n}z^n + \cdots, \quad z \in \mathbb{D}.$$

Thus, the upper bound of (2.10) is the best possible for each $n \in \mathbb{N}$ whenever $f = f_n$. $\square$

For the coefficients a_n of the functions $f \in \mathcal{S}^*_r(\alpha)$ of the form (1.1), we find the following sharp upper bounds.

Theorem 2.4 *If $f \in \mathcal{S}_r^*(\alpha)$, then*

$$|a_n| \leq \begin{cases} \prod_{k=0}^{n-2} \frac{|2(1-\alpha)+(1-2\alpha)k|}{k+1}, & \text{if } \alpha \in [0,1) \setminus \{1/2\}, \\ \frac{1}{(n-1)!}, & \text{if } \alpha = 1/2. \end{cases}$$

This inequality is sharp for the function k *given by* (2.4).

Proof Supposing that $f \in \mathcal{S}_r^*(\alpha)$, it follows that

$$\frac{f(z)}{zf'(z)} \prec \frac{1+(1-2\alpha)z}{1-z},$$

and, according to the definition of subordination, the above is equivalent to

$$\frac{zf'(z)}{f(z)} \prec \frac{1-z}{1+(1-2\alpha)z}.$$

If we take in the definition (1.6) the values $p := 1$, $A := -1$, $B := 1-2\alpha$, and $\beta := 0$, from this relation, using Lemma 1.3 for the above values, it follows that $f \in \mathcal{S}_r^*(\alpha)$, $\alpha \in [0,1)$, and hence

$$|a_n| \leq \prod_{k=0}^{n-2} \frac{|2(1-\alpha)+(1-2\alpha)k|}{k+1}, \quad n \geq 2.$$

In the third part of the proof of Theorem 2.1, we showed that the function $\mathrm{k} \in \mathcal{S}_r^*(\alpha)$. For $\alpha \in [0,1) \setminus \{1/2\}$, the sharpness of the result follows from the extremal function given in [7, p. 741] or could be proved directly, while for $\alpha = 1/2$ the sharpness is obvious. □

3 Conclusion

In the current paper, due to the major importance of the study of the logarithmic coefficients γ_n of the function $f \in \mathcal{S}$, we obtained sharp upper bounds for some expressions associated with the logarithmic coefficients γ_n of the functions that belong to the well-known classes like $\mathcal{S}_r^*(\alpha)$, $\mathcal{S}_{4\ell}^*$, and $\mathcal{S}_\ell^*$, and an upper bound for the functions in the class $\mathcal{S}_r^*(\alpha)$. All results are the best possible (sharp, i.e., cannot be improved), while the second one extended an earlier estimate obtained by the authors.

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The authors declare no competing interests.

Author contributions

E.A.A. made the original draft preparation. M.J. contributed analysis of the mathematical computation. T.B. and N.E.C. reviewed and edited the manuscript. All authors reviewed the manuscript. All authors read and approved the final manuscript.

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