Solving Eq. (3.9) for $q$, with $h = c_p T$,

$$q = \left(c_p T_2 + \frac{u_2^2}{2}\right) - \left(c_p T_1 + \frac{u_1^2}{2}\right) \tag{3.76}$$

From the definition of total temperature, Eq. (3.27), the terms on the right-hand side of Eq. (3.76) simply result in

$$q = c_p T_{o_2} - c_p T_{o_1} = c_p(T_{o_2} - T_{o_1}) \tag{3.77}$$

Equation (3.77) clearly indicates that *the effect of heat addition is to directly change the total temperature of the flow*. If heat is added, $T_o$ increases; if heat is extracted, $T_o$ decreases.

Let us proceed to find the ratios of properties between regions 1 and 2 in terms of the Mach numbers $M_1$ and $M_2$. From Eq. (3.5), and noting that

$$\rho u^2 = \rho a^2 M^2 = \rho \frac{\gamma p}{\rho} M^2 = \gamma p M^2$$

we obtain

$$p_2 - p_1 = \rho_1 u_1^2 - \rho_2 u_2^2 = \gamma p_1 M_1^2 - \gamma p_2 M_2^2$$

Hence,

$$\frac{p_2}{p_1} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2} \tag{3.78}$$

Also, from the perfect gas equation of state and Eq. (3.2),

$$\frac{T_2}{T_1} = \frac{p_2}{p_1}\frac{\rho_1}{\rho_2} = \frac{p_2}{p_1}\frac{u_2}{u_1} \tag{3.79}$$

From Eq. (3.20) and the definition of Mach number,

$$\frac{u_2}{u_1} = \frac{M_2}{M_1}\frac{a_2}{a_1} = \frac{M_2}{M_1}\left(\frac{T_2}{T_1}\right)^{1/2} \tag{3.80}$$

Substituting Eqs. (3.78) and (3.80) into (3.79),

$$\frac{T_2}{T_1} = \left(\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2 \tag{3.81}$$

Since $\rho_2/\rho_1 = (p_2/p_1)(T_1/T_2)$, Eqs. (3.78) and (3.81) yield

$$\frac{\rho_2}{\rho_1} = \left(\frac{1 + \gamma M_2^2}{1 + \gamma M_1^2}\right)\left(\frac{M_1}{M_2}\right)^2 \tag{3.82}$$



The ratio of total pressures is obtained directly from Eqs. (3.30) and (3.78),

$$\frac{p_{o_2}}{p_{o_1}} = \frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}\left(\frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_1^2}\right)^{\gamma/(\gamma - 1)} \tag{3.83}$$

The ratio of total temperatures is obtained directly from Eqs. (3.28) and (3.81),

$$\frac{T_{o_2}}{T_{o_1}} = \left(\frac{1 + \gamma M_1^2}{1 + \gamma M_2^2}\right)^2 \left(\frac{M_2}{M_1}\right)^2 \left(\frac{1 + \frac{\gamma - 1}{2} M_2^2}{1 + \frac{\gamma - 1}{2} M_1^2}\right) \tag{3.84}$$

Finally, the entropy change can be found from Eq. (1.36) with $T_2/T_1$ and $p_2/p_1$ given by Eqs. (3.81) and (3.78), respectively.

For convenience of calculation, we use sonic flow as a reference condition. Let $M_1 = 1$; the corresponding flow properties are denoted by $p_1 = p^*$, $T_1 = T^*$, $\rho_1 = \rho^*$, $p_{o_1} = p_o^*$, and $T_{o_1} = T_o^*$. The flow properties at any other value of $M$ are then obtained by inserting $M_1 = 1$ and $M_2 = M$ into Eq. (3.78) and Eqs. (3.81) to (3.84), yielding

$$\frac{p}{p^*} = \frac{1 + \gamma}{1 + \gamma M^2} \tag{3.85}$$

$$\frac{T}{T^*} = M^2 \left(\frac{1 + \gamma}{1 + \gamma M^2}\right)^2 \tag{3.86}$$

$$\frac{\rho}{\rho^*} = \frac{1}{M^2}\left(\frac{1 + \gamma M^2}{1 + \gamma}\right) \tag{3.87}$$

$$\frac{p_o}{p_o^*} = \frac{1 + \gamma}{1 + \gamma M^2}\left[\frac{2 + (\gamma - 1)M^2}{\gamma + 1}\right]^{\gamma/(\gamma - 1)} \tag{3.88}$$

$$\frac{T_o}{T_o^*} = \frac{(\gamma + 1)M^2}{(1 + \gamma M^2)^2}[2 + (\gamma - 1)M^2] \tag{3.89}$$

Equations (3.85) through (3.89) are tabulated as a function of $M$ for $\gamma = 1.4$ in Table A.3. Note that, for a given flow, no matter what the local flow properties are, the reference sonic conditions (the starred quantities) are constant values.

**Example 3.3** Air enters a constant area duct at $M_1 = 0.2$, $p_1 = 1$ atm, and $T_1 = 273$ K. Inside the duct, the heat added per unit mass is $q = 1.0 \times 10^6$ J/kg. Calculate the flow properties $M_2$, $p_2$, $T_2$, $\rho_2$, $T_{o_2}$, and $p_{o_2}$ at the exit of the duct.

SOLUTION From Table A.1: For $M_1 = 0.2$: $T_{o_1}/T_1 = 1.008$ and $p_{o_1}/p_1 = 1.028$. Hence

$$T_{o_1} = 1.008T_1 = 1.008(273) = 275.2 \text{ K}$$

$$p_{o_1} = 1.028p_1 = 1.028(1 \text{ atm}) = 1.028 \text{ atm}$$

$$c_p = \frac{\gamma R}{\gamma - 1} = \frac{(1.4)(287)}{0.4} = 1005 \text{ J/kg}\cdot\text{K}$$

From Eq. (3.77):

$$T_{o_2} = \frac{q}{c_p} + T_{o_1} = \frac{1.0 \times 10^6}{1005} + 275.2 = \boxed{1270 \text{ K}}$$

From Table A.3: For $M_1 = 0.2$: $T_1/T^* = 0.2066$, $p_1/p^* = 2.273$, $p_{o_1}/p_o^* = 1.235$, and $T_{o_1}/T_o^* = 0.1736$. Hence

$$\frac{T_{o_2}}{T_o^*} = \frac{T_{o_2}}{T_{o_1}}\frac{T_{o_1}}{T_o^*} = \frac{1270}{275.2}(0.1736) = 0.8013$$

From Table A.3, this corresponds to $\boxed{M_2 = 0.58}$.

Also from Table A3: For $M_2 = 0.58$: $T_2/T^* = 0.8955$, $p_2/p^* = 1.632$, $p_{o_2}/p_o^* = 1.083$. Hence

$$T_2 = \frac{T_2}{T^*}\frac{T^*}{T_1}T_1 = (0.8955)\left(\frac{1}{0.2066}\right)(273) = \boxed{1183 \text{ K}}$$

$$p_2 = \frac{p_2}{p^*}\frac{p^*}{p_1}p_1 = 1.632\frac{1}{2.273}1 \text{ atm} = \boxed{0.718 \text{ atm}}$$

$$p_{o_2} = \frac{p_{o_2}}{p_o^*}\frac{p_o^*}{p_{o_1}}p_{o_1} = 1.083\frac{1}{1.235}1.028 = \boxed{0.902 \text{ atm}}$$

Since 1 atm $= 1.01 \times 10^5$ N/m$^2$,

$$\rho_2 = \frac{p_2}{RT_2} = \frac{(0.718)(1.01 \times 10^5)}{(278)(1183)} = \boxed{0.214 \text{ kg/m}^3}$$

Certain physical trends reflected by the numbers obtained from such solutions are important, and are summarized below:

1. For *supersonic flow* in region 1, i.e., $M_1 > 1$, when heat is added
   *a.* Mach number decreases, $M_2 < M_1$
   *b.* Pressure increases, $p_2 > p_1$
   *c.* Temperature increases, $T_2 > T_1$
   *d.* Total temperature increases, $T_{o_2} > T_{o_1}$
   *e.* Total pressure decreases, $p_{o_2} < p_{o_1}$
   *f.* Velocity decreases, $u_2 < u_1$



2. For *subsonic* flow in region 1, i.e., $M_1 < 1$, when heat is added
   *a.* Mach number increases, $M_2 > M_1$
   *b.* Pressure decreases, $p_2 < p_1$
   *c.* Temperature increases for $M_1 < \gamma^{-1/2}$ and decreases for $M_1 > \gamma^{-1/2}$
   *d.* Total temperature increases, $T_{o_2} > T_{o_1}$
   *e.* Total pressure decreases, $p_{o_2} < p_{o_1}$
   *f.* Velocity increases, $u_2 > u_1$

For heat extraction (cooling of the flow), all of the above trends are opposite.

From the above, it is important to note that heat addition always drives the Mach numbers towards 1, decelerating a supersonic flow and accelerating a subsonic flow. This is emphasized in Fig. 3.11, which is a Mollier diagram (enthalpy versus entropy) of the one-dimensional heat-addition process. The curve in Fig. 3.11 is called the *Rayleigh curve*, and is drawn for a set of given initial conditions. If the conditions in region 1 are given by point 1 in Fig. 3.11, then the particular Rayleigh curve through point 1 is the locus of all possible states in region 2. Each point on the curve corresponds to a different value of $q$ added or taken away. Point $a$ corresponds to maximum entropy; also at point $a$ the flow is sonic. The lower branch of the Rayleigh curve below point $a$ corresponds to supersonic flow; the upper branch above point $a$ corresponds to

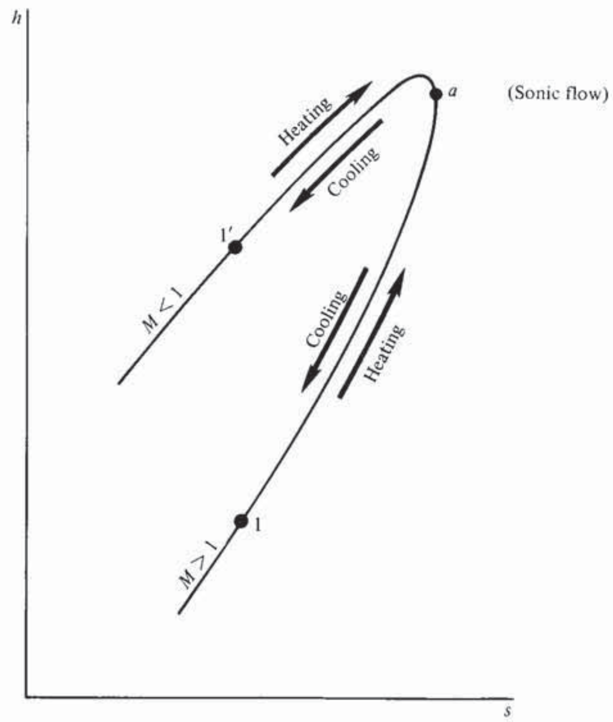


**Figure 3.11** The Rayleigh curve.