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LYAPUNOV DESIGN

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Keywords: Control Lyapunov Function, Lyapunov Design, Model Reference Adaptive Control, Adaptive Control, Backstepping Design.

Contents

Glossary

Lipschitz continuity: The function $\mathbf{f}(\mathbf{x}, t)$ is said to be Lipschitz continuous in $\mathbf{x}$, if for some $h > 0$, there exists $L \geq 0$ such that $|\mathbf{f}(\mathbf{x}_1, t) - \mathbf{f}(\mathbf{x}_2, t)| \leq L|\mathbf{x}_1 - \mathbf{x}_2|$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{B}(0, t)$, $t > 0$ where $\mathbf{B}(0, h)$ is the ball of radius h centered at 0. The constant L is called the Lipschitz constant.

Equilibrium point: $\mathbf{x}^*$ is said to be an equilibrium point of system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \mathbf{x}(t_0) = \mathbf{x}_0$ if $\mathbf{f}(\mathbf{x}^*, t) = \mathbf{0}$ for all $t \geq 0$.

Stable equilibrium: An equilibrium point where all solutions starting at nearby points stay nearby. The equilibrium state $\mathbf{x} = \mathbf{0}$ is said to be stable if, for any $\epsilon > 0$ and $t_0 \geq 0$, there exists a $\delta(\epsilon, t_0)$ such that if $|\mathbf{x}(t_0)| < \delta(\epsilon, t_0)$, then $|\mathbf{x}(t)| < \epsilon$ for all $t > t_0$.

Uniformly stable: The equilibrium point $\mathbf{x} = \mathbf{0}$ is uniformly stable if δ is the definition of stable equilibrium can be chosen independent of t_0.

Region of attraction: The set of all points with the property that the trajectories starting at these points asymptotically converge to the origin.

Asymptotic stability: The equilibrium point $\mathbf{x} = \mathbf{0}$ is an asymptotically stable equilibrium point of system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ if (i) $\mathbf{x} = \mathbf{0}$ is a stable equilibrium point; and (ii) $\mathbf{x} = \mathbf{0}$ is attractive, that is for all $t_0 \geq 0$ there exists $\delta(t_0)$ such that $|\mathbf{x}_0| < \delta \rightarrow \lim_{t\rightarrow\infty} |\mathbf{x}(t)| = 0$, i.e., all trajectories starting at nearby points approach the equilibrium point as time goes to infinity.

Hurwitz matrix: A square real matrix is Hurwitz if all its eigenvalues have negative real parts.

Positive (semi-) definite function: A scalar function of a vector argument $V(\mathbf{x})$ is positive (semi-) definite if it vanishes at the origin and is positive (nonnegative) for all points in the neighborhood of the origin, excluding the origin.

Positive (semi-) definite matrix: A square symmetric real matrix $\mathbf{P}$ is positive (semi-) definite if the quadratic form $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ is a positive (semi-) definite function.

Negative (semi-) definite function: A scalar function of a vector argument $V(\mathbf{x})$ is negative (semi-) definite if it vanishes at the origin and is negative (nonpositive) for all points in the neighborhood of the origin, excluding the origin.

Negative (semi-) definite matrix: A square symmetric real matrix $\mathbf{P}$ is negative (semi-) definite if the quadratic form $V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x}$ is a negative (semi-) definite function.

Lyapunov equation: A linear algebraic matrix equation of the form $\mathbf{PA}+\mathbf{A}^T\mathbf{P} = -\mathbf{Q}$, where $\mathbf{A}$ and $\mathbf{Q}$ are real square matrices and $\mathbf{Q}$ is symmetric and positive definite. The equation has a (unique) positive definite matrix solution $\mathbf{P}$ if and only if $\mathbf{A}$ is Hurwitz.

Lyapunov function (LF): A scalar positive definite function $V(\mathbf{x})$ of the states whose derivative along the trajectories of the system is negative definite.

Control Lyapunov Function: A scalar smooth positive definite and radially unbounded function $V(\mathbf{x}) : R^n \rightarrow R_+$ is called a Control Lyapunov Function (CLF) for control system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$ there exists $\mathbf{u}(\mathbf{x})$ such that $\frac{\partial V(\mathbf{x})}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) < 0 \ \forall \mathbf{x} \neq \mathbf{0}$.

Invariant Set: : A set Ω is an invariant set for a dynamic system if every system trajectory which starts from a point in Ω remains in Ω for all future time.

$\mathbf{L}_2^n$**:** Let $\mathbf{x}(t) \in R^n$ be a continuous or piecewise continuous vector function. Then the $2-$ norm space is defined as $\mathbf{L}_2^n =: \{\mathbf{x} : |\|\mathbf{x}\|_2 = \int_0^\infty \mathbf{x}^T\mathbf{x}dt < \infty\}$.

$\mathbf{L}_\infty^n$**:** Let $\mathbf{x}(t) \in R^n$ be a continuous or piecewise continuous vector function. Then the $\infty-$ norm space is defined as $\mathbf{L}_\infty^n =: \{\mathbf{x} : |\|\mathbf{x}\|_\infty = \max_{1\leq i\leq n} |x_i| < \infty\}$.

diag$[*]$**:** Diagonal matrix with given diagonal elements.

inf $\alpha(t)$**:** The largest number that is smaller than or equal to the minimum value of $\alpha(t)$.

$\lambda_{max}(\mathbf{A})$**:** The maximum eigenvalue of $\mathbf{A}$, where $\lambda_{max}(\mathbf{A})$ is real.

Summary

This article gives an overview on some state-of-the-art approaches of Lyapunov design by dividing systems into several distinct classes, though in general there is no systematic procedure in choosing a suitable Lyapunov function candidate for controller design to guarantee the closed-loop stability for a given nonlinear system. After a brief introduction and historic review, this article sequentially presents (i) the basic concepts of Lyapunov stability and control Lyapunov functions, (ii) Lyapunov equations and model reference adaptive control based on Lyapunov

design for matched systems, (iii) Lyapunov redesign, adaptive redesign and robust design for matched systems, (iv) adaptive backstepping design for unmatched nonlinear systems, (v) Lyapunov design by exploiting physical properties for special classes of systems, and (vi) design flexibilities and considerations in actual design.

1. Introduction

Lyapunov design has been a primary tool for nonlinear control system design, stability and performance analysis since its introduction in 1982. The basic idea is to design a feedback control law that renders the derivative of a specified Lyapunov function candidate negative definite or negative semi-definite. Lyapunov's direct method is a mathematical interpretation of the physical property that if a system's total energy is dissipating, then the states of the system will ultimately reach to an equilibrium point. The basic idea behind the method is that, if there exists a kind of continuous scalar "energy" function such that this "energy" diminishes along the system's trajectory, then the system is said to be asymptotically stable. Since there is no need to solve the solution of the differential equations governing the system in determining its stability, it is usually referred to as the direct method (see *Lyapunov Stability*).

Although Lyapunov direct method is efficient for stability analysis, it is of restricted applicability due to the difficulty in selecting a Lyapunov function. The situation is different when facing the controller design problem, where the control has not being specified, and the system under consideration is undetermined. Lyapunov functions have been effectively utilized in the synthesis of control systems. The basic idea is that, by first choosing a Lyapunov function candidate and then the feedback control law can be specified such that it renders the derivative of the specified Lyapunov function candidate negative definite, or negative semi-definite when invariance principle can be used to prove asymptotic stability. This way of designing control is called Lyapunov design. Lyapunov design depends on the selection of Lyapunov function candidates. Though the result is sufficient, it is a difficult problem to find a Lyapunov function (LF) satisfying the requirements of Lyapunov design. Fortunately, during the past several decades, many effective control design approaches have been developed for different classes of linear and nonlinear systems based on the basic ideas of Lyapunov design. Lyapunov functions are additive, like energy, i.e., Lyapunov functions for combinations of subsystems may be derived by adding the Lyapunov functions of the subsystems. This point can be seen clearly in the adaptive control design and backstepping design in this article.

Though Lyapunov design is a very powerful tools for control system design, stability and performance analysis, the construction of a Lyapunov function is not easy for general nonlinear systems, and it is usually a trial-and-error process and there is a lack of systematic methods. Different choices of Lyapunov functions may result in different control structures and control performance. Past experience shows that a good design of Lyapunov function should fully utilize the property of the studied systems. Lyapunov design is used in many contexts, such as dynamic feedback, output-feedback, estimation of region of attraction, and adaptive control, among others. The article is not meant to be comprehensive but to serve as an introduction to the state-of-the-art of full-state feedback design based on Lyapunov techniques for several typical classes of autonomous systems.

Intensive research in adaptive control was first motivated for the design of autopilots for high performance aircraft in the early 1950s because their dynamics change drastically when they fly from one operating point to another and constant gain feedback control cannot handle it effectively. The lack of stability theory and one disastrous flight test led to the diminishing interest in adaptive control in the late 1950s. The 1960s saw many advances in control theory and adaptive control in particular. Simultaneous development in computers and electronics that made the implementation of complex controllers possible, interest in adaptive control and its applications was renewed in the 1970s and several breakthrough results were made. The uncover of nonrobust behaviour of adaptive control subject to small disturbance and unmodelled dynamics in 1979 and the earlier 1980s, it led to better understanding of the instability mechanisms and the design of robust adaptive control in the later 1980s though it was very controversial initially. They were all systems satisfying the *matching condition*. In the later 1980s and earlier 1990s, the matching condition was relaxed to the *extended matching condition*, which for one period was regarded as the frontier that could not be crossed by Lyapunov design, and then further relaxed to the *strict-feedback systems* with general unmatched uncertainties through backstepping design, which is the state-of-the-art of adaptive control.

This article gives an overview of the state-of-the-art approaches of Lyapunov design and ways of choosing Lyapunov functions in the area of full-state adaptive control. Section 2 presents the concepts of Lyapunov stability analysis and control Lyapunov functions. In Section 3, Lyapunov functions for linear time invariant systems are presented first, then the results are utilized to solve Model Reference Adaptive Control (MRAC) problems for classes of linear and nonlinear systems which can be transformed to systems having stable linear portion. For this class of problems, the choice of Lyapunov functions is systematic and controller design is standard. In Section 4, after the presentation of Lyapunov Redesign, Adaptive Lyapunov Redesign, Robust Lyapunov Redesign for a class of matched systems, backstepping controller design is discussed for unmatched nonlinear systems. By exploiting the physical properties of the systems under study, Section 5 shows that different choices of Lyapunov functions and better controllers are possible. Section 6 discusses the design flexibilities and considerations in actual applications of Lyapunov design, and further research work.

2. Control Lyapunov Function

Though Lyapunov's method applies to nonautonomous systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, for clarity and simplicity, we shall restrain our discussion to time-invariant nonlinear systems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{1}$$

where $\mathbf{x} \in R^n$, and $\mathbf{f}(\mathbf{x}) : R^n \to R^n$ is continuous. The basic idea of Lyapunov direct method consists of (ii) choosing a radially unbounded positive definite Lyapunov function candidate $V(\mathbf{x})$, and (ii) evaluating its derivative $\dot{V}(\mathbf{x})$ along system dynamics (1) and checking its negativeness for stability analysis.

Lyapunov design refers to the synthesis of control laws for some desired closed-loop stability properties using Lyapunov functions for nonlinear control systems

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) \tag{2}$$

where $\mathbf{x} \in R^n$ is the state, $\mathbf{u} \in R^m$ is the control input, and $\mathbf{f}(\mathbf{x}, \mathbf{u})$ is locally Lipschitz on $(\mathbf{x}, \mathbf{u})$, and $\mathbf{f}(\mathbf{0}, \mathbf{0}) = \mathbf{0}$.

The usefulness of Lyapunov direct method for feedback control design $\mathbf{u}(\mathbf{x})$ can be seen as follows: Substituting $\mathbf{u} = \mathbf{u}(\mathbf{x})$ into (2), we have the autonomous closed-loop dynamics $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))$ and Lyapunov direct method can then be used for stability analysis.

In actually applications, Lyapunov design can be conceptually divided into two steps:

(a) choose a candidate Lyapunov function V for the system, and

(b) design a controller which renders its derivative $\dot{V}$ negative.

Sometimes, it may be more advantageous to reverse the order of operation, i.e., design a controller that is most likely to be able to stabilize the closed-loop system first by examining the properties of the system, and then choose a Lyapunov function candidate V for the closed-loop system to show that it is indeed a Lyapunov function. Lyapunov design is sufficient. Stabilizing controllers are obtained if the processes succeed. If the attempts fail, no conclusion can be drawn on the existence of a stabilizing controller.

Let function $V(\mathbf{x})$ be a Lyapunov function candidate. Thus the task is to search for $\mathbf{u}(\mathbf{x})$ to guarantee that, for all $\mathbf{x} \in R^n$, the time derivative of $V(\mathbf{x})$ along system (2) satisfy

$$\dot{V}(\mathbf{x}) = \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x})) \leq -W(\mathbf{x}) \tag{3}$$

where $W(\mathbf{x})$ is a positive definite function. For affine nonlinear system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}, \quad \mathbf{f}(\mathbf{0}) = \mathbf{0} \tag{4}$$

the inequality (3) becomes

$$\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \frac{\partial V}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x}) \mathbf{u}(\mathbf{x}) \leq -W(\mathbf{x}) \tag{5}$$

In general, this is a difficult task. A system for which a good choice of $V(\mathbf{x})$ and $W(\mathbf{x})$ exists is said to possess a control Lyapunov function. A smooth positive definite and radially unbounded function $V(\mathbf{x}) : R^n \to R_+$ is called a Control Lyapunov Function (CLF) for (2) if

$$\inf_{\mathbf{u} \in R^m} \{ \frac{\partial V(\mathbf{x})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}) \} < 0, \forall \mathbf{x} \neq \mathbf{0} \tag{6}$$

If $V(\mathbf{x})$ is a CLF for affine nonlinear system (4), then a particular stabilizing control law, $\mathbf{u}(\mathbf{x})$, smooth for all $\mathbf{x} \neq \mathbf{0}$, is given by the Artstein and Sontag's universal controller

$$\mathbf{u}(\mathbf{x}) = \begin{cases} -\dfrac{\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}) + \sqrt{(\frac{\partial V}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}))^2 + (\frac{\partial V}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x}))^4}}{\frac{\partial V}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x})}, & \dfrac{\partial V}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x}) \neq 0 \\ 0, & \dfrac{\partial V}{\partial \mathbf{x}} \mathbf{g}(\mathbf{x}) = 0 \end{cases}$$

The steps of Lyapunov design and concept of "Control Lyapunov Function" are used for systems with controls to differentiate the classical term "Lyapunov function" for systems without controls. As a design tool for general nonlinear systems, the main deficiency of the CLF concept is that a CLF is unknown. The task of finding an appropriate CLF may be as complex as that of designing a stabilizing feedback law. However, for several important classes of nonlinear systems, these two tasks can be solved simultaneously.

When $\dot{V}(\mathbf{x})$ is only negative semidefinite, asymptotic stability cannot be concluded from Lyapunov function method directly. However, if $\mathbf{x} = \mathbf{0}$ is shown to be the only solution for $\dot{V}(\mathbf{x}) = 0$, then asymptotic stability can still be drawn by evoking LaSalle's Invariance Principle, Invariant Set Theorem, which basically states that, if $\dot{V}(\mathbf{x}) \leq 0$ of a chosen Lyapunov function candidate $V(\mathbf{x})$, then all solutions asymptotically converge to the largest invariant set in the set $\{\mathbf{x} \mid \dot{V}(\mathbf{x}) = 0\}$ as $t \to \infty$. In fact, this approach has been frequently used in the proof of asymptotic stability of a closed-loop system.

Lemma 2.1 [**Barbalat**] *Consider the function* $\phi(t) : R_+ \to R$. *If* $\phi(t)$ *is uniformly continuous and* $\lim_{t\to\infty} \int_0^t \phi(\tau)d\tau$ *exists and is finite, then* $\lim_{t\to\infty} \phi(t) = 0$.

Theorem 2.1 [**LaSalle**] *Let (a)* Ω *be a positively invariant set of* $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, *(b)* $V(\mathbf{x}) : \Omega \to R_+$ *be a continuously differentiable function such that* $\dot{V}(\mathbf{x}) \leq 0, \forall \mathbf{x} \in \Omega$, *and (c)* $E = \{\mathbf{x} \in \Omega | \dot{V}(\mathbf{x}) = 0\}$, *and* M *be the largest invariant set contained in* E. *Then, every bounded solution* $\mathbf{x}(t)$ *starting in* Ω *converges to* M *as* $t \to \infty$.

To show that one variable is indeed converges to zero, Barbalat's Lemma is frequently used. If Ω is the whole space R^n, then the above local Invariant Set Theorem becomes the global one. To prove the asymptotic stability of the system, we only need to show that no solution other than $\mathbf{x}(t) \equiv \mathbf{0}$ can stay forever in E.

It should be noted that there may exist many Lyapunov functions for a given nonlinear system. Specific choices of Lyapunov functions may yield better, cleaner controllers than others. Usually, Lyapunov functions are chosen as quadratic form due to its elegancy of mathematical treatment. However, it is not exclusive. Other forms have also been used in the literature, such as energy-based Lyapunov functions, integral-type Lyapunov functions, which have been applied in the design of controllers for classes of uncertain nonlinear systems.

3. Lyapunov Design via Lyapunov Equation

Model Reference Adaptive Control (MRAC) was originally proposed to solve the problem in which the design specifications are given by a reference model, and the parameters of the controller are adjusted by an adaptation mechanism/law such that the closed-loop dynamics of the system are the same as the reference model which gives the desired response to a command signal. In solving this class of problems, the Lyapunov equation plays a very important role in choosing the Lyapunov function and deriving the feedback control and adaptation mechanism. In fact, the construction of Lyapunov functions is systematic and straightforward for the class of systems which can be transformed into two portions: (i) a stable linear portion so that linear stability results can be directly applied, and (ii) matched nonlinear portion which can be handled using different techniques such as adaptive or robust control techniques in different situations. Thus, MRAC can also be viewed as Lyapunov design based on Lyapunov equations. To explain the concepts clearly, Lyapunov equation and Lyapunov stability analysis are firstly presented for linear time-invariant systems, then adaptive control design for classes of unknown linear time invariant systems and unknown nonlinear systems are presented by utilizing Lyapunov equation.

3.1. Lyapunov Equation

Though linear systems are well understood, it is interesting to look at them in the Lyapunov language, and provide a basis of Lyapunov design for systems having linear portions. For simplicity, consider the following simple controllable Linear Time Invariant (LTI) systems described by

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{7}$$

where $\mathbf{x} \in R^n$, and $u \in R$ are the states, and control variable, respectively, $\mathbf{A} \in R^{n\times n}$ and $\mathbf{b} \in R^n$. It is well known that there is always a global quadratic LF, and the stabilizing controller can be obtained constructively. Let the state feedback control be

$$u = -\mathbf{k}\mathbf{x} \tag{8}$$

the resulting closed-loop system will be of the form

$$\dot{\mathbf{x}} = \mathbf{A}_m\mathbf{x}, \quad \mathbf{A}_m = \mathbf{A} - \mathbf{b}\mathbf{k} \tag{9}$$

From the linear system theory, there are many ways to design $\mathbf{k}$ for a desirable stable closed-loop system. The most intuitive and direct one might be the pole-placement method. In the context of this article, we shall look at the problem in the sense of Lyapunov design. Not surprisingly, Lyapunov functions can be systematically found to describe stable linear systems owing to the following theorem.

Theorem 3.1 *The LTI system $\dot{\mathbf{x}} = \mathbf{A}_m\mathbf{x}$ is asymptotically stable if and only if, given any symmetric positive-definite matrix $\mathbf{Q}$, there exists a symmetric positive-definite matrix $\mathbf{P}$, which is the unique solution of the so-called Lyapunov equation*

$$\mathbf{P}\mathbf{A}_m + \mathbf{A}_m^T\mathbf{P} = -\mathbf{Q} \tag{10}$$

For such a solution, the positive definite quadratic function of the form

$$V(\mathbf{x}) = \mathbf{x}^T\mathbf{P}\mathbf{x} \tag{11}$$

is a LF for the closed-loop system (9), since

$$\dot{V}(\mathbf{x}) = -\mathbf{x}^T\mathbf{Q}\mathbf{x} < 0, \quad \forall \mathbf{x} \neq \mathbf{0} \tag{12}$$

Another method to design $\mathbf{k}$ is the well known optimal linear quadratic (LQ) design method. To investigate the problem in the context of CLF, consider the following Lyapunov function candidate for (7)

$$V = \mathbf{x}^T \mathbf{P} \mathbf{x} \tag{13}$$

where $\mathbf{P} = \mathbf{P}^T > 0$. For $\mathbf{P}$ to define a CLF (6), the following inequality should hold

$$\inf_{u \in R} \{\mathbf{x}^T \mathbf{A}^T \mathbf{P} \mathbf{x} + \mathbf{x}^T \mathbf{P} \mathbf{A} \mathbf{x} + u^T \mathbf{b}^T \mathbf{P} \mathbf{x} + \mathbf{x}^T \mathbf{P} \mathbf{b} u\} < 0 \quad \forall \mathbf{x} \neq \mathbf{0} \tag{14}$$

which implies that $u(\mathbf{x})$ should take the form that $u(\mathbf{x}) = -\gamma \mathbf{b}^T \mathbf{P} \mathbf{x}$ with $\gamma > 0$ (the corresponding linear feedback gain $\mathbf{k} = \gamma \mathbf{b}^T \mathbf{P}$). Thus, $\mathbf{P}$ defines a CLF if

$$\inf_{\gamma \in R} \{\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} - 2\gamma \mathbf{P} \mathbf{b} \mathbf{b}^T \mathbf{P}\} < 0 \tag{15}$$

Such a $\mathbf{P}$ can always be found through the solution of algebraic Riccati equation

$$\mathbf{A}^T \mathbf{P} + \mathbf{P} \mathbf{A} + \mathbf{Q} - 2\gamma \mathbf{P} \mathbf{b} \mathbf{b}^T \mathbf{P} = \mathbf{0} \tag{16}$$

which is known to solve the optimal linear quadratic (LQ) state design which minimizes the cost function $J = \int_0^\infty [\mathbf{x}^T \mathbf{Q} \mathbf{x} + \frac{1}{2\gamma} u^2] dt$ and subject to the dynamic constraints imposed by (7) (see *Optimal Linear Quadratic Control (LQ)*). Equation (16) guarantees that $\dot{V} = -\mathbf{x}^T \mathbf{Q} \mathbf{x}$, and in turn asymptotic stability of the closed-loop systems.

The results are readily available for multi-input-multi-output (MIMO) systems. Techniques in dealing with linear systems in state space are well established. (see *Classical Design Methods for Continuous LTI-Systems, Design of State Space Controllers (Pole Placement) for SISO Systems, Pole Placement Control, Optimal Linear Quadratic Control (LQ)*).

3.2. MRAC for Linear Time Invariant Systems

To illustrate the basic steps in solving MRAC for linear time invariant systems, consider the following LTI plant described by the state-space model

$$\dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + g \mathbf{b} u \tag{17}$$

where $\mathbf{x} \in R^n$; $u \in R$ are the states and input respectively, $\mathbf{A} \in R^{n \times n}$ and $\mathbf{b} \in R^n$ are in the controller canonical form as

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_1 & -a_2 & \cdots & -a_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{18}$$

with unknown constants $a_i, i = 1, \cdots, n$, and control input gain $g > 0$ is an unknown constant. The objective is to drive $\mathbf{x}$ to follow some desired reference trajectory $\mathbf{x}_m \in R^n$ and guarantee closed-loop stability. Let the reference trajectory $\mathbf{x}_m$ be generated from a reference model specified by the LTI system

$$\dot{\mathbf{x}}_m = \mathbf{A}_m \mathbf{x}_m + g_m \mathbf{b} r \tag{19}$$

where $r \in R$ is a bounded reference input, $\mathbf{A}_m \in R^{n \times n}$ is a stable matrix given by

$$\mathbf{A}_m = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -a_{m1} & -a_{m2} & \cdots & -a_{mn} \end{bmatrix} \tag{20}$$

with $a_{mi}, i = 1, \cdots, n$, chosen such that $s^n + a_{mn} s^{n-1} + \cdots + a_{m1}$ is a Hurwitz polynomial. The reference model and input r are chosen such that $\mathbf{x}_m(t)$ represents a desired trajectory that $\mathbf{x}$ has to follow, i.e., $\mathbf{x} \to \mathbf{x}_m$ as $t \to \infty$.

Consider a general linear control law of the form

$$u = \mathbf{k}(t)\mathbf{x} + k_r(t) r \tag{21}$$

where $\mathbf{k}$ and k_r may be chosen freely. The closed-loop system then becomes

$$\dot{\mathbf{x}} = (\mathbf{A} + g\mathbf{b}\mathbf{k})\mathbf{x} + g k_r \mathbf{b} r \tag{22}$$

It is clear there exist constant parameters $\mathbf{k}^*$ and k_r^* such that the *matching conditions*

$$a_i + g k_i^* = a_{mi}, \qquad g k_r^* = g_m \tag{23}$$

hold, i.e., equations (19) and (22) are equivalent. Since a_i and g are unknown, so are $\mathbf{k}^*$ and k_r^*, which means that controller (21) with $\mathbf{k} = \mathbf{k}^*$ and $k_r = k_r^*$ is not feasible. This problem can be easily solved using on-line adaptive control techniques.

For ease of discussion, let $\boldsymbol{\theta} = [\mathbf{k}^*, k_r^*]^T$, $\hat{\boldsymbol{\theta}} = [\mathbf{k}(t), k_r(t)]^T$, define parameter estimation errors $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}} = [\tilde{\boldsymbol{\theta}}_{\mathbf{x}}^T, \tilde{\theta}_r]^T$ with $\tilde{\boldsymbol{\theta}}_x = \mathbf{k}^* - \mathbf{k}(t)$, $\tilde{\theta}_r = k_r^* - k_r(t)$ and denote $\boldsymbol{\phi} = [\mathbf{x}^T, r]^T$. Accordingly, equation (22) can be written as

$$\begin{aligned} \dot{\mathbf{x}} &= (\mathbf{A} + g\mathbf{b}\mathbf{k}^*)\mathbf{x} + g\mathbf{b}k_r^* r - g\mathbf{b}\tilde{\boldsymbol{\theta}}_x \mathbf{x} - g\mathbf{b}\tilde{\theta}_r r \\ &= \mathbf{A}_m \mathbf{x} + g_m \mathbf{b} r - g\mathbf{b}\boldsymbol{\phi}^T \tilde{\boldsymbol{\theta}} \end{aligned} \tag{24}$$

Define the tracking error $\mathbf{e} = \mathbf{x} - \mathbf{x}_m$. Comparing equations (19) and (24) give the closed-loop error equation

$$\dot{\mathbf{e}} = \mathbf{A}_m \mathbf{e} - g\mathbf{b}\boldsymbol{\phi}^T \tilde{\boldsymbol{\theta}} \tag{25}$$

which has a stable linear portion and an unknown parametric uncertainty input, which turns out to be easily solvable using the facts that (i) given any stable known matrix $\mathbf{A}_m$, for any symmetric positive-definite matrix $\mathbf{Q}$, there exists a unique symmetric positive-definite matrix $\mathbf{P}$ satisfying

$$\mathbf{P}\mathbf{A}_m + \mathbf{A}_m^T \mathbf{P} = -\mathbf{Q} \tag{26}$$

as detailed in Subsection 3.1, and (ii) the linear-in-the-parameter uncertainty $\mathbf{b}\boldsymbol{\phi}^T\tilde{\boldsymbol{\theta}}$ can be dealt with using adaptive techniques. Owing to the above observations, choose the Lyapunov function candidate by augmenting the Lyapunov function in (10) with a quadratic parameter estimation error term as follows

$$V(\mathbf{e}, \boldsymbol{\theta}) = \mathbf{e}^T\mathbf{P}\mathbf{e} + g\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}, \quad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T > \mathbf{0} \tag{27}$$

Noticing that $\dot{\tilde{\boldsymbol{\theta}}} = -\dot{\hat{\boldsymbol{\theta}}}$, the time derivative of V is given by

$$\begin{aligned} \dot{V} &= -\mathbf{e}^T\mathbf{Q}\mathbf{e} - 2g\mathbf{e}^T\mathbf{P}\mathbf{b}\boldsymbol{\phi}^T\tilde{\boldsymbol{\theta}} + 2g\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\theta}}} \\ &= -\mathbf{e}^T\mathbf{Q}\mathbf{e} - 2g\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}(\boldsymbol{\Gamma}\boldsymbol{\phi}\mathbf{e}^T\mathbf{P}\mathbf{b} - \dot{\hat{\boldsymbol{\theta}}}) \end{aligned} \tag{28}$$

Apparently, choosing the parameter adaptation law as

$$\dot{\hat{\boldsymbol{\theta}}} = -\boldsymbol{\Gamma}\boldsymbol{\phi}\mathbf{e}^T\mathbf{P}\mathbf{b} \tag{29}$$

leads to $\dot{V} = -\mathbf{e}^T\mathbf{Q}\mathbf{e} \leq 0$. Accordingly, the following conclusions are in order: (i) the boundedness of $\mathbf{e}$ and $\tilde{\boldsymbol{\theta}}$, (ii) the boundedness of $\mathbf{x}$ and $\hat{\boldsymbol{\theta}}(t)$ (i.e., $\mathbf{k}(t)$ and $k_r(t)$) by noting the boundedness of $\mathbf{x}_m$ and $\boldsymbol{\theta}$, and the boundedness of the control signal u, and (iii) the tracking error $\lim_{t\to\infty}\mathbf{e} \to \mathbf{0}$ using Barbalat Lemma 2.1 because (a) $\int_0^\infty \mathbf{e}^T\mathbf{e} < c$ with constant $c > 0$ obtainable from (29), and (b) $\mathbf{e}$ is uniformly continuous since $\dot{\mathbf{e}}$ is bounded as can be seen from equation (25).

The basic ideas are not only readily applicable to MIMO LTI systems, it can also be extended for a class of nonlinear systems as will be detailed next.

3.3. MRAC for Nonlinear Systems

Model reference adaptive control can also be extended to classes of nonlinear systems. To illustrate the basic ideas, consider the simple nonlinear system in the form

$$\dot{x}_i = x_{i+1}, 1 \leq i \leq n-1 \tag{30}$$

$$\dot{x}_n = f(\mathbf{x}) + u = \boldsymbol{\phi}_n^T(\mathbf{x})\boldsymbol{\theta} + u \tag{31}$$

where $\mathbf{x} \in R^n$ and $u \in R$ are the states and input of the system, $\boldsymbol{\theta} \in R^l$ is a vector of constant unknown parameters, and $\boldsymbol{\phi}_n(\mathbf{x}) \in R^l$ is a known nonlinear function regressor. Suppose that the desired reference model is in the control canonical form as

$$\dot{\mathbf{x}}_m = \mathbf{A}_m\mathbf{x}_m + \mathbf{b}r \tag{32}$$

where $\mathbf{A}_m$ and $\mathbf{b}$ are defined by (20) and (18).

Define the tracking error $\mathbf{e} = \mathbf{x} - \mathbf{x}_m$, and parameter estimation error $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$. Consider the controller given by

$$u = -\sum_{i=1}^{n} a_i x_i - \boldsymbol{\phi}_n^T(\mathbf{x})\hat{\boldsymbol{\theta}} + r \tag{33}$$

where the terms $\sum_{i=1}^{n} a_i x_i$ and r are to form a stable linear portion, and the certainty equivalence portion $\boldsymbol{\phi}_n^T(\mathbf{x})\hat{\boldsymbol{\theta}}$ is in the linear-in-the-parameters form and thus can be handled easily using adaptive techniques. The control objective is to find the adaptive (dynamic) state feedback such that (i) $\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}_m(t)\| = 0$, and (ii) all the signals in the closed-loop are bounded.

The closed-loop error dynamics are

$$\dot{\mathbf{e}} = \mathbf{A}_m \mathbf{e} + \mathbf{b}\boldsymbol{\phi}_n^T(\mathbf{x})\tilde{\boldsymbol{\theta}} \tag{34}$$

Comparing with error dynamics (25) for the linear time invariant case, we know that mathematically they are equivalent. Thus, the following Lyapunov function candidate is in order

$$V = \mathbf{e}^T\mathbf{P}\mathbf{e} + \tilde{\boldsymbol{\theta}}^T \boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}, \quad \boldsymbol{\Gamma}^T = \boldsymbol{\Gamma} > \mathbf{0} \tag{35}$$

where $\mathbf{P}$ is the solution of the Lyapunov equation

$$\mathbf{A}_m^T\mathbf{P} + \mathbf{P}\mathbf{A}_m = -\mathbf{Q} \tag{36}$$

with $\mathbf{Q}$ being any symmetric positive definite matrix.

Noting that $\dot{\tilde{\boldsymbol{\theta}}} = -\dot{\hat{\boldsymbol{\theta}}}$, the time derivative of V along (34) is

$$\dot{V} = -\mathbf{e}^T\mathbf{Q}\mathbf{e} + 2\mathbf{e}^T\mathbf{P}\mathbf{b}\boldsymbol{\phi}_n^T\tilde{\boldsymbol{\theta}} - 2\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\theta}}} \tag{37}$$

Similarly, by choosing the parameters adaptation law as

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\phi}_n(\mathbf{x})\mathbf{e}^T\mathbf{P}\mathbf{b} \tag{38}$$

we have $\dot{V} = -\mathbf{e}^T\mathbf{Q}\mathbf{e} \le 0$. Following the standard arguments of Lyapunov design, we can conclude that (i) $\lim_{t\to\infty} \|\mathbf{e}(t)\| = 0$, i.e., $\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{x}_m(t)\| = 0$, and (ii) all the signals in the closed-loop are bounded.

In this section, only simple single input systems have been used to present the basic ideas. The same idea can be easily extended to multi-input systems, and many other systems that can be transformed into standard forms that are known to be solvable using MRAC techniques. In general, for nonlinear systems

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}) + \sum_{i=1}^{p} \mathbf{q}_i(\mathbf{z})\theta_i + \mathbf{g}(\mathbf{z})v, \qquad \mathbf{z} \in R^n \tag{39}$$

where $\boldsymbol{\theta} = [\theta_1, \cdots, \theta_p]^T$ are unknown constants, and $\mathbf{f}(\mathbf{z}), \mathbf{g}(\mathbf{z}), \mathbf{q}_i(\mathbf{z})$ are known nonlinear functions. If the nominal system $(\mathbf{f}(\mathbf{z}), \mathbf{g}(\mathbf{z}))$ is globally state feedback linearizable, and the matching conditions $\mathbf{q}_j(\mathbf{z}) \in \mathbf{span}\{\mathbf{g}(\mathbf{z})\}, 1 \le j \le p, \forall \mathbf{z} \in R^n$ are satisfied, then there exists a global θ-independent state space diffeomorphism $\mathbf{x} = T(\mathbf{z})$ and a state feedback

$$v = k(\mathbf{z}) + \beta(\mathbf{z})u \tag{40}$$

transforming system (39) into (30)-(31), and thus the MRAC design steps for (30)-(31) can be directly utilized.

The advantage of the MRAC approach is systematic, constructive, and the design performance of the closed-loop system can be specified easily. The disadvantages include (i) the cancellation of all the nonlinearities of the systems indiscriminately, and (ii) the solving of the Lyapunov equation as it can be very complex for higher order systems.

4. Lyapunov Design for Matched and Unmatched Uncertainties

MRAC is a Lyapunov design based on Lyapunov equations. In essence, it is based on feedback linearization. Though it is systematic, it has limited applicabilities. It is natural to investigate the problem of Lyapunov design for general nonlinear systems without restricting to the use of Lyapunov equations. In the last two decades, Lyapunov design was firstly restricted to the class of systems with *matching condition*, then relaxed to ones with *extended matching condition* or *generalized matching condition*, finally extended to systems in strict feedback form via the so-called *backstepping design*. In this section, Lyapunov design, in particular, Lyapunov redesign, is firstly discussed for nonlinear systems with uncertainties satisfying the *matching condition*, and under the assumption that there have existed a feedback control $u_0(\mathbf{x})$ and a CLF for the nominal nonlinear system. Then, a simple nonlinear system with the extended matching condition is used to demonstrate the basic concepts of backstepping design, which can be seen to be readily applicable for higher order systems with general unmatched conditions. Since the systems with *extended matching condition* or *generalized matching condition* can be solved elegantly by the adaptive backstepping design procedure, no further discussion is made for these systems using other methods except for backstepping design.

4.1. Lyapunov Design for Systems with Matched Uncertainties

Consider an affine nonlinear system in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})[u + \delta(\mathbf{x})] \tag{41}$$

where $\mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x})$ are smooth functions ($\mathbf{f}(\mathbf{0}) = \mathbf{0}$), $\mathbf{x} \in R^n$ is the states and $u \in R$ is the control signal, and $\delta(\mathbf{x})$ is an uncertain function known only to lie within certain bound, i.e., $|\delta(\mathbf{x})| \leq \Delta(\mathbf{x})$ with $\Delta(\mathbf{x})$ being a known nonnegative continuous function. Obviously, the uncertain term $\delta(\mathbf{x})$ enters the state equation exactly at the point where the control variable enters, this structural property is said that the uncertain term satisfies the *matching condition*.

4.1.1. Lyapunov Redesign

For the system described by (41), one may begin with a Lyapunov function for a nominal closed-loop system and then use this Lyapunov function to construct a controller which guarantees robustness to the given uncertainty. Assume that the nominal system (by dropping the uncertainty term $\delta(\mathbf{x})$) of the following form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u \tag{42}$$

is (i) uniformly asymptotically stabilizable at origin $\mathbf{x} = \mathbf{0}$ by a continuously differentiable feedback control law $u = u_0(\mathbf{x})$ with $u_0(\mathbf{0}) = 0$, and (ii) the corresponding CLF $V_0(\mathbf{x})$ is known, which satisfies

$$\begin{aligned} c_1\|\mathbf{x}\|^2 &\leq V_0(\mathbf{x}) \leq c_2\|\mathbf{x}\|^2 && (43)\\ \dot{V}_0(\mathbf{x}) &= \frac{\partial V_0}{\partial \mathbf{x}}[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u_0(\mathbf{x})] \leq -c_0\|\mathbf{x}\|^2 && (44) \end{aligned}$$

where c_0, c_1 and c_2 are some positive constants.

An additional feedback control $u_r(\mathbf{x})$ is to be designed using the knowledge of the CLF $V_0(\mathbf{x})$ and uncertain function $\delta(\mathbf{x})$ such that the overall control

$$u(\mathbf{x}) = u_0(\mathbf{x}) + u_r(\mathbf{x}) \tag{45}$$

stabilizes the actual system (41) in the presence of the uncertainty $\delta(\mathbf{x})$.

Substituting control (45) into (41) leads to the closed-loop system as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u_0 + \mathbf{g}(\mathbf{x})(u_r + \delta(\mathbf{x})) \tag{46}$$

Evaluating the derivative of $V_0(\mathbf{x})$ along the trajectories of (46) gives

$$\begin{aligned} \dot{V}_0(\mathbf{x}) &= \frac{\partial V_0}{\partial \mathbf{x}}(\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u_0(\mathbf{x})) + \frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})(u_r(\mathbf{x}) + \delta(\mathbf{x})) \\ &\leq -c_0\|\mathbf{x}\|^2 + \frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})(u_r(\mathbf{x}) + \delta(\mathbf{x})) && (47) \end{aligned}$$

The first term on the right-hand side is due to the nominal closed-loop system (44), and the second term represents the effect of the control $u_r(\mathbf{x})$ and the uncertain term $\delta(\mathbf{x})$ on $\dot{V}_0(\mathbf{x})$. Due to the matching condition, the uncertain term $\delta(\mathbf{x})$ appears on the right-hand side exactly at the same point where $u_r(\mathbf{x})$ appears. It is possible to redesign $u_r(\mathbf{x})$ to cancel the effect of $\delta(\mathbf{x})$ on $\dot{V}_0(\mathbf{x})$ such that

$$\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})(u_r(\mathbf{x}) + \delta(\mathbf{x})) \leq 0 \tag{48}$$

Though there are many different choices for $u_r(\mathbf{x})$, we will show the adaptive redesign and robust redesign next.

4.1.2. Adaptive Lyapunov Redesign

If the uncertain term is known to be linear-in-the-parameters, adaptive control techniques can be conveniently used to achieve both closed-loop asymptotic stability and a smooth control law. Assume that uncertain term $\delta(\mathbf{x}) = \boldsymbol{\omega}^T(\mathbf{x})\boldsymbol{\theta}$, where $\boldsymbol{\omega}(\mathbf{x}) \in R^l$ is a vector of known functions (regressor), $\boldsymbol{\theta} \in R^l$ are unknown constants of the system. Then, the corresponding system is in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})[u + \boldsymbol{\omega}^T(\mathbf{x})\boldsymbol{\theta}] \tag{49}$$

If $\boldsymbol{\theta}$ were known, the control law $u = u_0(\mathbf{x}) - \boldsymbol{\omega}^T(\mathbf{x})\boldsymbol{\theta}$ would directly cancel the term associated with $\boldsymbol{\theta}$, and render $\dot{V} \leq -c_0\|\mathbf{x}\|^2$ as can be seen from (47).

When $\boldsymbol{\theta}$ is unknown, consider the following *certainty-equivalence* control law

$$u \;=\; u_0 - \boldsymbol{\omega}^T(\mathbf{x})\hat{\boldsymbol{\theta}} \tag{50}$$

where $\hat{\boldsymbol{\theta}}$ represents the estimate of unknown $\boldsymbol{\theta}$. Substituting (50) into (49) leads to the closed-loop system

$$\dot{\mathbf{x}} \;=\; \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u_0 + \mathbf{g}(\mathbf{x})\boldsymbol{\omega}^T(\mathbf{x})\tilde{\boldsymbol{\theta}}, \quad \tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \tag{51}$$

Consider the Lyapunov function candidate by augmenting $V_0(\mathbf{x})$ with a quadratic parameter estimation error term as follows

$$V(\mathbf{x}, \tilde{\boldsymbol{\theta}}) = V_0(\mathbf{x}) + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}, \quad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T > \mathbf{0} \tag{52}$$

Noting that $\dot{\boldsymbol{\theta}} = \mathbf{0}$, the time derivative of $V(\mathbf{x}, \tilde{\boldsymbol{\theta}})$ along system (51) takes the form

$$\begin{aligned} \dot{V} &= \frac{\partial V_0}{\partial \mathbf{x}}[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u_0 + \mathbf{g}\boldsymbol{\omega}^T(\mathbf{x})\tilde{\boldsymbol{\theta}}] + \tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\theta}}} \\ &= \frac{\partial V_0}{\partial \mathbf{x}}[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u_0] + \frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\boldsymbol{\omega}^T(\mathbf{x})\tilde{\boldsymbol{\theta}} - \tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\dot{\hat{\boldsymbol{\theta}}} \end{aligned} \tag{53}$$

Apparently, choosing the parameters adaptation algorithm as

$$\dot{\hat{\boldsymbol{\theta}}} \;=\; \boldsymbol{\Gamma}\boldsymbol{\omega}(\mathbf{x})\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x}) \tag{54}$$

leads to

$$\dot{V} \;=\; \frac{\partial V_0}{\partial \mathbf{x}}[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u_0] \leq -c_0\|\mathbf{x}\|^2 \tag{55}$$

which further leads to the asymptotic stability of the states and boundedness of $\hat{\boldsymbol{\theta}}$, u, and $\dot{\mathbf{x}}$.

4.1.3. Robust Lyapunov Redesign

When the uncertainty $\delta(\mathbf{x})$ is unstructured in the sense that it can assume any value or function within the size of the given bounding function $\Delta(\mathbf{x})$, robust control is frequently applied to guarantee the boundedness of $\mathbf{x}$ using the knowledge of the CLF $V_0(\mathbf{x})$ and function bound $\Delta(\mathbf{x})$.

If asymptotic tracking is essential, then, from (47), we know that robust control

$$u_r(\mathbf{x}) = -\Delta(\mathbf{x})\mathrm{sgn}(\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})) \tag{56}$$

guarantees $\dot{V}_0(\mathbf{x}) \leq -c_0\|\mathbf{x}\|^2$ and the asymptotic stability of the closed-loop system.

However, control law (56) is a discontinuous function of the state $\mathbf{x}$. The discontinuity causes some theoretical as well as practical implementation problems. Though any smoothing approach can be introduced to smooth the control law, the trade-off is of uniformly ultimately bounded (UUB) stability rather than asymptotic stability because the closed-loop system can only be shown to converge to a small residual of the origin when the smooth approximated control is applied.

For boundedness (rather than asymptotic stability) of $\mathbf{x}$, which is also often required in reality, there is another way to design robust control. In this case, it is relaxed to show that the CLF $V_0(\mathbf{x})$ satisfies the following inequality

$$\dot{V}_0(\mathbf{x}) \leq -c_0\|\mathbf{x}\|^2 + \epsilon \tag{57}$$

where $\epsilon > 0$ is constant, and $V_0(x)$ is the function given in (43)-(44)

Apparently, $\dot{V}_0 \leq 0$ whenever $\|\mathbf{x}\| \geq \sqrt{\epsilon/c_0}$, and the states will converge to a compact set denoted by

$$\Omega := \{\mathbf{x} : \|\mathbf{x}\| \leq \sqrt{\epsilon/c_0}\} \tag{58}$$

Thus, for the uncertain system (41), we are to choose a robust controller $u_r(\mathbf{x})$, rather than satisfying equation (48), but relaxed to the following inequality

$$\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})(u_r(\mathbf{x}) + \delta(\mathbf{x})) \leq \epsilon \tag{59}$$

as can be clearly seen from equation (47).

By utilising the known bounding function $\Delta(\mathbf{x}) \geq |\delta(\mathbf{x})|$, we can indeed show that the so-called saturation-type control

$$u(\mathbf{x}) = -\Delta(\mathbf{x})\frac{\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\Delta(\mathbf{x})}{\|\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\Delta(\mathbf{x})\| + \epsilon} \tag{60}$$

satisfies the robust design inequality (59) as follows

$$\begin{aligned}
\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})(u_r(\mathbf{x}) + \delta(\mathbf{x})) &= -\frac{\|\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\Delta(\mathbf{x})\|^2}{\|\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\Delta(\mathbf{x})\| + \epsilon} + \frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\delta(\mathbf{x}) \\
&\leq -\frac{\|\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\Delta(\mathbf{x})\|^2}{\|\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\Delta(\mathbf{x})\| + \epsilon} + \|\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\Delta(\mathbf{x})\| \\
&= \frac{\|\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\Delta(\mathbf{x})\|}{\|\frac{\partial V_0}{\partial \mathbf{x}}\mathbf{g}(\mathbf{x})\Delta(\mathbf{x})\| + \epsilon}\epsilon < \epsilon
\end{aligned}$$

It should be noted that, there are many kinds of robust controllers. The choice of controller (60) is only one of the many possible choices.

4.2. Backstepping Design for Systems with Unmatched Uncertainties

For all the control approaches presented, the systems under study must satisfy the matching conditions. In an attempt to overcome these restrictions, a recursive design procedure called backstepping design has been introduced for strict-feedback systems. Backstepping design presents a systematic way for the construction of Lyapunov Function and the controller derivation, at the same time solve the problem of unmatched conditions. The key idea of backstepping is to start with a subsystem which is stabilizable with a known feedback law for a known Lyapunov function, and then adds to its input an integrator. For the augmented subsystem a new stabilizing feedback law is explicitly designed and shown to be stabilizing for a new Lyapunov function. The process continues till the explicit construction of the controller and the CLF for the complete system. The approach is systematic, and recursive in the construction of the CLF and controller design for the actual system.

To introduce the backstepping design clearly, the *extended matching* system is used to show how to remove the restriction of *matching condition* by the backstepping design procedure. Consider the following second-order system which satisfies the *extended matching condition*, where the parametric uncertainty $\boldsymbol{\theta}$ enters the system one integrator before the control u does

$$\dot{x}_1 = \boldsymbol{\theta}^T \boldsymbol{\omega}(x_1) + x_2 \tag{61}$$

$$\dot{x}_2 = u \tag{62}$$

It is apparent that the uncertainty is not in the range, not in the same equation, of control u and it cannot be cancelled directly. In accordance with the backstepping design strategy, the design procedure can be described as a two-step process. Firstly, we can view x_2 as a "control variable" for (61) and an *embedded* control input $\alpha_1(x_1)$ is designed to stabilize $z_1 = x_1$ with perturbation $z_2 = x_2 - \alpha_1(x_1)$. For the $\dot{z}_2$ subdynamics, the physical control u appears (in this case) and is to be designed such that $z_2 \to 0$, i.e., x_2 in (62) tracks $\alpha_1(x_1)$, which in turn guarantees that x_1 be stabilized at the equilibrium $x_1 = 0$ by the fictitious control $\alpha_1(x_1)$.

4.2.1. Backstepping for Known Parameter Case

Step 1. For $z_1 = x_1$, equation (61) can be re-written as

$$\dot{z}_1 = \boldsymbol{\theta}^T \boldsymbol{\omega}(x_1) + \alpha_1(x_1, \boldsymbol{\theta}) + z_2 \tag{63}$$

for which the stabilizing virtual or fictitious control $\alpha_1(x_1, \boldsymbol{\theta})$ is to be designed. Since the parameters $\boldsymbol{\theta}$ are known, consider the virtual control

$$\alpha_1(x_1, \boldsymbol{\theta}) = -c_1 x_1 - \boldsymbol{\theta}^T \boldsymbol{\omega}(x_1), \quad c_1 > 0 \tag{64}$$

which leads to the closed-loop dynamics

$$\dot{z}_1 = -c_1 z_1 + z_2, \quad z_2 = x_2 - \alpha_1(x_1, \boldsymbol{\theta}) \tag{65}$$

and the Lyapunov function candidate $V_1 = \frac{1}{2} z_1^2$ for the (z_1)- subsystem. The derivative of V_1 along (65) is given by

$$\dot{V}_1 = -c_1 z_1^2 + z_1 z_2 \tag{66}$$

which leads to the conclusion of asymptotic stabilization of z_1 if $z_2 = 0$. According to the standard backstepping procedure, as x_2 is not the physical control, backstepping design has to proceed and the coupling term $z_1 z_2$ will be cancelled in the next step for global asymptotic stability of the whole system. In general, $z_2 \neq 0$, it must be compensated for in the step(s) to follow.

Step 2. Noting equations (62) and (64), the derivative of z_2 is given by

$$\dot{z}_2 = \dot{x}_2 - \dot{\alpha}_1 = u + c_1 x_2 + \frac{\partial \alpha_1(x_1, \boldsymbol{\theta})}{\partial x_1}\left[\boldsymbol{\theta}^T \boldsymbol{\omega}(x_1) + x_2\right] \tag{67}$$

which apparently includes the physical control u. To design the physical control u to stabilize the whole system, the (z_1, z_2)- system, consider the augmented Lyapunov function candidate

$$V_2 = V_1 + \frac{1}{2} z_2^2 \tag{68}$$

Its derivative along (65) and (67) is given by

$$\dot{V}_2 = -c_1 z_1^2 + z_1 z_2 + z_2\left[u + c_1 x_2 + \frac{\partial \alpha_1(x_1, \boldsymbol{\theta})}{\partial x_1}(\boldsymbol{\theta}^T \boldsymbol{\omega}(x_1) + x_2)\right] \tag{69}$$

It is clear that the control

$$u \;\; = \;\; -c_2 z_2 - z_1 - c_1 x_2 + \frac{\partial \alpha_1(x_1, \boldsymbol{\theta})}{\partial x_1}(x_2 + \boldsymbol{\theta}^T \boldsymbol{\omega}(x_1)) \tag{70}$$

leads to

$$\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 \tag{71}$$

which proves that equilibrium $z = 0$ is globally uniformly stable, and the boundedness of $\alpha_1(x_1, \boldsymbol{\theta})$ and u. Owing to the definition of z, it is clear that $x_1 \to 0$ and $x_2 \to -\boldsymbol{\theta}^T \boldsymbol{\omega}(0)$ as $t \to \infty$.

4.2.2. Adaptive Backstepping for Unknown Parameter Case

When the parameters $\boldsymbol{\theta}$ are unknown, both $\alpha_1(x_1, \boldsymbol{\theta})$ in (64) and u in (70) depend on unknown parameter $\boldsymbol{\theta}$, and u is not feasible. This problem can be elegantly and systematically solved using adaptive backstepping as detailed below.

Step 1. The same as for the known parameter case, we have equation (63) for which a virtual adaptive stabilizing control $\alpha_1(x_1, \hat{\boldsymbol{\theta}}_1)$ is to be designed, where $\hat{\boldsymbol{\theta}}_1$ is the estimate of $\boldsymbol{\theta}$. Consider the virtual certainty equivalence control

$$\alpha_1(x_1, \hat{\boldsymbol{\theta}}_1) = -c_1 x_1 - \hat{\boldsymbol{\theta}}_1^T \boldsymbol{\omega}(x_1), \quad c_1 > 0 \tag{72}$$

and the Lyapunov function candidate augmented by a quadratic parameter estimation error term as follows

$$V_{s1} = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\boldsymbol{\theta}}_1^T \boldsymbol{\Gamma}_1^{-1} \tilde{\boldsymbol{\theta}}_1, \quad \tilde{\boldsymbol{\theta}}_1 = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_1, \quad \boldsymbol{\Gamma}_1^T = \boldsymbol{\Gamma}_1 > \mathbf{0} \tag{73}$$

Substituting (72) into (63) gives the error equation

$$\dot{z}_1 = -c_1 z_1 + z_2 + \tilde{\boldsymbol{\theta}}_1^T \boldsymbol{\omega}(x_1), \quad z_2 = x_2 - \alpha(x_1, \hat{\boldsymbol{\theta}}_1) \tag{74}$$

Noting $\boldsymbol{\theta}$ is a constant, the first time derivative of V_{s1} along (74) is

$$\dot{V}_{s1} \;=\; z_1 z_2 - c_1 z_1^2 + \tilde{\boldsymbol{\theta}}_1^T \boldsymbol{\Gamma}_1^{-1}\left(\boldsymbol{\Gamma}_1 \boldsymbol{\omega}(x_1) z_1 - \dot{\hat{\boldsymbol{\theta}}}_1\right) \tag{75}$$

Choosing the parameter adaptation algorithm as

$$\dot{\hat{\boldsymbol{\theta}}}_1 \;=\; \boldsymbol{\Gamma}_1 \boldsymbol{\omega}(x_1) x_1 \tag{76}$$

leads to

$$\dot{V}_{s1} \;=\; -c_1 z_1^2 + z_1 z_2 \tag{77}$$

The same as for the known parameter case, backstepping design has to proceed as the physical control u has not appeared yet, and the coupling term $z_1 z_2$ can be cancelled in the next step for global asymptotic stability of the whole system.

Step 2. The derivative of z_2, noting the virtual adaptive control (72) and the parameter update law (76), is given by

$$\begin{aligned} \dot{z}_2 &= u - \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_1}{\partial \hat{\boldsymbol{\theta}}_1}\dot{\hat{\boldsymbol{\theta}}}_1 \\ &= u - \frac{\partial \alpha_1}{\partial x_1}(x_2 + \boldsymbol{\theta}^T \boldsymbol{\omega}(x_1)) - \frac{\partial \alpha_1}{\partial \hat{\boldsymbol{\theta}}_1}\boldsymbol{\Gamma}_1 \boldsymbol{\omega}(x_1) z_1 \end{aligned} \tag{78}$$

Due to the presence of unknown parameters $\boldsymbol{\theta}$ in (78), it is hard to design a stabilizing control, the physical control u in this case, by using the already introduced parameter estimate $\hat{\boldsymbol{\theta}}_1$. To overcome this difficulty, a second estimate, $\hat{\boldsymbol{\theta}}_2$, of $\boldsymbol{\theta}$ is needed. Consider the control as

$$u = -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \hat{\boldsymbol{\theta}}_1}\boldsymbol{\Gamma}_1 \boldsymbol{\omega}(x_1) z_1 + \frac{\partial \alpha_1}{\partial x_1}\hat{\boldsymbol{\theta}}_2^T \boldsymbol{\omega}(x_1) \tag{79}$$

Substituting (79) into (78) leads to

$$\dot{z}_2 = -c_2 z_2 - z_1 - \tilde{\boldsymbol{\theta}}_2^T \frac{\partial \alpha_1}{\partial x_1}\boldsymbol{\omega}(x_1), \quad \tilde{\boldsymbol{\theta}}_2 = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_2 \tag{80}$$

The presence of the new parameter estimate $\hat{\boldsymbol{\theta}}_2$ suggests the following augmented Lyapunov function candidate

$$V_{s2}(z_1, z_2, \hat{\boldsymbol{\theta}}_1, \hat{\boldsymbol{\theta}}_2) \;=\; V_1 + \frac{1}{2} z_2^2 + \frac{1}{2}\tilde{\boldsymbol{\theta}}_2^T \boldsymbol{\Gamma}_2^{-1}\tilde{\boldsymbol{\theta}}_2, \quad \boldsymbol{\Gamma}_2^T = \boldsymbol{\Gamma}_2 > \mathbf{0} \tag{81}$$

The derivative of V_{s2} along (74) and (80) is

$$\begin{aligned} \dot{V}_{s2} &= \dot{V}_1 + z_2 \dot{z}_2^2 - \tilde{\boldsymbol{\theta}}_2^T \boldsymbol{\Gamma}_2^{-1}\dot{\hat{\boldsymbol{\theta}}}_2 \\ &= -c_1 z_1^2 - c_2 z_2^2 - \tilde{\boldsymbol{\theta}}_2^T \boldsymbol{\Gamma}_2^{-1}\left(\boldsymbol{\Gamma}_2 \frac{\partial \alpha_1}{\partial x_1}\boldsymbol{\omega} z_2 - \dot{\hat{\boldsymbol{\theta}}}_2\right) \end{aligned} \tag{82}$$

Choosing the second parameter update law

$$\dot{\hat{\boldsymbol{\theta}}}_2 = -\boldsymbol{\Gamma}_2 \frac{\partial \alpha_1}{\partial x_1} \boldsymbol{\omega} z_2 \tag{83}$$

yields

$$\dot{V}_{s2} = -c_1 z_1^2 - c_2 z_2^2 \tag{84}$$

This implies the resulting adaptive system consisting of (61)-(62) with the control law (79) and the update laws (76), (83) is globally stable and, in addition $x_1(t) \to 0$ and $x_2(t) \to -\hat{\boldsymbol{\theta}}_1^T \boldsymbol{\omega}(0)$ as $t \to \infty$.

4.2.3. Adaptive Backstepping with Tuning Function

Though adaptive backstepping can elegantly solve the problem of unmatched uncertainties, multiple estimates of the same parameters are necessary, which is the so-called overparameterization problem. For easy implementation, it is much more desirable to have a minimum number of parameters to be tuned. The overparameterization problem can be relaxed using the so-called *tuning functions*. At each consecutive step, a tuning function is designed to compensate for the effect of parameter estimation transients. In contrast to standard adaptive backstepping, these intermediate update laws are not implemented, and only the final tuning function is used as the parameter update law.

Step 1. The same as for the standard adaptive backstepping in subsection 4.2.2, consider the virtual adaptive control

$$\alpha_1(x_1, \hat{\boldsymbol{\theta}}) = -c_1 x_1 - \hat{\boldsymbol{\theta}}^T \boldsymbol{\omega}(x_1), \quad c_1 > 0 \tag{85}$$

and the Lyapunov function candidate with a quadratic parameter estimation error term as follows

$$V_{s1} = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\boldsymbol{\theta}}^T \boldsymbol{\Gamma}^{-1} \tilde{\boldsymbol{\theta}}, \quad \tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}, \quad \boldsymbol{\Gamma}^T = \boldsymbol{\Gamma} > \mathbf{0} \tag{86}$$

Substituting (85) into (63) gives

$$\dot{z}_1 = -c_1 z_1 + z_2 + \tilde{\boldsymbol{\theta}}^T \boldsymbol{\omega}(x_1) \tag{87}$$

Since $\dot{\boldsymbol{\theta}} = \mathbf{0}$, the first time derivative of V_{s1} along (87) is

$$\dot{V}_{s1} \;\; = \;\; -c_1 z_1^2 + z_1 z_2 + \tilde{\boldsymbol{\theta}}^T \boldsymbol{\Gamma}^{-1} (\boldsymbol{\tau}_1 - \dot{\hat{\boldsymbol{\theta}}}), \quad \boldsymbol{\tau}_1 = \boldsymbol{\Gamma} \boldsymbol{\omega}(x_1) z_1 \tag{88}$$

As $\alpha_1(x_1, \hat{\boldsymbol{\theta}})$ in (85) is not the physical control yet, it is beneficial to leave the design of the parameter adaptation law for $\dot{\hat{\boldsymbol{\theta}}}$ undetermined, i.e., an additional degree of freedom for the controller design in the future step(s). The term $\boldsymbol{\tau}_1$ is commonly referred as the tuning function. Again, the coupling term $z_1 z_2$ will be cancelled in the next step.

Step 2. The derivative of $z_2 = x_2 - \alpha_1(x_1, \hat{\boldsymbol{\theta}})$, noting the virtual adaptive control (85), is given by

$$
\begin{aligned}
\dot{z}_2 &= u - \frac{\partial \alpha_1}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_1}{\partial \hat{\boldsymbol{\theta}}}\dot{\hat{\boldsymbol{\theta}}} \\
&= u - \frac{\partial \alpha_1}{\partial x_1}(x_2 + \boldsymbol{\theta}^T \boldsymbol{\omega}(x_1)) - \frac{\partial \alpha_1}{\partial \hat{\boldsymbol{\theta}}}\dot{\hat{\boldsymbol{\theta}}} \qquad (89)
\end{aligned}
$$

Unlike the standard adaptive backstepping in Subsection 4.2.2, there is no need to introduce another estimate for the presence of unknown parameters $\boldsymbol{\theta}$ in (89) since $\hat{\boldsymbol{\theta}}$ is not defined yet. Furthermore, since the quadratic parameter estimation error term has already been included in V_{s1} in (86), the Lyapunov function candidate, V_{s2}, for $(z_1, z_2)-$ system takes the following simple form

$$
V_{s2} = V_{s1} + \frac{1}{2}z_2^2 \qquad (90)
$$

Its derivative is then given by

$$
\begin{aligned}
\dot{V}_2 &= -c_1 z_1^2 + z_1 z_2 + (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \boldsymbol{\Gamma}^{-1}(\dot{\hat{\boldsymbol{\theta}}} - \boldsymbol{\tau}_1) + z_2[u - \frac{\partial \alpha_1}{\partial x_1}(x_2 + \boldsymbol{\theta}^T \boldsymbol{\omega}) - \frac{\partial \alpha_1}{\partial \hat{\boldsymbol{\theta}}}\dot{\hat{\boldsymbol{\theta}}}] \qquad (91) \\
&= -c_1 z_1^2 + z_2[z_1 + u - \frac{\partial \alpha_1}{\partial x_1}x_2 - \frac{\partial \alpha_1}{\partial \hat{\boldsymbol{\theta}}}\dot{\hat{\boldsymbol{\theta}}} - \hat{\boldsymbol{\theta}}^T \frac{\partial \alpha_1}{\partial x_1}\boldsymbol{\omega}] + (\hat{\boldsymbol{\theta}} - \boldsymbol{\theta})^T \boldsymbol{\Gamma}^{-1}[\dot{\hat{\boldsymbol{\theta}}} - \boldsymbol{\tau}_2]
\end{aligned}
$$

where the tuning function $\boldsymbol{\tau}_2 = \boldsymbol{\tau}_1 - \boldsymbol{\Gamma} z_2 \frac{\partial \alpha_1}{\partial x_1}\boldsymbol{\omega}$. It is apparent that the following control and parameter update laws

$$
\begin{aligned}
u &= -z_1 - c_2 z_2 + \frac{\partial \alpha_1}{\partial x_1}x_2 + \frac{\partial \alpha_1}{\partial \hat{\boldsymbol{\theta}}}\dot{\hat{\boldsymbol{\theta}}} + \hat{\boldsymbol{\theta}}^T \frac{\partial \alpha_1}{\partial x_1}\boldsymbol{\omega}, \quad c_2 > 0 \qquad (92) \\
\dot{\hat{\boldsymbol{\theta}}} &= \boldsymbol{\tau}_2 = \boldsymbol{\Gamma} z_1 \boldsymbol{\omega} - \boldsymbol{\Gamma} z_2 \frac{\partial \alpha_1}{\partial x_1}\boldsymbol{\omega} \qquad (93)
\end{aligned}
$$

lead to

$$
\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2 \leq 0 \qquad (94)
$$

This implies the resulting adaptive system consisting of (61)-(62) with the control law (92) and the update law (93) is globally stable and, in addition $x_1(t) \to 0$ and $x_2(t) \to -\hat{\boldsymbol{\theta}}^T \boldsymbol{\omega}(0)$ as $t \to \infty$.

The above two-step design is used to introduce the concept of integrator backstepping and adaptive backstepping for the *extended matching case*. In the first step, the first equation is considered and a stabilizing controller is designed. In the second step, by considering the augmented Lyapunov function, a controller is constructed for the augmented system, the actual system in this case. This step-by-step design can be repeated to obtain controllers for high order strict-feedback systems with general unmatched uncertainties. Backstepping is a recursive procedure that interlaces the choice of a CLF with the design of feedback control. It breaks a design problem for the full system into a sequence of design problems for lower-order systems. Backstepping design can be used to overcome the barrier of *matching condition*. Consider the following nonlinear system in strict feedback form

$$
\begin{aligned}
&\dot{x}_i = x_{i+1} + f_i(\bar{\mathbf{x}}_i), \quad 1 \leq i \leq n-1 \\
&\dot{x}_n = f_n(\mathbf{x}) + u,
\end{aligned} \qquad (95)
$$

where $\bar{\mathbf{x}}_i = [x_1, \cdots, x_i]^T$, $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in R^n, u \in R$, are the state variables, system input, respectively, and functions $f_i(\bar{\mathbf{x}}_i)$ are unknown smooth uncertainties in general. Note that the state equation for $\dot{x}_i$ depends only on $x_1, x_2, \cdots, x_i$ and affinely on x_{i+1}.

If $f_i(\bar{\mathbf{x}}_i) = \boldsymbol{\omega}_i^T(\bar{\mathbf{x}}_i)\boldsymbol{\theta}$ with $\boldsymbol{\omega}_i(\bar{\mathbf{x}}_i) \in R^p$ known smooth functions and $\boldsymbol{\theta} \in R^p$ unknown constants, then the system (95) is called a nonlinear system in *parametric strict-feedback form*. It can be easily shown that, through the recursive and systematic construction of diffeomorphism state transformations

$$z_i = x_i - \alpha_{i-1}(\bar{\mathbf{x}}_{i-1}, \hat{\boldsymbol{\theta}}_1, \ldots, \hat{\boldsymbol{\theta}}_{i-1}), \alpha_0 = 0, \quad 1 \le i \le n \tag{96}$$

and the choice of consecutively augmented Lyapunov functions

$$V_i = V_{i-1} + \frac{1}{2}(z_i^2 + \tilde{\boldsymbol{\theta}}_i^T \boldsymbol{\Gamma}_i^{-1} \tilde{\boldsymbol{\theta}}_i), \quad \boldsymbol{\Gamma}_i = \boldsymbol{\Gamma}_i^T > \mathbf{0}, \quad V_0 = 0, \quad 1 \le i \le n \tag{97}$$

which, for the parametric strict-feedback nonlinear system, lead to the following adaptive stabilizing controller

$$u = \alpha_n(\mathbf{x}, \hat{\boldsymbol{\theta}}, \ldots, \hat{\boldsymbol{\theta}}_n) \tag{98}$$

$$\dot{\hat{\boldsymbol{\theta}}}_i = \boldsymbol{\Gamma}_i \left(\boldsymbol{\omega}_i - \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j} \boldsymbol{\omega}_j \right) z_i, \quad i = 1, \ldots, n \tag{99}$$

where $\hat{\boldsymbol{\theta}}_i \in R^p$ are the multiple estimates of $\boldsymbol{\theta}$, and functions $\alpha_i, i = 1, \ldots, n-1$, are chosen to stabilize the augmented $(z_1, ..., z_i)$- subsystem.

Using the tuning function algorithm, the corresponding state transformations, Lyapunov functions, tuning functions, parameter update laws and control law are given by

$$z_i = x_i - \alpha_{i-1}(\bar{\mathbf{x}}_{i-1}, \hat{\boldsymbol{\theta}}_1, \ldots, \hat{\boldsymbol{\theta}}_{i-1}), \quad \alpha_0 = 0, \quad 1 \le i \le n \tag{100}$$

$$V_i = V_{i-1} + \frac{1}{2} z_i^2, \quad V_0 = 0, \quad 1 \le i \le n \tag{101}$$

$$\boldsymbol{\tau}_i = \boldsymbol{\tau}_{i-1} - \boldsymbol{\Gamma} z_i \left[\phi_i - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} \phi_{i+1} \right], \quad \tau_0 = 0, \quad 1 \le i \le n \tag{102}$$

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\tau}_n \tag{103}$$

$$\begin{aligned} u &= -z_{n-1} - c_n z_n + \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} x_{i+1} + \frac{\partial \alpha_{n-1}}{\partial \hat{\boldsymbol{\theta}}} \boldsymbol{\tau}_n \\ &\quad + \left[\sum_{i=1}^{n-2} z_{i+1} \frac{\partial \alpha_{n-1}}{\partial \hat{\boldsymbol{\theta}}} \boldsymbol{\Gamma} - \hat{\boldsymbol{\theta}} \right] \left[\phi_n - \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial x_i} \phi_{i+1} \right] \end{aligned} \tag{104}$$

Backstepping design, though effective in solving the unmatched uncertainties, the controller becomes very complex for higher order dynamic systems. Backstepping design techniques can also be extended to more general uncertain nonlinear systems than that in (95), including certain multi-input nonlinear counterparts, under certain assumptions. If the nonlinear functions are not linear-in-the-parameters, different techniques can be used to solve the problem. By making use of Young's inequality ($2ab \le \frac{1}{\epsilon}a^2 + \epsilon b^2$, $\forall a, b \in R$ and $\epsilon > 0$), robust control design with smoothed control laws is one choice for boundedness in the neighborhood of $\mathbf{z} = \mathbf{0}$.

5. Property-based Lyapunov Design

Most Lyapunov design methods presented have been obtained from a purely mathematical point of view, i.e., by examining the mathematical features of the given differential equations, a Lyapunov function candidates V is chosen and shown that $\dot{V}$ can be made negative. In addition, Lyapunov functions in quadratic forms have been frequently chosen due to its convenience of mathematical treatment. However, quadratic Lyapunov functions are not exclusive. Lyapunov functions of different forms should also be considered in practice as they may lead to better controllers in solving actual problems where nonlinear parameterization is common. In this section, we shall present the physically motived Lyapunov design first by considering the robotic systems as an application example, then an integral Lyapunov design is discussed by exploiting the positivity of the "inertia" functions of the systems.

5.1. Physically Motivated Lyapunov Design

The mathematical approach, though effective for simple systems, is often found of little practice use, sometime clumsy, for complicated dynamic equations. In comparison, if engineering insights and physical properties are properly exploited, elegant and more useful Lyapunov analysis is expected for very complex systems. As an example, consider the control problem of a rigid body robot having n degrees of freedom described by

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) = \mathbf{u} \tag{105}$$

where $\mathbf{q} \in R^n$, $\mathbf{u} \in R^n$ are the joint displacements and applied joint forces/torques, inertia matrix $\mathbf{D}(\mathbf{q}) \in R^{n\times n}$ is symmetric positive definite, Coriolis and centrifugal matrix $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) \in R^{n\times n}$ satisfies $\mathbf{v}^T[\dot{\mathbf{D}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})]\mathbf{v} = 0$, $\forall \mathbf{v} \in R^n$, if $\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})$ is defined using the Christoffel Symbols, and $\mathbf{G}(\mathbf{q}) \in R^n$ denotes gravitational forces, respectively. The dynamics can be expressed in the linear-in-the-parameters form

$$\mathbf{D}(\mathbf{q})\mathbf{a} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\mathbf{v} + \mathbf{G}(\mathbf{q}) = \boldsymbol{\phi}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}, \mathbf{a})\boldsymbol{\theta} \tag{106}$$

where $\boldsymbol{\phi}(\mathbf{q}, \dot{\mathbf{q}}, \mathbf{v}, \mathbf{a}) \in R^{n\times r}$ is the regressor matrix of known functions, and $\boldsymbol{\theta} \in R^r$ are the constant parameters of interest. By defining $\mathbf{x}_1 = \mathbf{q}$, $\mathbf{x}_2 = \dot{\mathbf{q}}$, system (105) can be easily transformed into the standard form (2) (see *Robot Kinematics and Dynamics*).

For such complicated dynamics, controller design becomes very difficult if no physical properties are utilized. The resulting controllers would be very clumsy and awkward as well. Though engineers have been routinely using PD (proportional plus derivative) controllers to control them, theoretical justification for closed-loop stability is only possible by evoking some physical properties of the system.

Consider a simple PD controller with gravity compensation given by

$$\mathbf{u} = -\mathbf{K}_D\dot{\mathbf{q}} - \mathbf{K}_P\mathbf{q} + \mathbf{G}(\mathbf{q}), \quad \mathbf{K}_D = \mathbf{diag}[k_{Dii}] > \mathbf{0}, \mathbf{K}_P = \mathbf{diag}[k_{Pii}] > \mathbf{0} \tag{107}$$

The resulting closed-loop error dynamics are

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{K}_D\dot{\mathbf{q}} + \mathbf{K}_P\mathbf{q} = \mathbf{0} \tag{108}$$

While linear system theory is of no use to prove its stability, it is almost impossible to use trial-and-error to search for a Lyapunov function for (108) because it contains hundreds of terms for higher order robotic arms commonly found in industry (see *Robot Control and Programming*).

With physical insight, however, a Lyapunov function can be found easily by noting that (i) the inertia matrix $\mathbf{D}(\mathbf{q})$ is symmetric positive definite and $\dot{\mathbf{q}}^T\mathbf{D}(\mathbf{q})\dot{\mathbf{q}}$ denotes the kinetic energy of the system, and (ii) the PD control mimicks a combination of dampers and springs, and $\mathbf{q}^T\mathbf{K}_P\mathbf{q}$ denotes the "artificial potential energy" associated with the virtual spring in the control. Thus, the following Lyapunov function candidate is in order

$$V = \frac{1}{2}[\dot{\mathbf{q}}^T\mathbf{D}(\mathbf{q})\dot{\mathbf{q}} + \mathbf{q}^T\mathbf{K}_P\mathbf{q}] \tag{109}$$

Noticing $\dot{\mathbf{q}}^T\dot{\mathbf{D}}(\mathbf{q})\dot{\mathbf{q}} = 2\dot{\mathbf{q}}^T\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}$, the derivative of V is given by

$$\dot{V} = \dot{\mathbf{q}}^T[\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}] + \dot{\mathbf{q}}^T\mathbf{K}_P\mathbf{q} \tag{110}$$

Observing (105), $\dot{V}$ becomes

$$\dot{V} = \dot{\mathbf{q}}^T[\mathbf{u} - \mathbf{G}(\mathbf{q}) + \mathbf{K}_P\mathbf{q}] \tag{111}$$

Substituting the controller (107) in gives

$$\dot{V} = -\dot{\mathbf{q}}^T\mathbf{K}_D\dot{\mathbf{q}} \tag{112}$$

which is negative semidefinite, and no conclusion can be drawn about its asymptotic stability. To prove asymptotic stability of the closed-loop system, LaSalle's Invariant Set Theorem 2.1 can be used by finding the largest invariant set in the set $\{\mathbf{x} \mid \dot{V} \equiv 0\}$. Noticing that $\dot{V} \equiv 0$ leads to $\dot{\mathbf{q}} \equiv \mathbf{0}$, which, subsequently, leads to $\ddot{\mathbf{q}} = \mathbf{0}$, then from the closed-loop error equation (108), the conclusion that $\mathbf{q} = \mathbf{0}$ follows for $\mathbf{K}_P > \mathbf{0}$. Accordingly, global asymptotic stability is achieved by evoking the LaSalle's Invariant Set Theorem 2.1 which states that all solutions globally asymptotically converge to the largest invariant set in the set $\{\mathbf{x} \mid \dot{V} = 0\}$ if $V(\mathbf{x})$ is radially unbounded and $\dot{V}(\mathbf{x}) \leq 0$ over the whole state space.

Through the simple position control example, we have demonstrated the benefit in constructing Lyapunov functions from physical consideration, and the power of Lyapunov method in solving stability problem of nonlinear systems. In this case, Lyapunov direct method is used to prove the closed-loop stability of a commonly used controller. However, from the concept of Control Lyapunov function, the controllers that render the $\dot{V}$ in (111) non-positive is not unique.

The above regulation controller can be easily extended to adaptive control for trajectory tracking, where the robot is actually required to follow a desired trajectory, rather than merely reaching a desired position. Assume that the desired position, velocity and acceleration are denoted by $\mathbf{q}_d(t)$, $\dot{\mathbf{q}}_d(t)$ and $\ddot{\mathbf{q}}_d(t)$, respectively, which are bounded and known. For convenience of analysis, define the tracking error as $\mathbf{e} = \mathbf{q} - \mathbf{q}_d$, the velocity reference signal and the filtered tracking error as

$$\dot{\mathbf{q}}_r = \dot{\mathbf{q}}_d - \mathbf{\Lambda}\mathbf{e} \tag{113}$$

$$\mathbf{s} = \dot{\mathbf{q}} - \dot{\mathbf{q}}_r = \dot{\mathbf{e}} + \mathbf{\Lambda}\mathbf{e}, \;\; \mathbf{\Lambda} = \mathbf{diag}[\lambda_i] > \mathbf{0} \tag{114}$$

It is clear that $\mathbf{e}$ is a filtered version of $\mathbf{s}$ since $\mathbf{e} = \mathbf{G}(p)\mathbf{s}$ where $\mathbf{G}(p) = \mathbf{diag}[\frac{1}{p+\lambda_i}]$, $p = d/dt$. Since $\mathbf{G}(p)$ is strictly proper and exponentially stable, it is well known from linear system theories that (i) if $\mathbf{s} \in \mathbf{L}_2^n$, then $\dot{\mathbf{e}} \in \mathbf{L}_2^n$, and continuous $\mathbf{e} \in \mathbf{L}_2^n \cap \mathbf{L}_\infty^n$, $\mathbf{e} \to \mathbf{0}$ as $t \to \infty$, (ii) if $\mathbf{s} \to \mathbf{0}$ as $t \to \infty$, then $\mathbf{e}$, $\dot{\mathbf{e}} \to \mathbf{0}$ as $t \to \infty$. Therefore, to prove the convergence of $\mathbf{e}$ and $\dot{\mathbf{e}}$, it is only necessary to prove the convergence of $\mathbf{s}$.

Noticing that $\mathbf{s} = \dot{\mathbf{q}} - \dot{\mathbf{q}}_r$ and $\dot{\mathbf{s}} = \ddot{\mathbf{q}} - \ddot{\mathbf{q}}_r$, the following equation holds

$$\mathbf{D}(\mathbf{q})[\dot{\mathbf{s}} + \mathbf{K}_1\mathbf{s}] + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{s} = \mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} - [\mathbf{D}(\mathbf{q})(\ddot{\mathbf{q}}_r - \mathbf{K}_1\mathbf{s}) + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}_r] \tag{115}$$

where matrix $\mathbf{K}_1 \geq \mathbf{0}$, and should be chosen such that $\mathbf{D}(\mathbf{q})\mathbf{K}_1 \geq \mathbf{0}$ which holds if and only if $\mathbf{D}(\mathbf{q})$ and $\mathbf{K}_1$ are commutative, i.e., $\mathbf{D}(\mathbf{q})\mathbf{K}_1 = \mathbf{K}_1\mathbf{D}(\mathbf{q})$. It suffices to choose $\mathbf{K}_1 = \alpha\mathbf{I}$ with constant $\alpha \geq 0$.

Substituting (105) into the above equation leads to

$$\mathbf{D}(\mathbf{q})\dot{\mathbf{s}} + \mathbf{D}(\mathbf{q})\mathbf{K}_1\mathbf{s} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{s} = \mathbf{u} - \mathbf{D}(\mathbf{q})(\ddot{\mathbf{q}}_r - \mathbf{K}_1\mathbf{s}) + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}_r + \mathbf{G}(\mathbf{q}) \tag{116}$$

Noticing the expression (106), equation (116) can be written as

$$\mathbf{D}(\mathbf{q})\dot{\mathbf{s}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{s} + \mathbf{D}(\mathbf{q})\mathbf{K}_1\mathbf{s} = \mathbf{u} - \boldsymbol{\phi}(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_r,\ddot{\mathbf{q}}_r - \mathbf{K}_1\mathbf{s})\boldsymbol{\theta} \tag{117}$$

For tracking control, consider the following certainty equivalence controller

$$\mathbf{u} = \boldsymbol{\phi}(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_r,\ddot{\mathbf{q}}_r - \mathbf{K}_1\mathbf{s})\hat{\boldsymbol{\theta}} + \mathbf{K}_2\mathbf{s}, \quad \mathbf{K}_2 > \mathbf{0} \tag{118}$$

Define $\tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}$. The closed-loop error equation becomes

$$\mathbf{D}(\mathbf{q})\dot{\mathbf{s}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{s} + \mathbf{K}\mathbf{s} = -\boldsymbol{\phi}(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_r,\ddot{\mathbf{q}}_r - \mathbf{K}_1\mathbf{s})\tilde{\boldsymbol{\theta}}, \quad \mathbf{K} = \mathbf{K}_2 + \mathbf{D}(\mathbf{q})\mathbf{K}_1 \tag{119}$$

To prove the stability of the closed-loop system, consider the Lyapunov function candidate

$$V = \frac{1}{2}\left[\mathbf{s}^T\mathbf{D}(\mathbf{q})\mathbf{s} + \tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\tilde{\boldsymbol{\theta}}\right], \quad \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T > \mathbf{0} \tag{120}$$

The particular choice of $\mathbf{s}^T\mathbf{D}(\mathbf{q})\mathbf{s}$ in V is motivated by the fact that the $\dot{\mathbf{q}}^T\mathbf{D}(\mathbf{q})\dot{\mathbf{q}}$ is the kinetic energy of the system and the following nice properties: (i) $\dot{\mathbf{D}}(\mathbf{q}) - 2\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ is skew symmetric, and (ii) $\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}}$ appears in $\dot{V}$ such that the possible controller singularity is avoided elegantly. The quadratic parameter estimation error term is introduced to accommodate unknown parameters. Noticing that $\mathbf{s}^T\dot{\mathbf{D}}(\mathbf{q})\mathbf{s} = 2\mathbf{s}^T\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{s}$, the derivative of V is given by

$$\dot{V} = \mathbf{s}^T[\mathbf{D}(\mathbf{q})\dot{\mathbf{s}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\mathbf{s} + \mathbf{K}\mathbf{q}] + \tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\theta}}} \tag{121}$$

Observing the closed-loop error equation (119) and the fact that $\dot{\boldsymbol{\theta}} = \mathbf{0}$, equation (121) becomes

$$\dot{V} = -\mathbf{s}^T\mathbf{K}\mathbf{s} + \tilde{\boldsymbol{\theta}}^T\boldsymbol{\phi}^T(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_r,\ddot{\mathbf{q}}_r - \mathbf{K}_1\mathbf{s})\mathbf{e} + \tilde{\boldsymbol{\theta}}^T\boldsymbol{\Gamma}^{-1}\dot{\tilde{\boldsymbol{\theta}}} \tag{122}$$

It is clear that the parameter adaptation law

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\phi}(\mathbf{q},\dot{\mathbf{q}},\dot{\mathbf{q}}_r,\ddot{\mathbf{q}}_r - \mathbf{K}_1\mathbf{s})\mathbf{s} \tag{123}$$

renders

$$\dot{V} = -\mathbf{s}^T \mathbf{K} \mathbf{s} \tag{124}$$

which leads to the following two inequalities

$$-V(0) \le V(t) - V(0) = \int_0^t \dot{V}(\tau) d\tau = -\int_0^t \mathbf{s}^T \mathbf{K} \mathbf{s} d\tau$$
$$\int_0^t \|\mathbf{s}(\tau)\|^2 d\tau \le \frac{V(0)}{\lambda_{max}(\mathbf{K})} < \infty$$

Accordingly, we have $\mathbf{s} \in \mathbf{L}_2$, $\dot{\mathbf{e}} \in \mathbf{L}_2^n$ and continous $\mathbf{e} \to \mathbf{0}$ as $t \to \infty$. The boundedness of $\mathbf{s}$ and $\tilde{\boldsymbol{\theta}}$ follow directly from the boundedness of V. The boundedness of $\hat{\boldsymbol{\theta}}$ is obtainable by noticing that $\boldsymbol{\theta}$ is constant. From (119), it is clear that $\dot{\mathbf{s}}$ is bounded. The combination of $\mathbf{s} \in \mathbf{L}_2$ and bounded $\dot{\mathbf{s}}$ leads to $\mathbf{s} \to \mathbf{0}$ as $t \to \infty$, which subsequently leads to $\lim_{t\to\infty} \mathbf{e} = \mathbf{0}$.

It is clear that with the different choice of Lyapunov function, there is no need to invoke the invariant set theorem for the conclusion of asymptotic stability.

5.2. Integral Lyapunov Function for Nonlinear Parameterizations

Lyapunov functions in quadratic form have been frequently chosen due to their convenience of mathematical treatment. In practice, other forms should also be considered, and in some cases, they may lead to better controllers in solving actual problems where nonlinear parameterization is common. In this section, an integral Lyapunov function is presented, which results in a controller structure that avoids the possible controller singularity problem encountered in feedback linearization completely without evoking any preventing measures such as the so-called $\sigma-$ modification, projection algorithms, and among others. For simplicity, consider the following simple nonlinear system as it can be easily generalized

$$m(x)\dot{x} \;\; = \;\; f(x) + u \tag{125}$$

where $x \in R$, $u \in R$, $y \in R$ are the state variable, system input and output, respectively; continuous functions $m(x) > 0$ and $f(x)$ can be expressed as

$$m(x) = \boldsymbol{\theta}^T \boldsymbol{\omega}_m(x), \quad f(x) = \boldsymbol{\theta}^T \boldsymbol{\omega}_f(x) \tag{126}$$

where $\boldsymbol{\theta} \in R^p$ is a vector of unknown constant parameters, $\boldsymbol{\omega}_m(x) \in R^p$ and $\boldsymbol{\omega}_f(x) \in R^p$ are known regressor vectors.

The objective is to control x to follow a given reference signal y_d. Define $e = x - y_d$. From (125), the time derivative of e can be written as

$$m(x)\dot{e} = f(x) + u - m(x)\dot{y}_d \tag{127}$$

Let $m_\alpha(x) = m(x)\alpha(x)$ where $\alpha(x)$: $R \to R_+$ is a smooth weighting function, and is used for appropriate choice of integral functions to be introduced. Noting that $x = e + y_d$, for ease of discussion, denote $m_\alpha(e + y_d) = m_\alpha(x)$.

Define the integral Lyapunov function candidate as

$$V_0 = \int_0^e \sigma m_\alpha(\sigma + y_d) d\sigma \tag{128}$$

which is made radially unbounded by choosing appropriate $\alpha(x)$.

The time derivative of V_0 given in (128) is

$$\begin{aligned}
\dot{V}_0 &= e m_\alpha(x)\dot{e} + \dot{y}_d \int_0^e \sigma \frac{\partial m_\alpha(\sigma + y_d)}{\partial y_d} d\sigma \\
&= e m_\alpha(x)\dot{e} + \dot{y}_d \int_0^e \sigma \frac{\partial m_\alpha(\sigma + y_d)}{\partial \sigma} d\sigma \\
&= e\alpha(x)m(x)\dot{e} + \dot{y}_d m_\alpha(x) e + \dot{y}_d \int_0^e m_\alpha(\sigma + y_d) d\sigma
\end{aligned} \tag{129}$$

Substituting (127) into (129) leads to

$$\dot{V}_0 = e\alpha(x)\left[f(x) + u + \frac{\dot{y}_d}{e\alpha(x)} \int_0^e m_\alpha(\sigma + y_d) d\sigma \right] \tag{130}$$

Noting the expressions in (126), equation (130) becomes

$$\dot{V}_0 = e\alpha(x)[\boldsymbol{\theta}^T \boldsymbol{\omega}(\mathbf{z}) + u] \tag{131}$$

where

$$\boldsymbol{\omega}(\mathbf{z}) \;=\; \boldsymbol{\omega}_f(x) + \frac{\dot{y}_d}{e\alpha(x)} \int_0^e \boldsymbol{\omega}_m(\sigma + y_d)\alpha(\sigma + y_d) d\sigma, \quad \mathbf{z} = [x,\ y_d,\ \dot{y}_d]^T \in R^3 \tag{132}$$

It can be checked that

$$\lim_{e \to 0} \boldsymbol{\omega}(\mathbf{z}) = \boldsymbol{\omega}_f(x) + \frac{\dot{y}_d \boldsymbol{\omega}_m(y_d)\alpha(y_d)}{\alpha(x)} \tag{133}$$

which means that $\boldsymbol{\omega}(\mathbf{z})$ is well defined. If the parameter vector $\boldsymbol{\theta}$ is available, a possible controller is $u = [-\frac{ke}{\alpha(x)} - \boldsymbol{\theta}^T \boldsymbol{\omega}(\mathbf{z})]$ with design parameter $k > 0$. For this controller, (131) becomes $\dot{V}_0 = -ke^2 < 0, \forall e \neq 0$. According to Definition 2.2, we conclude that V_0 is a CLF and $e \to 0$ as $t \to \infty$.

In the case of unknown parameter $\boldsymbol{\theta}$, consider its *certainty-equivalence* controller given by

$$u = \left[-\frac{ke}{\alpha(x)} - \hat{\boldsymbol{\theta}}^T \boldsymbol{\omega}(\mathbf{z}) \right] \tag{134}$$

where $\hat{\boldsymbol{\theta}}$ is the estimate of $\boldsymbol{\theta}$. Substituting (134) into (131) leads to

$$\dot{V}_0 = -ke^2 + \tilde{\boldsymbol{\theta}}^T \boldsymbol{\omega}(\mathbf{z})\alpha(x)e, \quad \tilde{\boldsymbol{\theta}} = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}} \tag{135}$$

The system stability is not clear at this stage yet because the last term in (135) is indefinite and contains unknown $\tilde{\boldsymbol{\theta}}$. To remove such an uncertain term with $\tilde{\boldsymbol{\theta}}$, consider the following augmented Lyapunov function candidate

$$V = V_0 + \frac{1}{2}\tilde{\boldsymbol{\theta}}^T \boldsymbol{\Gamma}^{-1} \tilde{\boldsymbol{\theta}}, \ \boldsymbol{\Gamma} = \boldsymbol{\Gamma}^T > \mathbf{0} \tag{136}$$

Its time derivative is

$$\dot{V} = -e^2 + \tilde{\boldsymbol{\theta}}^T \boldsymbol{\Gamma}^{-1}\left[\boldsymbol{\Gamma}\boldsymbol{\omega}(\mathbf{z})\alpha(x)e + \dot{\hat{\boldsymbol{\theta}}}\right] \tag{137}$$

The following adaptive law

$$\dot{\hat{\boldsymbol{\theta}}} = \boldsymbol{\Gamma}\boldsymbol{\omega}(\mathbf{z})\alpha(x)e \tag{138}$$

leads to

$$\dot{V} = -ke^2 \leq 0 \tag{139}$$

This guarantees that $V(0) \in L_\infty$ for any bounded initial values $x(0)$ and $y_d(0)$. Integrating (139), we have $\int_0^\infty e^2(\tau)d\tau \leq V(0) < \infty$ and $0 \leq V(t) \leq V(0)$. This implies that $e \in L_2 \cap L_\infty$ and $\hat{\boldsymbol{\theta}}(t)$ is bounded. Consequently, u and $\dot{e}$ are also bounded. Since $e \in L_2 \cap L_\infty$ and $\dot{e} \in L_\infty$, we conclude $\lim_{t\to\infty} e = 0$ by Barbalat's Lemma 2.1.

Different choices of weighting function $\alpha(x)$ may produce different control performance. For $m_\alpha(x) = \boldsymbol{\theta}^T\boldsymbol{\omega}_m(x)\alpha(x)$ with known functions $\boldsymbol{\omega}_m(x)$, it is not difficult to design $\alpha(x)$ such that $0 < c_1 \leq m_\alpha(x) \leq c_0$. For example, if $m(x) = \exp(-x^2)(\theta_1 + \theta_2 x^2)$ with constant parameters $\theta_1, \theta_2 > 0$, then one may take $\alpha(x) = \exp(x^2)/(1 + x^2)$ which leads to

$$\min(\theta_1, \theta_2) \leq m_\alpha(x) = \frac{\theta_1 + \theta_2 x^2}{1 + x^2} \leq \max(\theta_1, \theta_2)$$

Though the concepts are introduced through a simple first order system, it can be applied to a larger class of the system of the form directly

$$\begin{aligned} \dot{x}_i &= x_{i+1}\,, \quad i = 1, 2, \cdots, n-1 \\ \dot{x}_n &= \frac{f(\mathbf{x})}{m(\mathbf{x})} + \frac{1}{m(\mathbf{x})}u \end{aligned} \tag{140}$$

where $\mathbf{x} = [x_1 \;\; x_2 \;\; \cdots \;\; x_n]^T \in R^n$, $u \in R$, are the state variables, and input, respectively, functions $f(\mathbf{x}), m(\mathbf{x})$ may take the following forms

$$f(\mathbf{x}) = \boldsymbol{\theta}^T\boldsymbol{\omega}_f(\mathbf{x}) + f_0(\mathbf{x}), \qquad m(\mathbf{x}) = \boldsymbol{\theta}^T\boldsymbol{\omega}_m(\mathbf{x}) + m_0(\mathbf{x}) > 0 \tag{141}$$

where $\boldsymbol{\theta} \in R^p$ is a vector of unknown constant parameters, $\boldsymbol{\omega}_f(\mathbf{x}) \in R^p$ and $\boldsymbol{\omega}_m(\mathbf{x}) \in R^p$ are known regressor vectors, continuous functions $f_0(\mathbf{x})$ and $m_0(\mathbf{x})$ are known.

Furthermore, by combining function approximation, adaptive backstepping control design using the integral Lyapunov functions, semi-globally stable and singularity-free adaptive controller can be obtained for the following class of nonlinear systems in strict-feedback form

$$\begin{cases} \dot{x}_i = f_i(\bar{\mathbf{x}}_i) + g_i(\bar{\mathbf{x}}_i)x_{i+1}, 1 \leq i \leq n-1 \\ \dot{x}_n = f_n(\mathbf{x}) + g_n(\mathbf{x})u \end{cases} \tag{142}$$

where $\bar{\mathbf{x}}_i = [x_1, \cdots, x_i]^T$, $\mathbf{x} = [x_1, x_2, \cdots, x_n]^T \in R^n, u \in R$ are the state variables, and system input, respectively; $f_i(\cdot)$ and $g_i(\cdot) > 0, i = 1, 2, \cdots, n$ are unknown smooth functions and may not be linearly parameterized. It can also be applied to the following MIMO nonlinear system where the inputs appear in a triangular manner as follows

$$\begin{cases} \dot{x}_{1,i_1} = x_{1,i_1+1} \\ \dot{x}_{1,\rho_1} = f_1(\mathbf{X}) + g_{1,1}(x_1)u_1 \\ \dot{x}_{2,i_2} = x_{2,i_2+1} \\ \dot{x}_{2,\rho_2} = f_2(\mathbf{X}, u_1) + g_{2,2}(x_1, x_2)u_2 \\ \vdots \\ \dot{x}_{j,i_j} = x_{j,i_j+1} \\ \dot{x}_{j,\rho_j} = f_j(\mathbf{X}, u_1, \cdots, u_{j-1}) + g_{j,j}(x_1, x_2, \cdots, x_j)u_j \end{cases} \tag{143}$$

where $\mathbf{X} = [\mathbf{x}_1^T, \mathbf{x}_2^T, \cdots, \mathbf{x}_m^T]^T$ with $\mathbf{x}_j = [x_{j,1}, x_{j,2}, \cdots, x_{j,\rho_j}]^T \in R^{\rho_j}$ and $u_j \in R$ are the state variables, the system inputs, respectively; $f_j(\cdot)$ and $g_{j,i}(\cdot)$ are unknown smooth functions; ρ_j, m, $1 \le j \le m$ and $1 \le i_j \le \rho_j - 1$ are positive integers defining the order and the complexity of the system.

6. Design Flexibilities and Considerations

Except for the MRAC Lyapunov design where the selection of Lyapunov functions and control laws are systematic and straightforward, the degrees of freedom in Lyapunov design in general are numerous, case dependent, and allow for the careful shaping of the closed-loop performance. Though MRAC can specify the desired performance elegantly, it is in essence a *feedback linearization* design, in which it tries to replace all the linear/nonlinear terms indiscriminately of the system by the desired linear terms for closed-loop stability (see *Feedback Linearization*). As a results, some stabilizing nonlinearities are also been canceled by the control law. As an example, consider the stabilization problem of the following scalar dynamic system

$$\dot{x} = 2\cos(x) - x^3 + u \tag{144}$$

Following the technique of feedback linearization, and taking $A_m = 1$, the following control is in order

$$u = -2\cos(x) + x^3 - x \tag{145}$$

The stability can be seen by taking $Q = 1$ (corresponding to $W(x) = x^2$) and solve (26) for $P = 1/2$ which corresponds to Lyapunov function $V(x) = x^2/2$.

The control law (145) indiscriminately cancels both nonlinearities $\cos x$ and $-x^3$ altogether, and replaces them with $-x$ such that the resulting feedback system is a stable linear system, $\dot{x} = -x$. Though stability is guaranteed with the automatic construction of the controller, it is obviously irrational to cancel $-x^3$ without further analysis. For stabilization at $x = 0$, the negative feedback term $-x^3$ is helpful, especially when x is large. At the same time, the presence of x^3 in the control law (145) is harmful. It leads to a large magnitude of control u

that may cause many practical problems. It would be more reasonable to selectively cancel the nonlinearities. Consider $V(x) = \frac{1}{2}x^2$ as before. Its derivative is

$$\dot{V} = x(2\cos(x) - x^3 + u) = -x^4 + x(2\cos(x) - u) \tag{146}$$

It is clear that there is no need to cancel the $-x^3$ term at all as it is stabilizing. Choosing $u = -2\cos(x) - x$ which grows only linearly with x leads to $\dot{V} = -W(x) \leq 0$ where $W(x) = x^2 + x^4$.

As said earlier, Lyapunov design is not unique and problem dependent. Another control law can be obtained via universal control (7). Since it is based on the assumption that $f(0) = 0$, the system needs to be converted into the required form first. Consider $u = -2\cos(x) + v$ where v is a new control to be defined, the system is transformed into $\dot{x} = -x^3 + v$. Again, considering $V(x) = \frac{1}{2}x^2$ as the CLF with $f(x) = -x^3$ and $g(x) = 1$, the universal control v is given by

$$v = x^3 - x\sqrt{x^4 + 1} \tag{147}$$

A very nice property of (147) is that $v \to 0$ as $|x| \to \infty$, which means that for large $|x|$ the control law for u reduces to the term $-\cos x$ required to place the equilibrium at $x = 0$. It is easy to check that $u = -\cos x + v$ satisfies (5) with $W(x) = x^2\sqrt{x^4 + 1}$. This control law is superior because it requires less control effort than the other two.

From the simple example, it is clear that the choices for functions α_i in the iterative back-stepping design are by no means unique, nor is the final choice for the control law. In a more general setting, it is hard to choose the intermediate control α_i. In actual applications, it is problem dependent. From a theoretical point of view, optimal control or sub-optimal control may be considered according to some meaningful cost functions. At each step, wasteful cancellations of the beneficial nonlinearities can be avoided. However, the reasonable and beneficial intermediate-step controls, especially at the earlier steps, may not be proven beneficial for the final design for the physical system. Further research is needed.

The choices for V_i are not unique as have been demonstrated in the article earlier. Recall that given a Lyapunov function V_i and control α_i at the ith design step, the new Lyapunov function V_{i+1} is given by

$$V_i = V_{i-1} + \frac{1}{2}z_i^2 \tag{148}$$

This choice for V_{i+1} is not the only choice that will lead to a successful design. Another degree of freedom is available in the design procedure. In this article, several types of Lyapunov functions, quadratic, energy-based quadratic and integral, have been presented, and they can be blended together in actual controller design for the best result. For example, at one step, one type of Lyapunov functions can be used, at the next step, another type can be considered by examining the properties of the system. Note that the types of Lyapunov functions introduced in the article are by no means exclusive. Numerous other types and different variations can be explored. It has been shown that $\frac{1}{2}z_i^2$ can be replaced by the general expression $\displaystyle\int_{\alpha_i}^{x_{i+1}} \phi(\bar{\mathbf{x}}_i, s)ds$ for a suitable choice of the function ϕ. The former is a special case of the latter for $\phi = s - \alpha_i(\bar{\mathbf{x}}_i)$.

This degree of freedom in the choice of ϕ can be significant as it allows the extensions of the backstepping design to the non-smooth case, and reduce the unnecessarily large control gains often caused by the quadratic terms. In addition, weighting functions can also be introduced in the design procedure. For example, instead of (148), the following functions can be chosen

$$V_i = \kappa_i V_{i-1} + \frac{1}{2} z_i^2 \tag{149}$$

where κ_i is any differentiable positive definite weighting function on V_i which can have a large effect on the control laws obtainable in the future steps.

In dealing with practical systems, many physical properties can be exploited, and often are found very useful in the construction of Lyapunov functions and controller design. Yet another degree of freedom in the design is the exploitation of those structural properties that are "hidden" in the individual equations. Consider the following strict-feedback nonlinear system

$$\dot{x}_1 = \theta x_2 - \theta x_1 \tag{150}$$

$$\dot{x}_2 = -x_1 u \tag{151}$$

where x_1, x_2 are the states, u is the control, and $\theta > 0$ is the unknown constant. By exploiting the positivity of θ, and the flexibility in the design steps, stable adaptive control can be easily developed without the possible controller singularity problem caused by the possibility of $x_1 = 0$ for equation (151) when feedback linearization control of the form is considered $u = \frac{v}{x_1}$ where v is the new control to be defined.

According to the standard backstepping procedure, in Step 1, α_1 is to be designed to stabilize the $(z_1 = x_1)$ - subsystem

$$\dot{z}_1 = \theta z_2 + \theta \alpha_1 - \theta z_1, \quad z_2 = x_2 - \alpha_1 \tag{152}$$

Noticing that θz_1 is stabilizing though $\theta > 0$ is unknown, choose $V_1 = \frac{1}{2} z_1^2$ as the Lyapunov function candidate which is different from the standard function candidate $V_1 = \frac{1}{2} z_1^2 + \frac{1}{2\gamma}(\hat{\theta} - \theta)^2$ in backstepping design for unknown parametric strict-feedback systems. The derivative of V_1 is given by

$$\dot{V}_1 = -\theta z_1^2 + \theta z_1 \alpha_1 + \theta z_1 z_2 \tag{153}$$

Noticing that $\theta > 0$, and choosing $\alpha_1 = -c_0 z_1$, $c_0 > 0$ for convenience, we have

$$\dot{V}_1 = -c_1 z_1^2 + \theta z_1 z_2, \quad c_1 = c_0 \theta + \theta > 0 \tag{154}$$

In Step 2, the actual control $u = \alpha_2$ is to be designed to stabilize the (z_1, z_2)- system avoiding any possible technical difficulty caused by $x_1 = 0$. Noting $\alpha_1 = -c_0 x_1$, the derivative of $z_2 = x_2 - \alpha_1$ is

$$\dot{z}_2 = -c_0 \theta z_2 + (c_0^2 + c_0) \theta x_1 - x_1 u \tag{155}$$

Consider the following Lyapunov function candidate

$$V_2 = \kappa V_1 + \frac{1}{2} z_2^2 + \frac{1}{2\gamma} \tilde{\theta}^2, \quad \text{constants } \kappa, \ \gamma > 0, \ \tilde{\theta} = \theta - \hat{\theta} \tag{156}$$

Its time derivative is

$$\begin{aligned}\dot{V}_2 &= -\kappa c_1 z_1^2 - c_0\theta z_2^2 + z_2[(c_0^2 + c_0 + \kappa)\theta x_1 - x_1 u] + \frac{1}{\gamma}\tilde{\theta}\dot{\tilde{\theta}} \\ &= -\kappa c_1 z_1^2 - c_0\theta z_2^2 + z_2 x_1[(c_0^2 + c_0 + \kappa)\hat{\theta} - u] + z_2 x_1 (c_0^2 + c_0 + \kappa)\tilde{\theta} - \frac{1}{\gamma}\tilde{\theta}\dot{\hat{\theta}} \qquad (157)\end{aligned}$$

Apparently, the adaptive control laws

$$\begin{aligned} u &= (c_0^2 + c_0 + \kappa)\hat{\theta} \\ \dot{\hat{\theta}} &= \gamma(c_0^2 + c_0 + \kappa) z_2 x_1 \qquad (158)\end{aligned}$$

gives $\dot{V}_2 = -\kappa c_1 z_1^2 - c_0\theta z_2^2 < 0$ for $\mathbf{z} \neq 0$. It follows that $\mathbf{z} = 0$ is global asymptotically stable, and $\hat{\theta}(t)$ is global asymptotically bounded. In view of $z_1 = x_1$, $z_2 = x_2 - \alpha_1$ and $\alpha_1 = -c_0 z_1$, this implies that x_1 and x_2 go to zeros asymptotically.

Smooth uncertainties have been discussed so far. For non-smooth uncertainties, robust control strategies, such as sliding mode control, can be employed. If the resulting control laws are non-smooth, they can then be approximated with smooth functions. In these case, the asymptotic stability of the system may not be claimed except for the convergence to a small neighbourhood of the origin. In addition, the parameter adaptation algorithms may have to be modified as well to improve the robustness of the system. The commonly used schemes are the σ- modification, e_1- modifications, and projection algorithm, among others to guarantee the boundedness of the closed-loop systems. A bit off the track, Lyapunov design, as the primary tool for nonlinear system design, is used to establish the existence of sliding surface in the literature of Sliding Mode Control which has received a great deal of attention because of its simplicity, robustness to parametric uncertainties and disturbances, and guaranteed transient performance (see *Sliding Mode Control*)

In a more general case, on-line function approximators using neural networks, fuzzy logic systems, or polynomial techniques, can be used to parametrize the unknown over a compact set, and controller design based the Lyapunov design can then be constructed accordingly. The basic idea behind this is to use neural networks, fuzzy systems or any other function approximations, to parameterize the unknown $f(\mathbf{x})$ and $g(\mathbf{x})$ and then design adaptive laws based on Lyapunov analysis (see *Fuzzy Control Systems, Neural Control Systems*). In this case, of course, the resulting controllers only guarantee the stability of the systems locally or semi-globally.

In this article, Lyapunov design has been investigated for several classes of systems. As the topic is so much open ended and problem dependent, this article is never meant to be complete. In the following, some thoughts on further research are itemized below.

(a) Few results are available in the literature on adaptive control of general MIMO nonlinear systems. This article only addressed the control problem of a special class of MIMO systems – the robotic systems. While the backstepping design techniques can be applied to those MIMO systems that are either in the strict-feedback forms or the inputs appear in a triangular manner, further research is necessary for general MIMO nonlinear systems.

(b) There are ample results on control of affine nonlinear systems, and few results are available for non-affine nonlinear systems. Research on the control of non-affine nonlinear system based on Lyapunov design is an important and challenging problem in further investigation.

(c) All the elegant methods for continuous-time nonlinear systems in strict-feedback form are not directly applicable to discrete-time systems in strict-feedback form. Currently only preliminary results are available, and further results are necessary for controller design of discrete-time nonlinear systems with general unmatched nonlinearities.

(d) Hybrid dynamical systems that contain both discrete and continuous signals are very common in reality. Further research on Lyapunov design is also calling attention.

It is believed that pursuing different ways to utilize the structure properties of different systems would be beneficial in deriving stable control schemes for a large class of multivariable nonlinear systems.

7. Conclusions

In this article, an overview has been given on some state-of-the-art approaches of Lyapunov design, though in general there is no systematic procedure in choosing a suitable Lyapunov function candidate for controller design for a given nonlinear system. After a brief introduction and historic review, this article presented (i) the basic concepts of Lyapunov stability and control Lyapunov functions, (ii) Lyapunov equations and model reference adaptive control based on Lyapunov design for matched systems, (iii) Lyapunov redesign, adaptive redesign and robust design for matched systems, (iv) adaptive backstepping design for unmatched nonlinear systems, (v) Lyapunov design by exploiting physical properties for special classes of systems, and (vi) design flexibilities and considerations in actual design.

Acknowledgements

The author is grateful to H. Unbehauen for the opportunity to contribute to the EOLSS and to M. Krstic, F.L. Lewis, I.M.Y. Mareels and M.W. Spong for their helpful suggestions in the preparation of the manuscript.

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