

# APPENDIX A

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## REVIEW OF MATRIX ALGEBRA

We present here a brief review of some concepts that are assumed as background for the text. Good references include Gantmacher (1977), Brogan (1974), and Strang (1980).

### A.1 BASIC DEFINITIONS AND FACTS

The *determinant* of an  $n \times n$  matrix is symbolized as  $|A|$ . If  $A$  and  $B$  are both square, then

$$|A| = |A^T|, \quad (\text{A.1-1})$$

$$|AB| = |A| \cdot |B|, \quad (\text{A.1-2})$$

where the superscript T represents transpose. If  $A \in C^{m \times n}$  and  $B \in C^{n \times m}$  (where  $n$  can equal  $m$ ), then

$$\text{trace}(AB) = \text{trace}(BA) \quad (\text{A.1-3})$$

$$|I_m + AB| = |I_n + BA|. \quad (\text{A.1-4})$$

( $C$  represents the complex numbers.)

For any matrices  $A$  and  $B$ ,

$$(AB)^T = B^T A^T \quad (\text{A.1-5})$$

and if  $A$  and  $B$  are nonsingular, then

$$(AB)^{-1} = B^{-1} A^{-1}. \quad (\text{A.1-6})$$

The *Kronecker product* of two matrices  $A = [a_{ij}] \in C^{m \times n}$  and  $B = [b_{ij}] \in C^{p \times q}$  is

$$A \otimes B = [a_{ij}B] \in C^{mp \times nq}. \tag{A.1-7}$$

(It is sometimes defined as  $A \otimes B = [Ab_{ij}]$ .) If  $A = [a_1 a_2 \cdots a_n]$ , where  $a_i$  are the columns of  $A$ , the *stacking operator* is defined by

$$s(A) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}. \tag{A.1-8}$$

It converts  $A \in C^{m \times n}$  into a vector  $s(A) \in C^{mn}$ . An identity that is often useful is

$$s(ABD) = (D^T \otimes A)s(B). \tag{A.1-9}$$

If  $A \in C^{m \times m}$  and  $B \in C^{p \times p}$ , then

$$|A \otimes B| = |A|^p \cdot |B|^m. \tag{A.1-10}$$

See Brewer (1978) for other results.

If  $\lambda_i$  is an eigenvalue of  $A$  with eigenvector  $v_i$ , then  $1/\lambda_i$  is an eigenvalue of  $A^{-1}$  with the same eigenvector, for

$$Av_i = \lambda_i v_i \tag{A.1-11}$$

implies that

$$\lambda_i^{-1} v_i = A^{-1} v_i. \tag{A.1-12}$$

If  $\lambda_i$  is an eigenvalue of  $A$  with eigenvector  $\omega_i$ , and  $\mu_j$  is an eigenvalue of  $B$  with eigenvector  $w_j$ , then  $\lambda_i \mu_j$  is an eigenvalue of  $A \otimes B$  with eigenvector  $v_i \otimes w_j$  (Brewer 1978).

## A.2 PARTITIONED MATRICES

If

$$D = \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix}, \tag{A.2-1}$$

where  $A_{ij}$  are matrices, then we write  $D = \text{diag}(A_{11}, A_{22}, A_{33})$  and say that  $D$  is *block diagonal*. If the  $A_{ii}$  are square, then

$$|D| = |A_{11}| \cdot |A_{22}| \cdot |A_{33}|, \tag{A.2-2}$$

and if  $|D| \neq 0$ , then

$$D^{-1} = \text{diag}(A_{11}^{-1}, A_{22}^{-1}, A_{33}^{-1}). \tag{A.2-3}$$

If

$$D = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}, \quad (\text{A.2-4})$$

where  $A_{ij}$  are matrices, then  $D$  is *upper block triangular* and (A.2-2) still holds. *Lower block triangular* matrices have the form of the transpose of (A.2-4).

If

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad (\text{A.2-5})$$

we define the *Schur complement* of  $A_{22}$  as

$$D_{22} = A_{22} - A_{21}A_{11}^{-1}A_{12} \quad (\text{A.2-6})$$

and the *Schur complement* of  $A_{11}$  as

$$D_{11} = A_{11} - A_{12}A_{22}^{-1}A_{21}. \quad (\text{A.2-7})$$

The inverse of  $A$  can be written

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} + A_{11}^{-1}A_{12}D_{22}^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}D_{22}^{-1} \\ -D_{22}^{-1}A_{21}A_{11}^{-1} & D_{22}^{-1} \end{bmatrix}, \quad (\text{A.2-8})$$

$$A^{-1} = \begin{bmatrix} D_{11}^{-1} & -D_{11}^{-1}A_{12}A_{22}^{-1} \\ -A_{22}^{-1}A_{21}D_{11}^{-1} & A_{22}^{-1} + A_{22}^{-1}A_{21}D_{11}^{-1}A_{12}A_{22}^{-1} \end{bmatrix}, \quad (\text{A.2-9})$$

or

$$A^{-1} = \begin{bmatrix} D_{11}^{-1} & -A_{11}^{-1}A_{12}D_{22}^{-1} \\ -A_{22}^{-1}A_{21}D_{11}^{-1} & D_{22}^{-1} \end{bmatrix}, \quad (\text{A.2-10})$$

depending, of course, on whether  $|A_{11}| \neq 0$ ,  $|A_{22}| \neq 0$ , or both. These can be verified by checking that  $AA^{-1} = A^{-1}A = I$ . By comparing these various forms, we obtain the well-known *matrix inversion lemma*

$$(A_{11}^{-1} + A_{12}A_{22}A_{21})^{-1} = A_{11} - A_{11}A_{12}(A_{21}A_{11}A_{12} + A_{22}^{-1})^{-1}A_{21}A_{11}. \quad (\text{A.2-11})$$

The Schur complement arises naturally in the solution of linear simultaneous equations, for if

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ Z \end{bmatrix}, \quad (\text{A.2-12})$$

then from the first equation

$$X = -A_{11}^{-1}A_{12}Y,$$

and using this in the second equation yields

$$(A_{22} - A_{21}A_{11}^{-1}A_{12})Y = Z. \tag{A.2-13}$$

If  $A$  is given by (A.2-5), then

$$|A| = |A_{11}| \cdot |A_{22} - A_{21}A_{11}^{-1}A_{12}| = |A_{22}| \cdot |A_{11} - A_{12}A_{22}^{-1}A_{21}|. \tag{A.2-14}$$

Therefore, the determinant of  $A$  is the product of the determinant of  $A_{11}$  (or  $A_{22}$ ) and the determinant of the Schur complement of  $A_{22}$  (or  $A_{11}$ ).

### A.3 QUADRATIC FORMS AND DEFINITENESS

If  $x \in R^n$  is a vector, then the square of the Euclidean norm is

$$\|x\|^2 = x^T x. \tag{A.3-1}$$

If  $S$  is any nonsingular transformation, the vector  $Sx$  has a norm squared of  $(Sx)^T Sx = x^T S^T Sx$ . Letting  $P = S^T S$ , we write

$$\|x\|_P^2 = x^T P x \tag{A.3-2}$$

as the norm squared of  $Sx$ . We call  $\|x\|_P$  the *norm of  $x$*  with respect to  $P$ . We call

$$x^T Q x \tag{A.3-3}$$

a *quadratic form*. We shall assume  $Q$  is real.

Every real square matrix  $Q$  can be decomposed into a *symmetric part*  $Q_s$  (i.e.,  $Q_s^T = Q_s$ ) and an *antisymmetric part*  $Q_a$  (i.e.,  $Q_a^T = -Q_a$ ):

$$Q = Q_s + Q_a, \tag{A.3-4}$$

where

$$Q_s = (Q + Q^T)/2, \tag{A.3-5}$$

$$Q_a = (Q - Q^T)/2. \tag{A.3-6}$$

If the quadratic form  $x^T A x$  has  $A$  antisymmetric, then it must be equal to zero since  $x^T A x$  is a scalar, so that  $x^T A x = (x^T A x)^T = x^T A^T x = -x^T A x$ . For a general real square  $Q$ , then

$$x^T Q x = x^T (Q_s + Q_a) x = x^T Q_s x. \tag{A.3-7}$$

We can therefore assume without loss of generality that  $Q$  in (A.3-3) is symmetric. Let us do so.

We say  $Q$  is:

*Positive definite* ( $Q > 0$ ) if  $x^T Q x > 0$  for all nonzero  $x$ .

*Positive semi-definite* ( $Q \geq 0$ ) if  $x^T Q x \geq 0$  for all nonzero  $x$ .

*Negative semi-definite* ( $Q \leq 0$ ) if  $x^T Q x \leq 0$  for all nonzero  $x$ .

*Negative definite* ( $Q < 0$ ) if  $x^T Q x < 0$  for all nonzero  $x$ .

*Indefinite* if  $x^T Q x > 0$  for some  $x$ ,  $x^T Q x < 0$  for other  $x$ .

We can test for definiteness independently of the vectors  $x$ . If  $\lambda_i$  are the eigenvalues of  $Q$ , then

$$\begin{aligned} Q > 0 & \quad \text{if all } \lambda_i > 0, \\ Q \geq 0 & \quad \text{if all } \lambda_i \geq 0, \\ Q \leq 0 & \quad \text{if all } \lambda_i \leq 0, \\ Q < 0 & \quad \text{if all } \lambda_i < 0. \end{aligned} \tag{A.3-8}$$

Another test is provided as follows. Let  $Q = [q_{ij}] \in R^{n \times n}$ . The *leading minors* or  $Q$  are

$$\begin{aligned} m_1 &= q_{11}, \\ m_2 &= \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix}, \\ m_3 &= \begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix}, \dots, \\ m_n &= |Q|. \end{aligned} \tag{A.3-9}$$

In terms of the minors, we have

$$\begin{aligned} Q > 0 & \quad \text{if } m_i > 0, \text{ all } i, \\ Q \geq 0 & \quad \text{if all principal minor not only leading minors) \\ & \quad \text{are nonnegative.} \\ Q \leq 0 & \quad \text{if } -Q \geq 0, \\ Q < 0 & \quad \text{if } \begin{cases} m_i < 0, & \text{all odd } i \\ m_i > 0, & \text{all even } i \end{cases} \end{aligned} \tag{A.3-10}$$

Any positive semidefinite matrix  $Q$  can be factored into *square roots* either as

$$Q = \sqrt{Q} \sqrt{Q}^T \tag{A.3-11}$$

or as

$$Q = \sqrt{Q}^T \sqrt{Q}. \tag{A.3-12}$$

The (“left” and “right”) square roots in (A.3-11) and (A.3-12) are not in general the same. Indeed,  $Q$  may have several roots since each of these factorizations is not even unique. If  $Q > 0$ , then all square roots are nonsingular.

If  $P > 0$ , then (A.3-2) is a norm. If  $P \geq 0$ , it is called a *seminorm* since  $x^T P x$  may be zero even if  $x$  is not.

### A.4 MATRIX CALCULUS

Let  $x \in C^n = [x_1 \ x_2 \ \cdots \ x_n]^T$  be a vector,  $s \in C$  be a scalar, and  $f(x) \in C^m$  be an  $m$ -vector function of  $x$ . The differential in  $x$  is

$$dx = \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix}, \tag{A.4-1}$$

and the derivative of  $x$  with respect to  $s$  (which could be time) is

$$\frac{dx}{ds} = \begin{bmatrix} dx_1/ds \\ dx_2/ds \\ \vdots \\ dx_n/ds \end{bmatrix}. \tag{A.4-2}$$

If  $s$  is a function of  $x$ . Then the *gradient* of  $s$  with respect to  $x$  is the *column* vector

$$s_x \triangleq \frac{\partial s}{\partial x} = \begin{bmatrix} \partial s / \partial x_1 \\ \partial s / \partial x_2 \\ \vdots \\ \partial s / \partial x_n \end{bmatrix}. \tag{A.4-3}$$

(The gradient is defined as a row vector in some references.) Then the total differential in  $s$  is

$$ds = \left( \frac{\partial s}{\partial x} \right)^T dx = \sum_{i=1}^n \frac{\partial s}{\partial x_i} dx_i. \tag{A.4-4}$$

If  $s$  is a function of two vectors  $x$  and  $y$ , then

$$ds = \left( \frac{\partial s}{\partial x} \right)^T dx + \left( \frac{\partial s}{\partial y} \right)^T dy. \tag{A.4-5}$$

The *Hessian* of  $s$  with respect to  $x$  is the second derivative

$$s_{xx} \triangleq \frac{\partial^2 s}{\partial x^2} = \left[ \frac{\partial^2 s}{\partial x_i \partial x_j} \right], \tag{A.4-6}$$

which is a symmetric  $n \times n$  matrix. In terms of the gradient and the Hessian, the *Taylor series expansion* of  $s(x)$  about  $x_0$  is

$$s(x) = s(x_0) + \left( \frac{\partial s}{\partial x} \right)^T (x - x_0) + \frac{1}{2} (x - x_0)^T \frac{\partial^2 s}{\partial x^2} (x - x_0) + O(3), \tag{A.4-7}$$

where  $O(3)$  represents terms of order 3, and  $s_x$  and  $s_{xx}$  are evaluated at  $x_0$ .

The *Jacobian* of  $f$  with respect to  $x$  is the  $m \times n$  matrix

$$f_x \triangleq \frac{\partial f}{\partial x} = \left[ \frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \cdots \frac{\partial f}{\partial x_n} \right], \quad (\text{A.4-8})$$

so that the total differential of  $f$  is

$$df = \frac{\partial f}{\partial x} dx = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i. \quad (\text{A.4-9})$$

We shall use the shorthand notation

$$\frac{\partial f^T}{\partial x} \triangleq \left( \frac{\partial f}{\partial x} \right)^T \in C^{n \times m}. \quad (\text{A.4-10})$$

If  $y$  is a vector and  $A, B, D, Q$  are matrices, all with dimensions so that the following expressions make sense, then we have the following results:

$$\frac{d}{dt}(A^{-1}) = -A^{-1} \dot{A} A^{-1}. \quad (\text{A.4-11})$$

Some useful gradients are

$$\frac{\partial}{\partial x}(y^T x) = \frac{\partial}{\partial x}(x^T y) = y, \quad (\text{A.4-12})$$

$$\frac{\partial}{\partial x}(y^T A x) = \frac{\partial}{\partial x}(x^T A^T y) = A^T y, \quad (\text{A.4-13})$$

$$\frac{\partial}{\partial x}(y^T f(x)) = \frac{\partial}{\partial x}(f^T(x) y) = f_x^T y, \quad (\text{A.4-14})$$

$$\frac{\partial}{\partial x}(x^T A x) = A x + A^T x, \quad (\text{A.4-15})$$

and if  $Q$  is symmetric, then

$$\frac{\partial}{\partial x}(x^T Q x) = 2Qx, \quad (\text{A.4-16})$$

$$\frac{\partial}{\partial x}(x - y)^T Q(x - y) = 2Q(x - y). \quad (\text{A.4-17})$$

The chain rule for two vector functions becomes

$$\frac{\partial}{\partial x}(f^T y) = f_x^T y + y_x^T f. \quad (\text{A.4-18})$$

Some useful Hessians are

$$\frac{\partial^2 x^T A x}{\partial x^2} = A + A^T, \quad (\text{A.4-19})$$

and if  $Q$  is symmetric

$$\frac{\partial^2 x^T Q x}{\partial x^2} = 2Q, \tag{A.4-20}$$

$$\frac{\partial^2}{\partial x^2} (x - y)^T Q (x - y) = 2Q. \tag{A.4-21}$$

Some useful Jacobians are

$$\frac{\partial}{\partial x} (Ax) = A \tag{A.4-22}$$

(contrast this with (A.4-12)), and the chain rule

$$\frac{\partial}{\partial x} (sf) = \frac{\partial}{\partial x} (fs) = sf_x + fs_x^T \tag{A.4-23}$$

(contrast this with (A.4-18)).

Some useful derivatives involving the trace and determinant are

$$\frac{\partial}{\partial A} \text{trace}(A) = I, \tag{A.4-24}$$

$$\frac{\partial}{\partial A} \text{trace}(BAD) = B^T D^T, \tag{A.4-25}$$

$$\frac{\partial}{\partial A} \text{trace}(ABA^T) = 2AB, \text{ if } B = B^T \tag{A.4-26}$$

$$\frac{\partial}{\partial A} |BAD| = |BAD|A^{-T}, \tag{A.4-27}$$

where  $A^{-T} \triangleq (A^{-1})^T$ .

### A.5 THE GENERALIZED EIGENVALUE PROBLEM

Consider the generalized eigenvalue problem

$$Gz = \mu Fz, \tag{A.5-1}$$

where

$$\det(\mu F - G) \equiv 0. \tag{A.5-2}$$

Then the finite generalized eigenvalues are the roots of  $\det(\mu F - G)$ . Let  $\mu_i$  be the roots of  $\det(\mu F - G)$  and define

$$\eta_i = \dim \ker(\mu_i F - G). \tag{A.5-3}$$

Then the rank 1 finite generalized eigenvectors are defined by

$$(\mu_i F - G)z_{ij}^1 = 0, j \in \hat{\eta}_i \tag{A.5-4}$$



(where  $\hat{\eta}_i = \{1, 2, \dots, \eta_i\}$ ) and the rank  $k$  finite eigenvectors for  $k > 1$  and each  $i$  and  $j$  by

$$(\mu_i F - G)z_{ij}^{k+1} = -Fz_{ij}^k, \quad k \geq 1. \quad (\text{A.5-5})$$

If  $F$  is nonsingular, the above equation can be used to solve recursively for the  $z_{ij}^k$  beginning with the highest rank eigenvector in each chain. In that case this construction provides the eigenstructure of  $F^{-1}G$ . In the case where  $F$  is singular, the above equation cannot generally be used to recursively generate the  $z_{ij}^k$ . Furthermore, there exist eigenvalues at infinity and corresponding eigenvectors that can be constructed as follows. Define  $\eta = \dim \ker(F)$ . Then the rank 1 infinite eigenvectors are defined by

$$Fz_{\infty j}^1 = 0, \quad j = \hat{\eta} \quad (\text{A.5-6})$$

and the rank  $k$  infinite eigenvectors for  $k > 1$  and each  $j$  by

$$Fz_{\infty j}^{k+1} = Gz_{\infty j}^k, \quad k \geq 1. \quad (\text{A.5-7})$$

By arranging the eigenvectors as the columns of two nonsingular matrices according to

$$Z = [z_{ij}^k | z_{\infty j}^k], \quad W = [Fz_{ij}^k | Gz_{\infty j}^k] \quad (\text{A.5-8})$$

with  $i, j, k$  incrementing in odometer order, then

$$W^{-1}FV = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \quad W^{-1}GV = \begin{bmatrix} M & 0 \\ 0 & I \end{bmatrix}, \quad (\text{A.5-9})$$

where  $M$  is a Jordan form matrix containing the finite generalized eigenvalues of  $(G, F)$  and  $N$  is a nilpotent Jordan matrix representing the infinite generalized eigenvalues. The above canonical form is also known as the *Weierstrass form*.

## REFERENCES

- Abu-Khalaf, M., and F. L. Lewis, “Nearly optimal control laws for nonlinear systems with saturating actuators using a neural network HJB approach,” *Automatica*, **41**, 779–791 (2005).
- Abu-Khalaf, M., F. L. Lewis, and Jie Huang, “Policy iterations on the Hamilton-Jacobi-Isaacs equation for  $H_\infty$  state feedback control with input saturation,” *IEEE Trans. Automatic Control*, **51** (12), 1989–1995 (2006).
- Abu-Khalaf, M., J. Huang, and F. L. Lewis, *Nonlinear  $H_2/H$ -Infinity Constrained Feedback Control: A Practical Design Approach Using Neural Networks*, Berlin: Springer-Verlag, 2006.
- Abu-Khalaf, M., F. L. Lewis, and J. Huang, “Neurodynamic programming and zero-sum games for constrained control systems,” *IEEE Trans. Neural Networks*, **19** (7), 1243–1252 (2008).
- Al-Tamimi, A., F. L. Lewis, and M. Abu-Khalaf, “Discrete-time nonlinear HJB solution using approximate dynamic programming: convergence proof,” *IEEE Trans. Systems, Man, Cybernetics, Part B*, **38** (4), 943–949 (2008).
- Anderson, B. D. O., and Y. Liu, “Controller reduction: concepts and approaches,” *IEEE Trans. Automatic Control*, AC-34, 802–812 (1989).
- Anderson, B. D. O., and J. B. Moore, *Linear Optimal Control*, Englewood Cliffs, NJ: Prentice-Hall, 1971.
- Armstrong, E. S., *ORACLs, A Design System for Linear Multivariable Control*, New York: Dekker, 1980.
- Åström, K. J., and B. Wittenmark, *Computer Controlled Systems*, Englewood Cliffs, NJ: Prentice-Hall, 1984.
- Athans, M., “A tutorial on the LQG/LTR method,” *Proc. Am. Control Conf.*, 1289–1296 (1986).

- Athans, M., and P. Falb, *Optimal Control*, New York: McGraw-Hill, 1966.
- Athans, M., P. Kapsouris, E. Kappos, and H. A. Spang III, "Linear quadratic Gaussian with loop-transfer recovery methodology for the F-100 engine," *J. Guid.*, 9, 45–52 (1986).
- Baird, L., "Reinforcement learning in continuous time: advantage updating," *Proc. International Conference on Neural Networks*, Orlando, FL, June 1994.
- Balakrishnan, S. N., J. Ding, and F. L. Lewis, "Issues on stability of ADP feedback controllers for dynamical systems," *IEEE Trans. Systems, Man, Cybernetics, Part B*, **38** (4), 913–917 (2008).
- Bardi, M., and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*, Boston: Birkhauser, 1997.
- Bartels, R. H., and G. W. Stewart, "Solution of the matrix equation  $AX + XB = C$ ," *Commun. ACM* 15 (6), 820–826 (1984).
- Barto, A. G., R. S. Sutton, and C. Anderson. "Neuron-like adaptive elements that can solve difficult learning control problems," *IEEE Trans. Systems, Man Cybernetics*, **SMC-13**, 834–846 (1983).
- Başar, T., and G. J. Olsder, *Dynamic Noncooperative Game Theory*, 2nd ed., Philadelphia, PA: SIAM, 1999.
- Bell, R. F., E. W. Johnson, R. V. Whitaker, and R. V. Wilcox, "Head positioning in a large disk drive," *Hewlett Packard J.*, pp. 14–20, Jan. 1984.
- Bellman, R. E., *Dynamic Programming*, Princeton, NJ: Princeton University Press, 1957.
- Bellman, R. E., and S. E. Dreyfus, *Applied Dynamic Programming*, Princeton, NJ, Princeton University Press, 1962.
- Bellman, R. E., and R. E. Kalaba, *Dynamic Programming and Modern Control Theory*, Orlando, FL Academic Press, 1965.
- Bertsekas, D. P., and J. N. Tsitsiklis, *Neuro-dynamic Programming*, Athena Scientific, Cambridge, MA, 1996.
- Bierman, G. J., *Factorization Methods for Discrete Sequential Estimation*, Orlando FL: Academic Press, 1977.
- Bittanti, S., A. J. Laub, and J. C. Willems, *The Riccati Equation*, New York: Springer-Verlag, 1991.
- Blakelock, J. H., *Automatic Control of Aircraft and Missiles*, New York: Wiley, 1965.
- Bradtke, S., B. Ydstie, and A. Barto, *Adaptive Linear Quadratic Control Using Policy Iteration*, report CMPSCI-94-49, University of Massachusetts, June 1994.
- Brewer, J. W., "Kronecker products and matrix calculus in system theory," *IEEE Trans. Circuits Systems*, CAS-25 (9), 772–781 (1978).
- Brogan, W. L., *Modern Control Theory*, New York: Quantum, 1974.
- Broussard, J., and N. Halyo, "Active flutter control discrete optimal constrained dynamic compensators," *Proc. Am. Control Conf.*, 1026–1034 (1983).
- Bryson, A. E., Jr., and Y. C. Ho, *Appl. Optimal Control*, New York: Hemisphere, 1975.
- Businger, P., and G. H. Golub, "Linear least squares solution by householder transformations," *Numer. Math.*, 7, 269–276 (1965).
- Busoniu, L., R. Babuska, B. De Schutter, and D. Ernst, *Reinforcement Learning and Dynamic Programming Using Function Approximators*, Boca Raton, FL: CRC, 2009.
- Cao, X., *Stochastic Learning and Optimization*, Berlin: Springer-Verlag, 2007.

- Casti, J., *Dynamical Systems and Their Applications: Linear Theory*, Orlando, FL: Academic Press, 1977.
- Casti, J., "The linear quadratic control problem: some recent results and outstanding problems," *SIAM Rev.*, 22 (4), 459–485 (1980).
- Chang, S. S. L., *Synthesis of Optimum Control Systems*, New York: McGraw-Hill, 1961.
- Chen, B. M., Z. Lin, and Y. Shamash, *Linear Systems Theory: a Structural Decomposition Approach*, Boston: Birkhauser, 2004.
- Clarke, D. W., and P. J. Gawthrop, "Self-tuning controller," *Proc. IEE*, 122 (9), 929–934 (1975).
- Darwin, C., *On the Origin of Species by Means of Natural Selection*, London: J. Murray, 1859.
- Davison, E. J., and I. J. Ferguson, "The design of controllers for the multivariable robust servomechanism problem using parameter optimization methods," *IEEE Trans. Automatic Control*, AC-26, 93–110 (1981).
- Doya, K. "Reinforcement learning in continuous time and space," *Neural Computation*, vol. 12, pp. 219–245, MIT Press, 2000.
- Doya, K., H. Kimura, and M. Kawato, "Neural mechanisms for learning and control," *IEEE Control Systems Magazine*, 42–54 (2001).
- Doyle, J. C., "Guaranteed margins for LQG regulators," *IEEE Trans. Automatic Control*, AC-23, 756–757 (1978).
- Doyle, J. C., and G. Stein, "Robustness with observers," *IEEE Trans. Automatic Control*, AC-24, 607–611 (1979).
- Doyle, J. C., and G. Stein, "Multivariable feedback design: concepts for a classical/modern synthesis," *IEEE Trans. Automatic Control*, AC-26, 4–16 (1981).
- Doyle, J. C., K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard  $H_2$  and  $H_\infty$  control problems," *IEEE Trans. Automatic Control*, AC-34, 831–847 (1989).
- Dyer, P., and S. R. McReynolds, "Extension of square root filtering to include process noise," *J. Optimiz. Theory Applic.*, 3 (6), 444 (1969).
- Elbert, T. F., *Estimation and Control of Systems*, New York: Van Nostrand Reinhold, 1984.
- Francis, B. A., *A Course in  $H_\infty$  Control Theory*, Springer Verlag, Lecture notes in *Control and Info. Sci.*, 88, (1986).
- Francis, B. A., and J. C. Doyle, "Linear control theory with an  $H_\infty$  optimality criterion," *SIAM J. Control Optim.*, 815–844 (1987).
- Francis, B. A., J. W. Helton, and G. Zames, " $H_\infty$ -optimal feedback controllers for linear multivariable systems," *IEEE Trans. Automatic Control*, AC-29, 888–900 (1984).
- Franklin, G. F., and J. D. Powell, *Digital Control of Dynamic Systems*, Reading, MA: Addison-Wesley, 1980.
- Franklin, G. F., J. D. Powell, and A. Emami-Naeini, *Feedback Control of Dynamic Systems*, Reading, MA: Addison-Wesley, 1986.
- Fulks, W., *Advanced Calculus*, New York: Wiley, 1967.
- Gangsaas, D., K. R. Bruce, J. D. Blight, and U.-L. Ly, "Application of modern synthesis to aircraft control: three case studies," *IEEE Trans. Automatic Control*, AC-31, 995–1014 (1986).
- Gantmacher, F. R., *The Theory of Matrices*, New York: Chelsea, 1977.

- Gawthrop, P. J., "Some interpretations of the self-tuning controller," *Proc. IEEE Control Sci.*, 124 (10), 889–894 (1977).
- Gelb, A., ed., *Applied Optimal Estimation*, Cambridge, MA: MIT Press, 1974.
- Golub, G. H., S. Nash, and C. Van Loan, "A Hessenberg-Schur method for the matrix problem  $AX + XB = C$ ," *IEEE Trans. Automatic Control*, AC-24, 909–913 (1979).
- Green, M., and D. Limebeer, *Robust Control Theory*, Englewood Cliffs, NJ: Prentice-Hall, 1993.
- Grimble, M. J., and M. A. Johnson, *Optimal Control and Stochastic Estimation: Theory and Applications*, vol. 1, New York: Wiley, 1988.
- Hanselmann, T., L. Noakes, and A. Zaknich, "Continuous-time adaptive critics," *IEEE Trans. Neural Networks*, 18 (3), 631–647 (2007).
- Harvey, C. A., and G. Stein, "Quadratic weights for asymptotic regulator properties," *IEEE Trans. Automatic Control*, AC-23, 378–387 (1978).
- Hewer, G. A., "An iterative technique for the computation of steady state gains for the discrete optimal regulator," *IEEE Trans. Automatic Control*, 16 (4), 382–384 (1971).
- IMSL, *Library Contents Document*, 8th ed., International Mathematical and Statistical Libraries, Inc., 7500 Bellaire Blvd., Houston, Texas, 77036, 1980.
- Ioannou, P., and B. Fidan, *Adaptive Control Tutorial*, Philadelphia: SIAM Press, 2006.
- Jadbabaie, A., J. Lin, and S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Automatic Control*, **48** (6), 988–1001 (2003).
- Kailath, T., *Linear Systems*, Englewood Cliffs, NJ: Prentice-Hall, 1980.
- Kalman, R. E., "New methods in Wiener filtering Theory," *Proceedings of the Symposium on Engineering Applications of Random Function Theory and Probability*, New York: Wiley, 1963.
- Kalman, R. E., and R. S. Bucy, "New results in linear filtering and prediction theory," *Trans. ASME J. Basic Eng.*, 83, 95–108 (1961).
- Kaminski, P. G., A. E. Bryson, and S. F. Schmidt, "Discrete square root filtering: a survey of current techniques," *IEEE Trans. Automatic Control*, AC-16 (6), 727–736 (1971).
- Kimura, H., Y. Lu, and R. Kawatani, "On the structure of  $H_\infty$  control systems and related extensions," *IEEE Trans. Automatic Control*, AC-36, 653–667 (1991).
- Kirk, D. E., *Optimal Control Theory*, Englewood Cliffs, NJ: Prentice-Hall, 1970.
- Kleinman, D. L., "On an iterative technique for Riccati equation computations," *IEEE Trans. Automatic Control*, **AC-13** (1), 114–115. (1968).
- Knobloch, H. W., A. Isidori, and D. Flokcerzi, *Topics in Control Theory*, Berlin: Springer-Verlag, 1993.
- Koivo, H. N., "A multivariable self-tuning controller," *Automatica*, 16, 351–366 (1980).
- Kreindler, E., and D. Rothschild, "Model-following in linear quadratic regulator," *AIAA J.*, 14 (7), 835–842 (1976).
- Kučera, V., *Discrete Linear Control, The Polynomial Equation Approach*, New York: Wiley, 1979.
- Kwakernaak, H., and R. Sivan, *Linear Optimal Control Systems*, New York: Wiley-Interscience, 1972.
- Lancaster, P., and L. Rodman, *Algebraic Riccati Equations*, Oxford University Press, UK, 1995.

- Laub, A. J., "A Shur Method for Solving Algebraic Riccati Equations," *IEEE Trans. Automatic Control*, AC-24, 913–921 (1979).
- Laub, A. J., "Efficient Multivariable Frequency Response Computations," *IEEE Trans. Automatic Control*, AC-26, 407–408 (1981).
- Letov, A. M., "Analytical Controller Design, I, II," *Autom. Remote Control*, 21, 303–306 (1960).
- Levine, W. S., and M. Athans, "On the Determination of the Optimal Constant Output Feedback Gains for Linear Multivariable Systems," *IEEE Trans. Automatic Control*, AC-15, 44–48 (1970).
- Lewis, F. L., *Optimal Estimation*, New York: Wiley, 1986.
- Lewis, F. L., and D. Vrabie, "Reinforcement learning and adaptive dynamic programming for feedback control," *IEEE Circuits Systems Mag.*, 32–38 (2009).
- Lewis, F. L., L. Xie, and D. Popa, *Optimal & Robust Estimation: With an Introduction to Stochastic Control Theory*, 2nd ed., Boca Raton, FL: CRC Press, 2007.
- Lewis, F. L., G. Lendaris, and Derong Liu, "Special issue on approximate dynamic programming and reinforcement learning for feedback control," *IEEE Trans. Systems, Man Cybernetics, Part B*, 38 (4) (2008).
- Li, Z. H., and M. Krstic, "Optimal design of adaptive tracking controllers for nonlinear systems," *Automatica*, 33 (8), 1459–1473 (1997).
- Ljung, L., *System Identification*, Englewood Cliffs, NJ: Prentice-Hall, 1999.
- Luenberger, D. G., *Optimization by Vector Space Methods*, New York: Wiley, 1969.
- Luenberger, D. G., *Introduction to Dynamic Systems*, New York: Wiley, 1979.
- MacFarlane, A. G. J., "Return difference and return-ratio matrices and their use in the analysis and design of multivariable feedback control systems," *Proc. IEE*, 117, 2037–2049 (1970).
- MacFarlane A. G. J., and B. Kouvaritakis, "A Design Technique for Linear Multivariable Feedback Systems," *Int. J. Control*, 25, 837–874 (1977).
- Marion, J. B., *Classical Dynamics of Particles and Systems*, Orlando, FL: Academic Press, 1965.
- MATLAB, The MathWorks, Inc., Cochituate Place, 24 Prime Parkway, Natick, MA 01760, 1992.
- McClamroch, N. H., *State Models of Dynamic Systems*, New York: Springer-Verlag, 1980.
- McFarlane, D., and K. Glover, "A loop shaping design procedure using  $H_\infty$  synthesis," *IEEE Trans. Automatic Control* AC-37, 759–769 (1992).
- McReynolds, S. R., Ph.D. thesis, Harvard University, Cambridge, MA, 1966.
- Medanic, J., "Closed-loop Stackelberg strategies in linear quadratic problems," *IEEE Trans. Automatic Control*, AC-23, 632–637 (1978).
- Mehta, P., and S. Meyn, "Q-learning and Pontryagin's minimum principle," *Proc. IEEE Conf. Decision and Control*, 3598–3605. (2009).
- Mendel, J. M., and R. W. MacLaren, "Reinforcement learning control and pattern recognition systems," in *Adaptive, Learning, and Pattern Recognition Systems: Theory and Applications*, ed. Mendel, J. M., and K. S. Fu, pp. 287–318, New York: Academic Press, 1970.
- Mil. Spec. 1797, *Flying Qualities of Piloted Vehicles*, 1987.

- Moerder, D. D., and A. J. Calise, "Convergence of a numerical algorithm for calculating optimal output feedback gains," *IEEE Trans. Automatic Control*, AC-30, 900–903 (1985).
- Moore, B. C., "Principal component analysis in linear systems: controllability, observability and model reduction," *IEEE Trans. Automatic Control*, AC-26, 17–32 (1982).
- Moore, K. L., *Iterative Learning Control for Deterministic Systems*, London: Springer-Verlag, 1993.
- Morari, M., and E. Zafiriou, *Robust Process Control*, Englewood, NJ: Prentice-Hall, 1989.
- Morf, M., and T. Kailath, "Square root algorithms for least-squares estimation," *IEEE Trans. Automatic Control*, AC-20 (4), 487–497 (1975).
- Murray, J., C. Cox, R. Saeks, and G. Lendaris, "Globally convergent approximate dynamic programming applied to an autolander," *Proc. Am. Control Conf.*, pp. 2901–2906, Arlington, VA, 2001.
- Nelder, J. A., and R. Mead, "A simplex method for function minimization," *Comput. J.* 7, 308–313 (1964).
- Neustic, V., and J. Primbs, *Constrained Nonlinear Optimal Control: A Converse HJB Approach*, Technical Report 96-021, California Institute of Technology, 1996.
- O'Brien, M. J., and J. B. Broussard, "Feedforward control to track the output of a forced model," *Proc. IEEE Conference on Decision and Control*, Dec. 1978.
- Olfati-Saber, R., and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Trans. Automatic Control*, 49 (9), 1520–1533 (2004).
- Papavassilopoulos, G. P., and J. B. Cruz, Jr., "On the existence of solutions to coupled matrix Riccati differential equations in linear quadratic Nash games," *IEEE Trans. Automatic Control*, AC-24, 127–129 (1979).
- Papoulis, A., *Probability, Random Variables, and Stochastic Processes*, 2nd ed., New York: McGraw Hill, 1984.
- Pappas, T., A. J. Laub, and N. R. Sandell, "On the numerical solution of the discrete-time algebraic Riccati equation," *IEEE Trans. Automatic Control*, AC-25, 631–641 (1980).
- Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, New York: Wiley-Interscience, 1962.
- Postlethwaite, I., J. M. Edmunds, and A. G. J. MacFarlane, "Principal gains and principal phases in the analysis of linear multivariable systems," *IEEE Trans. Automatic Control*, AC-26, 32–46 (1981).
- Powell, W. B., *Approximate Dynamic Programming*, Hoboken, NJ: Wiley, 2007.
- Press, W. H., B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes: The Art of Scientific Computing*, New York: Cambridge University Press, 1986.
- Rosenbrock, H. H., *Computer-aided Control System Design*, New York: Academic Press, 1974.
- Safanov, M. G., and M. Athans, "Gain and phase margin for multiloop LQG regulators," *IEEE Trans. Automatic Control*, AC-22, 173–178 (1977).
- Safanov, M. G., A. J. Laub, and G. L. Hartmann, "Feedback properties of multivariable systems: the role and use of the return difference matrix," *IEEE Trans. Automatic Control*, AC-26, 47–65 (1981).
- Sandell, W. R., "Decomposition vs. decentralization in large-scale system theory," *Proc. Conf. Dec. Control*, 1043–1046 (1976).



- Schmidt, S. F., "Estimation of state with acceptable accuracy constraints," TR 67-16, Analytical Mechanics Assoc., Palo Alto, California, 1967.
- Schmidt, S. F., "Computational techniques in Kalman filtering," *Theory and Applications of Kalman Filtering*, Chap. 3, NATO Advisory Group for Aerospace Research and Development, AGARDograph 139, Feb. 1970.
- Schultz, D. G., and J. L. Melsa, *State Functions and Linear Control Systems*, New York: McGraw-Hill, 1967.
- Schultz, W., "Neural coding of basic reward terms of animal learning theory, game theory, microeconomics and behavioral ecology," *Neurobiology*, **14**, 139–147 (2004).
- Sewell, G., "IMSL software for differential equations in one space variable," IMSL Tech. Report Series, No. 8202, 1982. IMSL, Inc., 7500 Bellaire Blvd., Houston, TX 77036.
- Shin, V., and C. Chen, "On the Weighting Factors of the Quadratic Criterion in Optimal Control," *Int. J. Control*, **19**, 947–955 (1974).
- Si, J., A. Barto, W. Powell, and D. Wunsch, *Handbook of Learning and Approximate Dynamic Programming*, IEEE Press, USA, 2004.
- Söderström, T., "On some algorithms for design of optimal constrained regulators," *IEEE Trans. Automatic Control*, **AC-23**, 1100–1101 (1978).
- Southworth, R. W., and S. L. Deleew, *Digital Computation and Numerical Methods*, New York: McGraw-Hill, 1965.
- Stein, G., and M. Athans, "The LQR/LTR procedure for multivariable feedback control design," *IEEE Trans. Automatic Control*, **AC-32**, 105–114 (1987).
- Stevens, B. L., and F. L. Lewis, *Aircraft Control and Simulation*, New York: Wiley, 1992.
- Strang, G., *Linear Algebra and Its Applications*, 2nd ed. Orlando, FL: Academic Press, 1980.
- Sutton, R. S., and A. G. Barto, *Reinforcement Learning—An Introduction*, Cambridge, MA: MIT Press, 1998.
- Tsitsiklis, J., *Problems in Decentralized Decision Making and Computation*, Ph.D. dissertation, Dept. Elect. Eng. and Comput. Sci., Cambridge, MA: MIT, 1984.
- Vamvoudakis, K. G., and F. L. Lewis, "Online actor-critic algorithm to solve the continuous-time infinite horizon optimal control problem," *Automatica*, **46** (5), 878–888 (2010a).
- Vamvoudakis, K. G., and F. L. Lewis, "Online solution of nonlinear two-player zero-sum games using synchronous policy iteration", *Proc. IEEE Conf. Decision & Control*, 3040–3047 (2010b).
- Vamvoudakis, K. G., and F. L. Lewis, "Multi-player non-zero sum games: online adaptive learning solution of coupled Hamilton-Jacobi equations," *Automatica*, **47** (8), 1556–12011.
- van der Schaft, A. J., " $L_2$ -Gain analysis of nonlinear systems and nonlinear state feedback  $H_\infty$  control," *IEEE Trans. Automatic Control*, **37** (6), 770–784 (1992).
- Vaughan, D. R., "A nonrecursive algebraic solution to the discrete Riccati equation," *IEEE Trans. Automatic Control*, **AC-15**, 597–599 (1970).
- Verriest, E. I., and F. L. Lewis, "On the linear quadratic minimum-time problem," *IEEE Trans. Automatic Control*, **AC-36**, pp. 859–863, July 1991.
- Vrabie, D., and F. L. Lewis, "Neural network approach to continuous-time direct adaptive optimal control for partially-unknown nonlinear systems," *Neural Networks*, **22** (3), 237–246 (2009).



- Vrabie, D., and F. L. Lewis, "Adaptive dynamic programming algorithm for finding online the equilibrium solution of the two-player zero-sum differential game," *Proc. Int. Joint Conf. Neural Networks*, 1–8 (2010a).
- Vrabie, D., and F. L. Lewis, "Integral reinforcement learning for online computation of feedback Nash strategies of nonzero-sum differential games," *Proc. IEEE Conf. Decision Control*, 3066–3071 (2010b).
- Vrabie, D., O. Pastravanu, M. Abu-Khalaf, and F. L. Lewis, "Adaptive optimal control for continuous-time linear systems based on policy iteration," *Automatica*, **45**, 477–484 (2009).
- Wang, F. Y., H. Zhang, D. Liu, "Adaptive dynamic programming: an introduction," *IEEE Computational Intelligence Magazine*, 39–47 (2009).
- Watkins, C., *Learning from Delayed Rewards*, Ph.D. Thesis, Cambridge, UK: Cambridge University, 1989.
- Watkins, C., and P. Dayan, "Q-learning," *Machine Learning*, **8**, 279–292 (1992).
- Werbos, P. J., "Neural networks for control and system identification," *Proc. IEEE Conf. Decision and Control*, Florida, 1989.
- Werbos., P. J., "A menu of designs for reinforcement learning over time," *Neural Networks for Control*, pp. 67–95, ed. W. T. Miller, R. S. Sutton, and P. J. Werbos, Cambridge, MA: MIT Press, 1991.
- Werbos, P. J., "Approximate dynamic programming for real-time control and neural modeling," *Handbook of Intelligent Control*, ed. D. A. White and D. A. Sofge, New York: Van Nostrand Reinhold, 1992.
- Wheeler, R. M., and K. S. Narendra, "Decentralized learning in finite Markov chains," *IEEE Trans. Automatic Control*, **31** (6), 1986.
- Wolovich, W. A., *Linear Multivariable Systems*, New York: Springer-Verlag, 1974.
- Zames, G., "Feedback and optimal sensitivity: model reference transformations, multiplicative seminorms and approximate inverses," *IEEE Trans. Automatic Control*, **26**, 301–320 (1981).
- Zhang, H., J. Huang, and F. L. Lewis, "Algorithm and stability of ATC receding horizon control," *Proc. IEEE Symp. ADPRL*, pp. 28–35, Nashville, TN, Mar. 2009.

# INDEX

- Abnormal function-of-final-state-fixed regulator problem, 185, 204
- Ackermann's formula, 91, 100, 161
- Action-dependent heuristic dynamic programming, 501
- Actor-critic implementation (of DT optimal adaptive control), 500
- Actor-critic structures, 463, 475
- Adaptive control, optimal:
  - for discrete-time systems, 490–491
  - using policy iteration algorithm, 494–495
  - using value iteration algorithm, 495–496
- Adaptive controllers, 461–462
  - direct vs. indirect, 463–464
- Adaptive dynamic programming, 464
- Adjoint system, 313
  - continuous, 114
  - discrete, 23, 34, 194, 205
- Admissible controls, 439, 491
- Admissible cost, 265
- Affine state-variable feedback, 179, 194
- Aircraft:
  - longitudinal autopilot, example, 166–167, 293–295
  - routing, example, 261–262
- Arithmetic mean, 18
- Asymptotic properties of the LQR, 307
- Asynchronous value iteration, 478–479
- Augmented state description, 343
- Bandwidth, 356, 366
- Bang-bang control, 234
- Bang-off-bang control, 246–248
- Bellman equation, 439–440, 446, 455, 462
  - and dynamic programming, 468–469
  - policy evaluation/improvement by, 474
- Bellman Ford algorithm, 479
- Bellman's principle of optimality, 260–261
  - for continuous systems, 277–278
  - for discrete systems, 263–264
  - and dynamic programming, 260–261, 467–468
- Bilinear system:
  - continuous optimal control, 169
  - discrete optimal control, 102
  - dynamic programming, 283–284
  - perturbation control, 212
- Bilinear tangent control law, 126
- Bode magnitude plot, 357, 359
- Bode multivariable plot, 361–363
- Bode singular value plot, 364–365
- Boundary conditions, 20, 23
- Bounded  $L_2$ -gain problem, 450–452
- Brachistochrone problem, 220–224
- Cargo loading, 285
- Chain rule for differentiation, 525
- Chang-Letov equation, 99–100, 165
- Closed-loop, poles, 297
- Closed-loop control, 41–53,
  - 143–146, 190, 195, 307
- Closed-loop Markov chain, 472–473
- Closed-loop system, 90, 97, 144, 154, 179, 194, 300
  - adjoint, 179, 185, 194, 205
  - characteristic polynomial, 97, 164, 291
  - locus of poles, 78, 93, 167
  - optimal steady-state poles, 91, 100, 108, 161, 165
  - for polynomial regulator, 291
- Closest point of approach, 16
- Command-generator tracker (CGT), 332–338
- Complementary sensitivity, *see* Cosensitivity
- Computer simulation, 21
  - bang-bang control, 243–245
  - bang-off-bang control, 253

- Computer simulation (*continued*)  
 digital control, 54  
 harmonic oscillator, 106, 152  
 linear quadratic regulator, 146, 149–150  
 linear quadratic tracker, 180  
 Newton's system, 77, 243–245, 253  
 preliminary analysis for, 49, 61  
 scalar optimal control, 30–31, 149–150  
 scalar system, 49, 149–150  
 scalar tracker, 181
- Conjugate gradient method, 15
- Conservation of energy, 117
- Constant output feedback, 316
- Constraints:  
 on control, 232, 262, 263–264  
 on state, 262, 263–264
- Constraint equation, 4–5
- Continuous-time systems:  
 integral reinforcement learning for optimal adaptive control of, 503–505  
 online implementation, 507–508  
 using policy iteration, 506  
 using value iteration, 506  
 synchronous optimal adaptive control for, 513–514
- Control delay, 288
- Controllability, *see* Reachability
- Control-weighting matrix, 299
- Convergence, conditions for, 304–305
- Cooperative control systems, 481
- Coriolis force, 224
- Cosensitivity, 360  
 and sensitivity, 357–361
- Costate, 23  
 equation, 24, 32–35, 115, 135
- Coupled nonlinear matrix equation, 302, 346–347
- Critical point, 2
- Cubic equation for optimal solution, 16, 124
- Curse of dimensionality, 274
- Curvature matrix:  
 constrained, 9, 210  
 continuous, 145–146, 189  
 discrete, 46, 210  
 unconstrained, 2
- Curve, length between two points, 117
- Cycloid, 224
- Damping ratio, 158
- Deadbeat control, 104
- Dead-zone function, 248
- Decentralized control, 343–344  
 linear quadratic regulator, 345–347
- Definiteness of matrices, 521–523
- Delay operator, 198, 288
- Descriptor systems, 102
- Design parameters, 305  
 tuning the, 305
- Detectability, 70, 156
- Deterministic policies, 465
- Deviation system, 317–319
- Deyst filter, 401
- Digital control, 30–31, 50–51, 53–54, 272–274, 292  
 harmonic oscillator, 106  
 Newton's system, example, 59–63  
 RC circuit, example, 55–58
- Diophantine equation, 290
- Discretization:  
 of continuous performance index, 271–274  
 of continuous system, 53, 271–274  
 of transfer function, 292
- Disturbance(s):  
 discrete system with, 182–183  
 and performance robustness, 356
- Dual optimization problems, 17, 18
- Dynamic(s):  
 augment the, 368  
 compensator, 314  
 optimization, 299–300
- Dynamic programming, 462, 467–468
- Eigenstructure assignment design of steady-state regulator, 90–92, 160–161
- Eigenvalues:  
 of inverse of a matrix, 519  
 of Kronecker product, 519
- Eigenvectors:  
 of Hamiltonian matrix, 34, 81, 90, 108, 159  
 of Kronecker product, 519  
 of optimal closed-loop system, 90–91
- Ergodic Markov chains, 466
- Estimation error, 396
- Euclidean norm, 298
- Euler's approximation, 273
- Euler's equation, 117, 131  
 via HJB equation, 281
- Explicit model-following design, 338–343
- Feedback:  
 output, 291  
 state, *see* State-variable feedback  
 suboptimal, 65–68
- Fictitious follower, 349
- Fictitious output, 70, 98, 156, 164
- Field of extremals, 269, 284
- Filters, washout, 313
- Final state:  
 fixed, 24, 25–28, 38–40, 46, 119–120, 138–141, 170, 201  
 free, 24, 28–30, 41–53, 143–146, 170, 201  
 on moving point, 214, 228  
 on surface, 215
- Fixed-final-state control, 38–40, 46, 138–141  
 state feedback formulation, 149, 171, 183–185, 194–195
- Free-final-state control, 44, 120–121
- Frequency domain design of linear quadratic regulator, 164–167
- Frequency-domain techniques, 355
- Functional equation of dynamic programming, 264
- Gain(s):  
 optimal, 375  
 scheduling, 311–313
- Game theory, 344, 438–439.  
*See also* Zero-sum games
- Generalized state-space systems, discrete optimal control, 102
- Geometric mean, 18
- Gradient, 1  
 based solution, 321  
 minimization algorithm, 321  
 numerical methods, 15  
 vector, 523
- Gramian, *see* Observability gramian; Reachability gramian
- $H_\infty$  control, application of zero-sum games to, 450–453
- $H_\infty$  design, 357, 430–435
- Hamiltonian, 348
- Hamiltonian function, 6, 22, 32, 113, 135, 278
- Hamiltonian matrix:  
 continuous, 136, 159  
 discrete, 34, 80, 90, 101

- eigenvectors, 34, 81, 90, 108, 159, 175
- Hamiltonian system:
  - continuous, 131, 136, 158–159, 172, 178
  - discrete, 34, 80, 90, 192
- Hamilton-Jacobi-Bellman equation, 277–279, 440–441, 443, 449, 468–469
- Hamilton's equations of motion, 117
- Hamilton's principle, 116–117
- Harmonic oscillator:
  - digital control of, 106
  - eigenstructure design, 92–95, 161–163
  - linear quadratic regulator, 150–153
  - minimum-fuel control, 259
  - minimum-time control, 259
  - root-locus design, 99
  - steady-state regulator, 99
  - zero input cost, 172
- Helicopter longitudinal autopilot, 293–295
- Hessian matrix, 1, 523
- Hewer's algorithm, 482–483
- Indirect adaptive controllers, 462, 463
- Infinite horizon optimal control problem, 75, 93, 157, 180, 196
- Integral reinforcement learning for optimal adaptive control of continuous-time systems, 503–505
  - online implementation, 507–508
  - using policy iteration, 506
  - using value iteration, 506
- Intercept problem, 122, 127–128, 215–216, 235–239
- Interpolation for discrete dynamic programming, 274–276
- Inverse of partitioned matrix, 520
- Iterative learning control, 488
- Jacobian, 5, 524
- Kalman filter, 391–404
- Kalman gain:
  - continuous dynamic, 144
  - discrete dynamic, 43, 105, 193, 203
  - static, 17
  - steady-state, 69, 74, 90, 161, 174
- Kernel matrix, 36, 46, 137
- Kronecker product, 103, 172, 519
- Lagrange multiplier, 6–7, 22, 113, 200, 301, 399
- Lagrange's equations of motion, 116–117
- Lagrangian for a dynamical system, 116
- Leading minors of matrix, 522
- Learning:
  - Monte Carlo, 488
  - reinforcement, 462–464, 503–505
  - temporal difference, 489–490
- Leibniz's rule, 111
- Linearized plant model, 355
- Linear minimum-energy problem, 254–257
- Linear minimum-fuel problem, 246–248
- Linear minimum-time problem, 213–214, 228–230
- Linear quadratic
  - Gaussian/loop-transfer recovery (LQG/LTR), 357
- Linear quadratic multiplayer games, 459–460
- Linear quadratic problem:
  - continuous dynamic systems, 135–157
  - discrete dynamic systems, 32
  - static, example, 11
- Linear quadratic regulator, 443–444
  - continuous state-costate formulation, 135
  - continuous state feedback formulation, 144–145
  - derivation by dynamic programming, 270–271
  - derivation via HJB equation, 281–283
  - discrete state-costate formulation, 32, 99
  - discrete state feedback formulation, 96
  - eigenstructure design, 108, 160–161
  - frequency domain design, 96–100, 165–166
  - with function of final state fixed, 185, 204–205
  - perturbation, 188, 206–209
  - for polynomial systems, 290
  - root-locus design, 99, 108, 165
  - steady-state, 90, 96–97, 156
  - suboptimal, 65–66, 154
  - with weighting of state-input inner product, 52–53, 104, 153–154
- Linear quadratic tracker:
  - continuous affine feedback formulation, 179
  - continuous state-costate formulation, 178–179
  - derivation via HJB equation, 286
  - discrete affine feedback formulation, 194
  - discrete state-costate formulation, 192
  - formulated as regulator, 183, 198–199
  - for polynomial systems, 288
  - suboptimal, 180, 195–196
  - time-invariant, 196
- Linear quadratic zero-sum games, 452–453
- Linear tangent control law, 217
- Loop gain, 97, 164, 356
  - singular value, 357
- Loop transfer recovery (LTR), 408–430
- Low frequency specifications, 367
- LQ solution algorithm, 304–305
- LQ tracker with output feedback, 322
- Lyapunov equation, 184, 300, 347, 350
  - algebraic, 37, 103, 136
  - closed-loop, 66, 74, 154
  - continuous observability, 136, 175
  - continuous reachability, 140, 175
  - discrete observability, 36, 46
  - discrete reachability, 36
  - online solution, 496
  - scalar, 48
  - solution, 36, 40, 46, 136, 137, 140
  - and value iteration, 483
  - as vector equation, 103, 172
- Markov chain, closed-loop, 472–473
- Markov decision processes, 464–473
- Matrix design equation, 297
- Matrix inversion lemma, 360, 520
- Maximum, 2
- Microprocessors, 53
- Minimum, *see* Necessary conditions for minimum; Sufficient conditions for minimum
- Minimum-energy problem, 21, 25, 38–39, 139
  - constrained, 254–257
- Minimum-fuel problem, 261–262
  - linear, 246–248
  - normal, 248

- Minimum-time problem,
  - 213–214, 228–230
  - with control weighting, 257–258
  - linear, 234
  - normal, 238
- Model-following control, 289, 296
- Model-following regulator, 340–343
- Modelling errors, 355
- Model reduction, 373–378
- Monte Carlo learning, 488
- Multi-Input Multi-Output (MIMO) systems, 297, 356
- Multiplayer games, linear quadratic, 459–460
- Multiplayer non-zero-sum games, 453
- Multiplicative uncertainties, 372
- Multivariable Bode plot, 361–363
- Nash equilibrium, 454, 456
- Nash game, 347–348
- Natural frequency, 158
- Necessary conditions for minimum, 185
  - continuous linear quadratic regulator, 144–145, 185
  - continuous linear quadratic tracker, 179
  - discrete linear quadratic regulator, 34, 205
  - discrete linear quadratic tracker, 194
  - general continuous systems, 115
  - general discrete systems, 23
  - static, 2, 6, 8
- Negative definite, 522
- Neighboring optimal solutions, 14, 187, 208
- Neural networks, 497–498
- Neuro-dynamic programming, 493
- Newton's system, examples:
  - bang-bang control, 239–245
  - bang-off-bang control, 239–245
  - constrained minimum-energy control, 248–253
  - digital control, 59–63
  - limiting control, 77–80
  - open-loop control, 141–143
  - optimal control, problem, 171
  - optimal control via HJB equation, 286
  - optimal steady-state poles, 157, 173
  - steady-state control, 78, 157, 173
  - suboptimal control, 78, 174
  - tracker, 196–197, 211
  - zero-input cost, 172
- Nominal trajectory, 209
- Nonlinear matrix equations, coupled, 302
- Nonlinear non-zero-sum games, 453–458
- Nonlinear systems:
  - discretization of, 271–274
  - function-of-final-state-fixed regulator, 199–201
  - optimal control, 24, 102, 115, 131–132, 439–441
  - optimal control by approximation, 168–169
  - optimal control using dynamic programming, 264–274, 276
  - perturbation control, 186–187, 206–209
  - tracking problem, 177–178, 190–191
- Non-zero-sum games, 453
  - cooperative/competitive aspects of, 459
- Norm, 521
- Normal function-of-final-state-fixed regulator problem, 185, 204
- Normal minimum-fuel problem, 248
- Normal minimum-time problem, 238–239
- Normal system, 238
- Numerical solution methods:
  - gradient, 15
  - steepest descent, 15
- Observability, 37, 70, 156, 305–306
  - canonical, 32, 341, 342
  - time-varying plant, 96, 163–164, 175
- Observability gramian:
  - continuous, 137
  - discrete, 37
  - time-varying plant, 96, 163–164
- Observer(s):
  - design, 384–387
  - filter, ARE, 387
  - the Kalman filter, 383–408
  - output-input, 386
- Observer, state, 291
- Open-loop control:
  - continuous, 136–137, 143
  - discrete, 40
  - with function of final state fixed, 172
  - scalar system, example, 27, 40–41, 135
- Operator gain, 366–367
- Optimal adaptive control
  - (continuous-time systems):
    - hybrid controller, 507–508
    - integral reinforcement learning for, 503–505
- Optimal adaptive control
  - (discrete-time systems), 490–491
  - actor-critic implementation, 500
  - Q learning, 501–503
- Optimal control problem:
  - constrained minimum-energy, 254–257
  - continuous, 112
  - continuous linear quadratic, 135–136
  - discrete, 19
  - discrete linear quadratic, 32
  - infinite horizon, 75, 93, 157, 180
  - linear minimum-fuel, 246–248
  - linear minimum-time, 234, 257–258
  - nonlinear systems, 19, 112, 439–441
  - solution via HJB equation, 279–280
- Optimal feedback gain, 305, 321
- Optimal gains, 311–313
- Optimal output feedback solution algorithm, 304
- Optimal policy, 467
- Optimal quality function, 484–485
- Optimal value, 467
- Optimization:
  - constrained, 301
  - dual problem, 17
  - by scalar manipulations, 4
- Orbit injection, minimum-time, example, 224–226
- Output feedback, 298
  - in decentralized control, 343–344
  - design, 302, 351–352
  - dynamic, 291
  - gain, 302
  - in game theory, 343–344
  - LQR with, 302–303
  - problem, 305
  - step-response shaping, 313
  - theory, 356
- Output injection, 70
- Output stabilization, 340

- Parameters, design, 305
  - PBH rank test, 72
  - Performance index, 319–320, 344–345
    - for continuous dynamic systems, 112, 131, 135–157
    - cubic, 104
    - for discrete dynamic systems, 20, 32
    - discretization of, 271–274
    - infinite horizon, 75, 93, 157
    - linear, 104
    - minimum fuel, 21
    - minimum-time, 20, 213–214
    - minimum-time with control weighting, 257–258
    - model-following, 289
    - optimal value of, 45, 140, 145, 220, 223
    - for polynomial systems, 288
    - quadratic, 21, 32, 131, 135
    - robustness, disturbances and, 356
    - specification, 365, 367
      - high frequency, 370–373
      - low frequency, 367
    - suboptimal, 66, 154
    - for tracking problem, 179, 194
    - for unconstrained optimization, 1
    - with zero input, 35–37, 136–137
  - Perturbation control, 186–187, 206–209
    - of bilinear system, 212
  - Perturbation state equation, 186, 207
  - Phase margin, 356, 357
  - Phase plane, 242, 250
  - Plant parameter variations, 355
  - Policy evaluation, 474
  - Policy improvement, 474
  - Policy iteration:
    - algorithm, 475–476
    - generalized, 484
    - implementation models, 488
    - integral reinforcement learning
      - for optimal adaptive control using, 506
    - optimal adaptive control using, 494–495
    - temporal difference learning
      - using, 492
    - using Q function, 487
  - Polynomials:
    - continuous decomposition, 176
    - discrete decomposition, 108
    - mirror image, 108
    - reciprocal, 108
  - Polynomial regulator:
    - continuous, 295–296
    - discrete, 291
  - Polynomial techniques, 430–431
  - Pontryagin's minimum principle, 232, 260, 281, 440
  - Positive definite, 521
  - Precompression, 368
    - balancing, 370
    - zero steady-state, 370
  - Predictive formulation of
    - polynomial system, 290
  - Preliminary analysis for computer simulation, 49, 61
  - Proportional navigation, 125
  - Q function, 484–485
    - defined, 484
    - policy iteration using, 487
    - value iteration using, 487
  - Q learning, for optimal adaptive control, 501–503
  - Quadratic equation for optimal final time, 130
  - Quadratic form, 521–523
  - Quadratic performance index, 298–299
  - Quadratic surfaces, example, 2, 9–10
  - Quantization for discrete dynamic programming, 274–276
  - Reachability, 40, 70, 139, 146, 238
    - time-varying plant, 96, 163–164, 175
  - Reachability gramian:
    - continuous, 139
    - discrete, 39–40, 46, 205
    - time-varying plant, 96, 163–164
  - Reachability matrix, 39–40, 72, 91, 238
  - Reachable canonical form, 292
  - Rectangle:
    - inside ellipse, 18
    - of maximum area, 17
  - Reference-input tracking, 313–314
  - Regulator redesign, 328–329
    - model-following regulator, 340–343
  - Reinforcement learning, 462–464
    - integral, for optimal adaptive control of continuous-time systems, 503–505
  - Rendezvous problem, 60, 122
  - Return difference, 97, 164
  - Riccati equation, 434–435
    - algebraic, 69, 97, 155, 174, 175
  - analytic solution, 80–84, 107–108, 155, 158–160
  - continuous, 144, 283
  - destabilizing solution, 175
  - discrete, 43, 194, 202
  - information formulation, 105
  - Joseph formulation, 44, 66, 103, 144, 154, 172, 271
  - limiting behavior, 70–71, 96, 153, 155–156
  - online solution, 483–484, 496, 508–509
  - solution from Hamiltonian system solutions, 172
  - square-root formulations, 105
  - stabilizing solution, 175
  - as vector equation, 173
- Robust design, 313, 356–357, 380–383
- Rollout algorithms, 479
- Root locus, 78, 94, 99, 108, 165
- Runge-Kutta integrator, 40
- Saddle point, 2
- Satellite, 113
- Saturated control, 236
- Saturation function, 257
- Scalar system, examples:
  - constrained minimum-energy control, 59
  - digital control, 55–58
  - dynamic programming, 264–270, 274–276
  - linear quadratic regulator, 47
  - minimum-fuel control, 59
  - minimum-time control, 258–259
  - open-loop control, 40–41, 141
  - optimal control, 25, 148–151
  - optimal control via HJB
    - equation, 279–280
  - steady-state control, 75–77, 174
  - steady-state tracker, 181–182
  - suboptimal control, 66–67, 173–174
  - tracker, example, 180–182
  - uncontrolled, 138
- Schur complement, 6, 520
- Seminorm, 523
- Sensitivity:
  - and cosensitivity, 357–361
  - function, 356
- Sensitivity matrix for initial costate, 132
- Separation of variables, 148, 280
- Separation principle, 404–408
- Sequential decision problems, 465–467
- Servo compensator, 334–335

- Shortest distance:
  - from point to line, 16, 226–228
  - between two points, 16, 117
- Shortest path problems, 479
- Signum function, 237
- Singular control interval, 238, 248
- Singular point, 3
- Singular value:
  - maximum, 362
  - minimum, 363
- Solution, gradient-based, 321
- Square root of matrix, 522
- Stability augmentation systems (SAS), 297
- Stability robustness, 355, 356, 371
- Stabilizability, 70, 156
- Stabilization of multi-input plant, 74, 157
- Stackelberg games, 348–351
- Stacking operator, 103, 172, 519
- Stage cost, 465
- Standard-feedback configuration, 358
- State observer, 291
- State trajectories, temporal
  - difference learning along, 489–490
- State transition matrix, 54, 95, 163
  - of adjoint closed-loop system, 184, 205
- State-variable feedback, 43–44, 70, 278
  - affine, 179, 194
  - for bang-bang control, 247–245
  - graphical, 219, 223, 261, 268
  - implicit, 219, 223
  - nonlinear, 219, 223, 258
- Stationarity condition, 7, 12, 23, 33, 115, 136, 281
- Stationary point, 2
- Steady-state control, 75–77, 90, 99, 155, 165–166, 180
- Steady-state cost, 37, 137, 149
- Steepest descent algorithm, 15
- Stochastic strategies or policies, 465
- Suboptimal control, 66, 78, 180, 195–196
- Suboptimal cost, 7
- Suboptimal feedback gain, 65–66, 154
- Sufficient conditions for
  - minimum:
    - continuous systems, 187–190
    - discrete systems, 208
    - static, 2, 9
- Sweep method:
  - continuous, 143, 179, 184
  - discrete, 42, 192, 202
- Switching:
  - curve, 242
  - function, 237
  - time, 240–241
- Symmetric part:
  - of matrix, 521
  - of polynomial, 108, 176
- Symplectic matrix, 81
- Synchronous optimal adaptive (continuous-time systems), 513–514
- Target set of final states, 215, 227
- Taylor series, 1, 8, 186, 206, 278, 523
- Temperature control, example, 118–121
- Temporal difference error, 482, 489
- Temporal difference learning, 489–490
  - policy iteration using, 492
  - value iteration using, 492–493
- Thrust angle programming, 126
  - in gravitational field, 257
- Time-varying systems:
  - limiting control of, 95–96, 163–164
  - observability, 95–96, 163–164
  - reachability, 95–96, 163–164
- Total differential, 524
- Tracker problem, 316–317
- Tracking:
  - with disturbance rejection, 337
  - reference-input, 313–314
  - a unit step, 329–330
- Tracking error, 183, 198, 319–320
- Transversality condition, 214–215, 239
- Two-degrees-of-freedom regulator, 291
- Two-player zero-sum games, 444–449
- Two-point boundary-value problem:
  - continuous, 114
  - discrete, 23
  - unit solution method, 170
- Unit solution method, 170–171
- Unmodelled dynamics, 355
- Utility, 466
- Value (of a policy), 466
  - optimal, 467
- Value function (of a policy), 466
- Value function approximation, 493
- Value iteration:
  - algorithm, 477–479
  - asynchronous, 478–479
  - implementation models, 488
  - integral reinforcement learning
    - for optimal adaptive control using, 506
    - optimal adaptive control using, 495–496
    - temporal difference learning using, 492–493
    - using Q function, 487
- Variation in function, 111
- Washout filters, 313
- White noise, 395
- Zermelo’s problem, 216–220, 258
- Zero-input cost:
  - continuous, 136–137
  - discrete, 35–37
  - scalar system, example, 137
- Zero-order hold, 53
- Zero-sum games:
  - application to  $H_\infty$  control, 450–453
  - Bellman equation, 446
  - defined, 445
  - linear quadratic, 452–453
  - two-player, 444–449