Fluid Mechanics and Thermodynamics of Turbomachinery

Seventh Edition
Dedication

In memory of Avril (22 years) and baby Paul.
Preface to the Seventh Edition

This book was originally conceived as a text for students in their final year reading for an honors degree in engineering that included turbomachinery as a main subject. It was also found to be a useful support for students embarking on postgraduate courses at masters level. The book was written for engineers rather than for mathematicians, although some knowledge of mathematics will prove most useful. Also, it is assumed from the start that readers will have completed preliminary courses in fluid mechanics. The stress is placed on the actual physics of the flows and the use of specialized mathematical methods is kept to a minimum.

Compared to the sixth edition, this new edition has had a large number of changes made in terms of presentation of ideas, new material, and additional examples. In Chapter 1, following the definition of a turbomachine, the fundamental laws of flow continuity, the energy and entropy equations are introduced as well as the all-important Euler work equation. In addition, the properties of working fluids other than perfect gases are covered and a steam chart is included in the appendices. In Chapter 2, the main emphasis is given to the application of the “similarity laws,” to dimensional analysis of all types of turbomachine and their performance characteristics. Additional types of turbomachine are considered and examples of high-speed characteristics are presented. The important ideas of specific speed and specific diameter emerge from these concepts and their application is illustrated in the Cordier Diagram, which shows how to select the machine that will give the highest efficiency for a given duty. Also, in this chapter the basics of cavitation are examined for pumps and hydraulic turbines.

The measurement and understanding of cascade aerodynamics is the basis of modern axial turbomachine design and analysis. In Chapter 3, the subject of cascade aerodynamics is presented in preparation for the following chapters on axial turbines and compressors. This chapter was completely reorganized in the previous edition. In this edition, further emphasis is given to compressible flow and on understanding the physics that constrain the design of turbomachine blades and determine cascade performance. In addition, a completely new section on computational methods for cascade design and analysis has been added, which presents the details of different numerical approaches and their capabilities.

Chapters 4 and 5 cover axial turbines and axial compressors, respectively. In Chapter 4, new material has been added to give better coverage of steam turbines. Sections explaining the numerous sources of loss within a turbine have been added and the relationships between loss and efficiency are further detailed. The examples and end-of-chapter problems have also been updated. Within this chapter, the merits of different styles of turbine design are considered including the implications for mechanical design such as centrifugal stress levels and cooling in high-speed and high temperature turbines. Through the use of some relatively simple correlations, the trends in turbine efficiency with the main turbine parameters are presented.

In Chapter 5, the analysis and preliminary design of all types of axial compressors are covered. Several new figures, examples, and end-of-chapter problems have been added. There is new coverage of compressor loss sources and, in particular, shock wave losses within high-speed rotors are explored in detail. New material on off-design operation and stage matching in multistage compressors has been added, which enables the performance of large compressors to be quantified.
Several new examples and end-of-chapter problems have also been added that reflect the new material on design, off-design operation, and compressible flow analysis of high-speed compressors.

Chapter 6 covers three-dimensional effects in axial turbomachinery and it possibly has the most new features relative to the sixth edition. There are extensive new sections on three-dimensional flows, three-dimensional design features, and three-dimensional computational methods. The section on through-flow methods has also been reworked and updated. Numerous explanatory figures have been added and there are new worked examples on vortex design and additional end-of-chapter problems.

Radial turbomachinery remains hugely important for a vast number of applications, such as turbocharging for internal combustion engines, oil and gas transportation, and air liquefaction. As jet engine cores become more compact there is also the possibility of radial machines finding new uses within aerospace applications. The analysis and design principles for centrifugal compressors and radial inflow turbines are covered in Chapters 7 and 8. Improvements have been made relative to the fifth edition, including new examples, corrections to the material, and reorganization of some sections.

Renewable energy topics were first added to the fourth edition of this book by way of the Wells turbine and a new chapter on hydraulic turbines. In the fifth edition, a new chapter on wind turbines was added. Both of these chapters have been retained in this edition as the world remains increasingly concerned with the very major issues surrounding the use of various forms of energy. There is continuous pressure to obtain more power from renewable energy sources and hydroelectricity and wind power have a significant role to play. In this edition, hydraulic turbines are covered in Chapter 9, which includes coverage of the Wells turbine, a new section on tidal power generators, and several new example problems. Chapter 10 covers the essential fluid mechanics of wind turbines, together with numerous worked examples at various levels of difficulty. In this edition, the range of coverage of the wind itself has been increased in terms of probability theory. This allows for a better understanding of how much energy a given size of wind turbine can capture from a normally gusting wind. Instantaneous measurements of wind speeds made with anemometers are used to determine average velocities and the average wind power. Important aspects concerning the criteria of blade selection and blade manufacture, control methods for regulating power output and rotor speed, and performance testing are touched upon. Also included are some very brief notes concerning public and environmental issues, which are becoming increasingly important as they, ultimately, can affect the development of wind turbines.

To develop the understanding of students as they progress through the book, the expounded theories are illustrated by a selection of worked examples. As well as these examples, each chapter contains problems for solution, some easy, some hard. See what you make of them—answers are provided in Appendix F!
Acknowledgments

The authors are indebted to a large number of people in publishing, teaching, research, and manufacturing organizations for their help and support in the preparation of this volume. In particular, thanks are given for the kind permission to use photographs and line diagrams appearing in this edition, as listed below:

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I would like to give my belated thanks to the late Professor W.J. Kearton of the University of Liverpool and his influential book *Steam Turbine Theory and Practice*, who spent a great deal of time and effort teaching us about engineering and instilled in me an increasing and life-long interest in turbomachinery. This would not have been possible without the University of Liverpool’s award of the W.R. Pickup Foundation Scholarship supporting me as a university student, opening doors of opportunity that changed my life.

Also, I give my most grateful thanks to Professor (now Sir) John H. Horlock for nurturing my interest in the wealth of mysteries concerning the flows through compressors and turbine blades during his tenure of the Harrison Chair of Mechanical Engineering at the University of Liverpool. At an early stage of the sixth edition some deep and helpful discussions of possible additions to the new edition took place with Emeritus Professor John P. Gostelow (a former undergraduate student of mine). There are also many members of staff in the Department of Mechanical Engineering during my career who helped and instructed me for which I am grateful.

Also, I am most grateful for the help given to me by the staff of the Harold Cohen Library, University of Liverpool, in my frequent searches for new material needed for the seventh edition.
Last, but by no means least, to my wife Rosaleen, whose patient support and occasional suggestions enabled me to find the energy to complete this new edition.

S. Larry Dixon

I would like to thank the University of Cambridge, Department of Engineering, where I have been a student, researcher, and now lecturer. Many people there have contributed to my development as an academic and engineer. Of particular importance is Professor John Young who initiated my enthusiasm for thermofluids through his excellent teaching of the subject. I am also very grateful to Rolls-Royce plc, where I worked for several years. I learned a huge amount about compressor and turbine aerodynamics from my colleagues there and they continue to support me in my research activities.

Almost all the contributions I made to this new edition were written in my office at King’s College, Cambridge, during a sabbatical. As well as providing accommodation and food, King’s is full of exceptional and friendly people who I would like to thank for their companionship and help during the preparation of this book.

As a lecturer in turbomachinery, there is no better place to be based than the Whittle Laboratory. I would like to thank the members of the laboratory, past and present, for their support and all they have taught me. I would like to make a special mention of Dr. Tom Hynes, my Ph.D. supervisor, for encouraging my return to academia from industry and for handing over the teaching of a turbomachinery course to me when I started as a lecturer. During my time in the laboratory, Dr. Rob Miller has been a great friend and colleague and I would like to thank him for the sound advice he has given on many technical, professional, and personal matters. Several laboratory members have also helped in the preparation of suitable figures for this book. These include Dr. Graham Pullan, Dr. Liping Xu, Dr Martin Goodhand, Vicente Jerez-Fidalgo, Ewan Gunn, and Peter O’Brien.

Finally, special personal thanks go to my parents, Hazel and Alan, for all they have done for me. I would like to dedicate my work on this book to my wife Gisella and my son Sebastian.

Cesare A. Hall
## List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
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<td>$A$</td>
<td>area</td>
</tr>
<tr>
<td>$a$</td>
<td>sonic velocity</td>
</tr>
<tr>
<td>$a, a'$</td>
<td>axial-flow induction factor, tangential flow induction factor</td>
</tr>
<tr>
<td>$b$</td>
<td>axial chord length, passage width, maximum camber</td>
</tr>
<tr>
<td>$C_c, C_t$</td>
<td>chordwise and tangential force coefficients</td>
</tr>
<tr>
<td>$C_{L}, C_{D}$</td>
<td>lift and drag coefficients</td>
</tr>
<tr>
<td>$C_F$</td>
<td>capacity factor ($= \frac{P_W}{P_R}$)</td>
</tr>
<tr>
<td>$C_p$</td>
<td>specific heat at constant pressure, pressure coefficient, pressure rise coefficient</td>
</tr>
<tr>
<td>$C_v$</td>
<td>specific heat at constant volume</td>
</tr>
<tr>
<td>$C_X, C_Y$</td>
<td>axial and tangential force coefficients</td>
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<td>$c$</td>
<td>absolute velocity</td>
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<tr>
<td>$c_o$</td>
<td>spouting velocity</td>
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<tr>
<td>$d$</td>
<td>internal diameter of pipe</td>
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<tr>
<td>$D$</td>
<td>drag force, diameter</td>
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<tr>
<td>$D_h$</td>
<td>hydraulic mean diameter</td>
</tr>
<tr>
<td>$D_s$</td>
<td>specific diameter</td>
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<tr>
<td>$DF$</td>
<td>diffusion factor</td>
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<td>$E, e$</td>
<td>energy, specific energy</td>
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<tr>
<td>$F$</td>
<td>force, Prandtl correction factor</td>
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<tr>
<td>$F_c$</td>
<td>centrifugal force in blade</td>
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<tr>
<td>$f$</td>
<td>friction factor, frequency, acceleration</td>
</tr>
<tr>
<td>$g$</td>
<td>gravitational acceleration</td>
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<tr>
<td>$H$</td>
<td>blade height, head</td>
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<tr>
<td>$H_E$</td>
<td>effective head</td>
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<td>$H_f$</td>
<td>head loss due to friction</td>
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<td>$H_G$</td>
<td>gross head</td>
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<td>$H_S$</td>
<td>net positive suction head (NPSH)</td>
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<tr>
<td>$h$</td>
<td>specific enthalpy</td>
</tr>
<tr>
<td>$I$</td>
<td>rothaply</td>
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<tr>
<td>$i$</td>
<td>incidence angle</td>
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<tr>
<td>$J$</td>
<td>wind turbine tip—speed ratio</td>
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<td>$j$</td>
<td>wind turbine local blade-speed ratio</td>
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<tr>
<td>$K, k$</td>
<td>constants</td>
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<tr>
<td>$L$</td>
<td>lift force, length of diffuser wall</td>
</tr>
<tr>
<td>$l$</td>
<td>blade chord length, pipe length</td>
</tr>
<tr>
<td>$M$</td>
<td>Mach number</td>
</tr>
<tr>
<td>$m$</td>
<td>mass, molecular mass</td>
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<tr>
<td>$N$</td>
<td>rotational speed, axial length of diffuser</td>
</tr>
<tr>
<td>$n$</td>
<td>number of stages, polytropic index</td>
</tr>
<tr>
<td>$o$</td>
<td>throat width</td>
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<tr>
<td>$P$</td>
<td>power</td>
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$P_R$ rated power of wind turbine
$\bar{P}_W$ average wind turbine power
$p$ pressure
$p_a$ atmospheric pressure
$p_v$ vapor pressure
$q$ quality of steam
$Q$ heat transfer, volume flow rate
$R$ reaction, specific gas constant, diffuser radius, stream tube radius
$Re$ Reynolds number
$R_H$ reheat factor
$R_o$ universal gas constant
$r$ radius
$S$ entropy, power ratio
$s$ blade pitch, specific entropy
$T$ temperature
$t$ time, thickness
$U$ blade speed, internal energy
$u$ specific internal energy
$V, v$ volume, specific volume
$W$ work transfer, diffuser width
$\Delta W$ specific work transfer
$W_x$ shaft work
$w$ relative velocity
$X$ axial force
$x, y$ dryness fraction, wetness fraction
$x, y, z$ Cartesian coordinate directions
$Y$ tangential force
$Y_p$ stagnation pressure loss coefficient
$Z$ number of blades, Zweifel blade loading coefficient
$\alpha$ absolute flow angle
$\beta$ relative flow angle, pitch angle of blade
$\Gamma$ circulation
$\gamma$ ratio of specific heats
$\delta$ deviation angle
$\varepsilon$ fluid deflection angle, cooling effectiveness, drag—lift ratio in wind turbines
$\zeta$ enthalpy loss coefficient, incompressible stagnation pressure loss coefficient
$\eta$ efficiency
$\theta$ blade camber angle, wake momentum thickness, diffuser half angle
$\kappa$ angle subtended by log spiral vane
$\lambda$ profile loss coefficient, blade loading coefficient, incidence factor
$\mu$ dynamic viscosity
$\nu$ kinematic viscosity, hub—tip ratio, velocity ratio
$\xi$ blade stagger angle
$\rho$ density
\( \sigma \) slip factor, solidity, Thoma coefficient
\( \sigma_b \) blade cavitation coefficient
\( \sigma_c \) centrifugal stress
\( \tau \) torque
\( \phi \) flow coefficient, velocity ratio, wind turbine impingement angle
\( \psi \) stage loading coefficient
\( \Omega \) speed of rotation
\( \Omega_S \) specific speed
\( \Omega_{SP} \) power specific speed
\( \Omega_{SS} \) suction specific speed
\( \omega \) vorticity

**Subscripts**

0 stagnation property
b blade
c compressor, centrifugal, critical
cr critical value
d design
D diffuser
e exit
h hydraulic, hub
i inlet, impeller
id ideal
m mean, meridional, mechanical, material
max maximum
min minimum
N nozzle
n normal component
o overall
opt optimum
p polytropic, pump, constant pressure
R reversible process, rotor
r radial
ref reference value
rel relative
s isentropic, shroud, stall condition
ss stage isentropic
t turbine, tip, transverse
ts ts total-to-static
tt total-to-total
List of Symbols

\( \mathbf{v} \) velocity
\( x, y, z \) Cartesian coordinate components
\( \theta \) tangential

Superscripts

\( \cdot \) time rate of change
\( - \) average
\( ' \) blade angle (as distinct from flow angle)
\( * \) nominal condition, throat condition
\( ^\wedge \) nondimensionalized quantity
Introduction: Basic Principles

Take your choice of those that can best aid your action.
Shakespeare, Coriolanus

1.1 Definition of a turbomachine

We classify as turbomachines all those devices in which energy is transferred either to, or from, a continuously flowing fluid by the *dynamic action* of one or more moving blade rows. The word *turbo* or *turbinis* is of Latin origin and implies that which spins or whirls around. Essentially, a rotating blade row, a *rotor* or an *impeller* changes the stagnation enthalpy of the fluid moving through it by doing either positive or negative work, depending upon the effect required of the machine. These enthalpy changes are intimately linked with the pressure changes occurring simultaneously in the fluid.

Two main categories of turbomachine are identified: first, those that *absorb* power to increase the fluid pressure or head (ducted and unducted fans, compressors, and pumps); second, those that *produce* power by expanding fluid to a lower pressure or head (wind, hydraulic, steam, and gas turbines). Figure 1.1 shows, in a simple diagrammatic form, a selection of the many varieties of turbomachines encountered in practice. The reason that so many different types of either pump (compressor) or turbine are in use is because of the almost infinite range of service requirements. Generally speaking, for a given set of operating requirements one type of pump or turbine is best suited to provide optimum conditions of operation.

Turbomachines are further categorized according to the nature of the flow path through the passages of the rotor. When the path of the *through-flow* is wholly or mainly parallel to the axis of rotation, the device is termed an *axial flow turbomachine* (e.g., Figures 1.1(a) and (e)). When the path of the *through-flow* is wholly or mainly in a plane perpendicular to the rotation axis, the device is termed a *radial flow turbomachine* (e.g., Figure 1.1(c)). More detailed sketches of radial flow machines are given in Figures 7.3, 7.4, 8.2, and 8.3. *Mixed flow turbomachines* are widely used. The term *mixed flow* in this context refers to the direction of the through-flow at the rotor outlet when both radial and axial velocity components are present in significant amounts. Figure 1.1(b) shows a mixed flow pump and Figure 1.1(d) a mixed flow hydraulic turbine.

One further category should be mentioned. All turbomachines can be classified as either *impulse* or *reaction* machines according to whether pressure changes are absent or present, respectively, in the flow through the rotor. In an impulse machine all the pressure change takes place in one or more nozzles, the fluid being directed onto the rotor. The Pelton wheel, Figure 1.1(f), is an example of an impulse turbine.
The main purpose of this book is to examine, through the laws of fluid mechanics and thermodynamics, the means by which the energy transfer is achieved in the chief types of turbomachines, together with the differing behavior of individual types in operation. Methods of analyzing the flow processes differ depending upon the geometrical configuration of the machine, whether the fluid can be regarded as incompressible or not, and whether the machine absorbs or produces work. As far as possible, a unified treatment is adopted so that machines having similar configurations and function are considered together.

1.2 Coordinate system

Turbomachines consist of rotating and stationary blades arranged around a common axis, which means that they tend to have some form of cylindrical shape. It is therefore natural to use a
cylindrical polar coordinate system aligned with the axis of rotation for their description and analysis. This coordinate system is pictured in Figure 1.2. The three axes are referred to as axial $x$, radial $r$, and tangential (or circumferential) $r\theta$.

In general, the flow in a turbomachine has components of velocity along all three axes, which vary in all directions. However, to simplify the analysis it is usually assumed that the flow does not vary in the tangential direction. In this case, the flow moves through the machine on axi-symmetric stream surfaces, as drawn on Figure 1.2(a). The component of velocity along an axi-symmetric stream surface is called the meridional velocity,

$$c_m = \sqrt{c_x^2 + c_r^2}$$  \hspace{1cm} (1.1)
In purely axial flow machines the radius of the flow path is constant and, therefore, referring to Figure 1.2(c) the radial flow velocity will be zero and \( c_m = c_x \). Similarly, in purely radial flow machines the axial flow velocity will be zero and \( c_m = c_r \). Examples of both of these types of machines can be found in Figure 1.1.

The total flow velocity is made up of the meridional and tangential components and can be written

\[
\sqrt{c_x^2 + c_r^2 + c_\theta^2} = \sqrt{c_m^2 + c_\theta^2} \tag{1.2}
\]

The swirl, or tangential, angle is the angle between the flow direction and the meridional direction:

\[
\alpha = \tan^{-1}(c_\theta/c_m) \tag{1.3}
\]

**Relative velocities**

The analysis of the flow-field within the rotating blades of a turbomachine is performed in a frame of reference that is stationary relative to the blades. In this frame of reference the flow appears as steady, whereas in the absolute frame of reference it would be unsteady. This makes any calculations significantly easier, and therefore the use of relative velocities and relative flow quantities is fundamental to the study of turbomachinery.

The relative velocity \( w \) is the vector subtraction of the local velocity of the blade \( U \) from the absolute velocity of the flow \( c \), as shown in Figure 1.2(c). The blade has velocity only in the tangential direction, and therefore the components of the relative velocity can be written as

\[
w_\theta = c_\theta - U, \quad w_x = c_x, \quad w_r = c_r \tag{1.4}
\]

The relative flow angle is the angle between the relative flow direction and the meridional direction:

\[
\beta = \tan^{-1}(w_\theta/c_m) \tag{1.5}
\]

By combining Eqs. (1.3), (1.4), and (1.5) a relationship between the relative and absolute flow angles can be found:

\[
\tan \beta = \tan \alpha - U/c_m \tag{1.6}
\]

**Sign convention**

Equations (1.4) and (1.6) suggest that negative values of flow angles and velocities are possible. In many turbomachinery courses and texts, the convention is to use positive values for tangential velocities that are in the direction of rotation (as they are in Figure 1.2(b) and (c)), and negative values for tangential velocities that are opposite to the direction of rotation. The convention adopted in this book is to ensure that the correct vector relationship between the relative and absolute velocities is applied using only positive values for flow velocities and flow angles.
**Velocity diagrams for an axial flow compressor stage**

A typical stage of an axial flow compressor is shown schematically in Figure 1.3 (looking radially inwards) to show the arrangement of the blading and the flow onto the blades.

The flow enters the stage at an angle $\alpha_1$ with a velocity $c_1$. This inlet velocity is set by whatever is directly upstream of the compressor stage: an inlet duct, another compressor stage or an inlet guide vane (IGV). By vector subtraction the relative velocity entering the rotor will have a magnitude $w_1$ at a relative flow angle $\beta_1$. The rotor blades are designed to smoothly accept this relative flow and change its direction so that at outlet the flow leaves the rotor with a relative velocity $w_2$ at a relative flow angle $\beta_2$. As shown later in this chapter, work will be done by the rotor blades on the gas during this process and, as a consequence, the gas stagnation pressure and stagnation temperature will be increased.

By vector addition the absolute velocity at rotor exit $c_2$ is found at flow angle $\alpha_2$. This flow should smoothly enter the stator row which it then leaves at a reduced velocity $c_3$ at an absolute angle $\alpha_3$. The diffusion in velocity from $c_2$ to $c_3$ causes the pressure and temperature to rise further. Following this the gas is directed to the following rotor and the process goes on repeating through the remaining stages of the compressor.

The purpose of this brief explanation is to introduce the reader to the basic fluid mechanical processes of turbomachinery via an axial flow compressor. It is hoped that the reader will follow the description given in relation to the velocity changes shown in Figure 1.3 as this is fundamental to understanding the subject of turbomachinery. Velocity triangles will be considered in further detail for each category of turbomachine in later chapters.
EXAMPLE 1.1

The axial velocity through an axial flow fan is constant and equal to 30 m/s. With the notation given in Figure 1.3, the flow angles for the stage are $\alpha_1$ and $\beta_2$ are $23^\circ$ and $\beta_1$ and $\alpha_2$ are $60^\circ$.

From this information determine the blade speed $U$ and, if the mean radius of the fan is 0.15 m, find the rotational speed of the rotor.

Solution

The velocity components are easily calculated as follows:

\[ w_{\theta 1} = c_x \tan \beta_1 \quad \text{and} \quad c_{\theta 1} = c_x \tan \alpha_1 \]

\[ \therefore U_m = c_{\theta 1} + w_{\theta 1} = c_x (\tan \alpha_1 + \tan \beta_1) = 64.7 \text{ m/s} \]

The speed of rotation is

\[ \Omega = \frac{U_m}{r_m} = 431.3 \text{ rad/s} \quad \text{or} \quad 431.3 \times \frac{30}{\pi} = 4119 \text{ rpm} \]

1.3 The fundamental laws

The remainder of this chapter summarizes the basic physical laws of fluid mechanics and thermodynamics, developing them into a form suitable for the study of turbomachines. Following this, the properties of fluids, compressible flow relations and the efficiency of compression and expansion flow processes are covered.

The laws discussed are

i. the continuity of flow equation;
ii. the first law of thermodynamics and the steady flow energy equation;
iii. the momentum equation;
iv. the second law of thermodynamics.

All of these laws are usually covered in first-year university engineering and technology courses, so only the briefest discussion and analysis is given here. Some textbooks dealing comprehensively with these laws are those written by Çengel and Boles (1994), Douglas, Gasiorek and Swaffield (1995), Rogers and Mayhew (1992), and Reynolds and Perkins (1977). It is worth remembering that these laws are completely general; they are independent of the nature of the fluid or whether the fluid is compressible or incompressible.

1.4 The equation of continuity

Consider the flow of a fluid with density $\rho$, through the element of area $dA$, during the time interval $dt$. Referring to Figure 1.4, if $c$ is the stream velocity the elementary mass is $dm = \rho c d\theta dA \cos \theta$, where $\theta$ is the angle subtended by the normal of the area element to the stream direction.
The element of area perpendicular to the flow direction is $dA_n = da \cos \theta$ and so $dm = \rho c dA_n dt$. The elementary rate of mass flow is therefore

$$dm = \frac{dm}{dt} = \rho c A_n$$  \hspace{0.5cm} (1.7)

Most analyses in this book are limited to one-dimensional steady flows where the velocity and density are regarded as constant across each section of a duct or passage. If $A_{n1}$ and $A_{n2}$ are the areas normal to the flow direction at stations 1 and 2 along a passage respectively, then

$$\dot{m} = \rho_1 c_1 A_{n1} = \rho_2 c_2 A_{n2} = \rho c A_n$$  \hspace{0.5cm} (1.8)

since there is no accumulation of fluid within the control volume.

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1.5 The first law of thermodynamics

The first law of thermodynamics states that, if a system is taken through a complete cycle during which heat is supplied and work is done, then

$$\oint (dQ - dW) = 0$$  \hspace{0.5cm} (1.9)

where $\oint dQ$ represents the heat supplied to the system during the cycle and $\oint dW$ the work done by the system during the cycle. The units of heat and work in Eq. (1.9) are taken to be the same.

During a change from state 1 to state 2, there is a change in the energy within the system:

$$E_2 - E_1 = \int_{1}^{2} (dQ - dW)$$  \hspace{0.5cm} (1.10a)

where $E = U + (1/2)mc^2 + mgz$.

For an infinitesimal change of state,

$$dE = dQ - dW$$  \hspace{0.5cm} (1.10b)
The steady flow energy equation

Many textbooks, e.g., Çengel and Boles (1994), demonstrate how the first law of thermodynamics is applied to the steady flow of fluid through a control volume so that the steady flow energy equation is obtained. It is unprofitable to reproduce this proof here and only the final result is quoted. Figure 1.5 shows a control volume representing a turbomachine, through which fluid passes at a steady rate of mass flow \( \dot{m} \), entering at position 1 and leaving at position 2. Energy is transferred from the fluid to the blades of the turbomachine, positive work being done (via the shaft) at the rate \( \dot{W}_x \). In the general case positive heat transfer takes place at the rate \( \dot{Q} \), from the surroundings to the control volume. Thus, with this sign convention the steady flow energy equation is

\[
\dot{Q} - \dot{W}_x = \dot{m} \left[ (h_2 - h_1) + \frac{1}{2} (c_2^2 - c_1^2) + g(z_2 - z_1) \right], \tag{1.11}
\]

where \( h \) is the specific enthalpy, \( 1/2c^2 \), the kinetic energy per unit mass and \( gz \), the potential energy per unit mass.

For convenience, the specific enthalpy, \( h \), and the kinetic energy, \( 1/2c^2 \), are combined and the result is called the stagnation enthalpy:

\[
h_0 = h + \frac{1}{2} c^2 \tag{1.12}
\]

Apart from hydraulic machines, the contribution of the \( g(z_2 - z_1) \) term in Eq. (1.11) is small and can usually be ignored. In this case, Eq. (1.11) can be written as

\[
\dot{Q} - \dot{W}_x = \dot{m}(h_{02} - h_{01}) \tag{1.13}
\]

The stagnation enthalpy is therefore constant in any flow process that does not involve a work transfer or a heat transfer. Most turbomachinery flow processes are adiabatic (or very nearly so) and it is permissible to write \( \dot{Q} = 0 \). For work producing machines (turbines) \( \dot{W}_x > 0 \), so that

\[
\dot{W}_x = \dot{W}_t = \dot{m}(h_{01} - h_{02}) \tag{1.14}
\]

For work absorbing machines (compressors) \( \dot{W}_x < 0 \), so that it is more convenient to write

\[
\dot{W}_c = -\dot{W}_x = \dot{m}(h_{02} - h_{01}) \tag{1.15}
\]
1.6 The momentum equation

One of the most fundamental and valuable principles in mechanics is Newton’s second law of motion. The momentum equation relates the sum of the external forces acting on a fluid element to its acceleration, or to the rate of change of momentum in the direction of the resultant external force. In the study of turbomachines many applications of the momentum equation can be found, e.g., the force exerted upon a blade in a compressor or turbine cascade caused by the deflection or acceleration of fluid passing the blades.

Considering a system of mass \( m \), the sum of all the body and surface forces acting on \( m \) along some arbitrary direction \( x \) is equal to \( \text{the time rate of change of the total } x\text{-momentum of the system} \), i.e.,

\[
\sum F_x = \frac{d}{dt}(mc_x) \quad (1.16a)
\]

For a control volume where fluid enters steadily at a uniform velocity \( c_{x1} \) and leaves steadily with a uniform velocity \( c_{x2} \), then

\[
\sum F_x = \dot{m}(c_{x2} - c_{x1}) \quad (1.16b)
\]

Equation (1.16b) is the one-dimensional form of the steady flow momentum equation.

Moment of momentum

In dynamics useful information can be obtained by employing Newton’s second law in the form where it applies to the moments of forces. This form is of central importance in the analysis of the energy transfer process in turbomachines.

For a system of mass \( m \), the vector sum of the moments of all external forces acting on the system about some arbitrary axis \( A \) fixed in space is equal to the time rate of change of angular momentum of the system about that axis, i.e.,

\[
\tau_A = m \frac{d}{dt}(rc_\theta) \quad (1.17a)
\]

where \( r \) is distance of the mass center from the axis of rotation measured along the normal to the axis and \( c_\theta \) the velocity component mutually perpendicular to both the axis and radius vector \( r \).

For a control volume the law of moment of momentum can be obtained. Figure 1.6 shows the control volume enclosing the rotor of a generalized turbomachine. Swirling fluid enters the control volume at radius \( r_1 \) with tangential velocity \( c_{\theta1} \) and leaves at radius \( r_2 \) with tangential velocity \( c_{\theta2} \). For one-dimensional steady flow,

\[
\tau_A = \dot{m}(r_2c_{\theta2} - r_1c_{\theta1}) \quad (1.17b)
\]

which states that the sum of the moments of the external forces acting on fluid temporarily occupying the control volume is equal to the net time rate of efflux of angular momentum from the control volume.
For a pump or compressor rotor running at angular velocity $\Omega$, the rate at which the rotor does work on the fluid is

$$W_c = \tau_A \Omega = \dot{m}(U_2 c_{\theta_2} - U_1 c_{\theta_1})$$  \hspace{1cm} (1.18a)

where the blade speed $U = \Omega r$.

Thus, the work done on the fluid per unit mass or specific work is

$$\Delta W_c = \frac{\dot{W}_c}{\dot{m}} = \frac{\tau_A \Omega}{\dot{m}} = U_2 c_{\theta_2} - U_1 c_{\theta_1} > 0$$  \hspace{1cm} (1.18b)

This equation is referred to as Euler’s pump or compressor equation.

For a turbine the fluid does work on the rotor and the sign for work is then reversed. Thus, the specific work is

$$\Delta W_t = \frac{\dot{W}_t}{\dot{m}} = U_1 c_{\theta_1} - U_2 c_{\theta_2} > 0$$  \hspace{1cm} (1.18c)

Equation (1.18c) is referred to as Euler’s turbine equation.

Note that, for any adiabatic turbomachine (turbine or compressor), applying the steady flow energy equation, Eq. (1.13), gives

$$\Delta W_x = (h_{01} - h_{02}) = U_1 c_{\theta_1} - U_2 c_{\theta_2}$$  \hspace{1cm} (1.19a)

Alternatively, this can be written as

$$\Delta h_0 = \Delta(U c_{\theta})$$  \hspace{1cm} (1.19b)

Equations (1.19a) and (1.19b) are the general forms of the Euler work equation. By considering the assumptions used in its derivation, this equation can be seen to be valid for adiabatic flow for any streamline through the blade rows of a turbomachine. It is applicable to both viscous and inviscid flow, since the torque provided by the fluid on the blades can be exerted by pressure forces or frictional forces. It is strictly valid only for steady flow but it can also be applied to time-averaged unsteady flow provided the averaging is done over a long enough time period. In all cases, all of the torque from the fluid must be transferred to the blades. Friction on the hub and casing of a
turbomachine can cause local changes in angular momentum that are not accounted for in the Euler work equation.

Note that for any stationary blade row, $U = 0$ and therefore $h_0 = \text{constant}$. This is to be expected since a stationary blade cannot transfer any work to or from the fluid.

**Rothalpy and relative velocities**

The Euler work equation, Eq. (1.19), can be rewritten as

$$I = h_0 - UC_\theta$$

where $I$ is a constant along the streamlines through a turbomachine. The function $I$ was first introduced by Wu (1952) and has acquired the widely used name *rothalpy*, a contraction of rotational stagnation enthalpy, and is a fluid mechanical property of some importance in the study of flow within rotating systems. The rothalpy can also be written in terms of the static enthalpy as

$$I = h + \frac{1}{2}c^2 - UC_\theta$$

The Euler work equation can also be written in terms of relative quantities for a rotating frame of reference. The relative tangential velocity, as given in Eq. (1.4), can be substituted in Eq. (1.20b) to produce

$$I = h + \frac{1}{2}(w^2 + U^2 + 2Uw_\theta) - U(w_\theta + U) = h + \frac{1}{2}w^2 - \frac{1}{2}U^2$$

Defining a relative stagnation enthalpy as $h_{0,\text{rel}} = h + (1/2)w^2$, Eq. (1.21a) can be simplified to

$$I = h_{0,\text{rel}} - \frac{1}{2}U^2$$

This final form of the Euler work equation shows that, for rotating blade rows, the relative stagnation enthalpy is constant through the blades provided the blade speed is constant. In other words, $h_{0,\text{rel}} = \text{constant}$, if the radius of a streamline passing through the blades stays the same. This result is important for analyzing turbomachinery flows in the relative frame of reference.

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**1.7 The second law of thermodynamics—entropy**

The *second law of thermodynamics*, developed rigorously in many modern thermodynamic textbooks, e.g., Çengel and Boles (1994), Reynolds and Perkins (1977), and Rogers and Mayhew (1992), enables the concept of entropy to be introduced and ideal thermodynamic processes to be defined.

An important and useful corollary of the second law of thermodynamics, known as the *Inequality of Clausius*, states that, for a system passing through a cycle involving heat exchanges,

$$\int \frac{dQ}{T} \leq 0$$

(1.22a)
where $dQ$ is an element of heat transferred to the system at an absolute temperature $T$. If all the processes in the cycle are reversible, then $dQ = dQ_R$, and the equality in Eq. (1.22a) holds true, i.e.,

$$\int \frac{dQ_R}{T} = 0 \quad (1.22b)$$

The property called *entropy*, for a finite change of state, is then defined as

$$S_2 - S_1 = \int_{1}^{2} \frac{dQ_R}{T} \quad (1.23a)$$

For an incremental change of state

$$dS = mds = \frac{dQ_R}{T} \quad (1.23b)$$

where $m$ is the mass of the system.

With steady one-dimensional flow through a control volume in which the fluid experiences a change of state from condition 1 at entry to 2 at exit,

$$\int_{1}^{2} \frac{d\dot{Q}}{T} \leq \dot{m}(s_2 - s_1) \quad (1.24a)$$

Alternatively, this can be written in terms of an entropy production due to irreversibility, $\Delta S_{irrev}$:

$$\dot{m}(s_2 - s_1) = \int_{1}^{2} \frac{d\dot{Q}}{T} + \Delta S_{irrev} \quad (1.24b)$$

If the process is adiabatic, $d\dot{Q} = 0$, then

$$s_2 \geq s_1 \quad (1.25a)$$

If the process is *reversible* as well, then

$$s_2 = s_1 \quad (1.25b)$$

Thus, for a flow undergoing a process that is both adiabatic and reversible, the entropy will remain unchanged (this type of process is referred to as *isentropic*). Since turbomachinery is usually adiabatic, or close to adiabatic, an isentropic compression or expansion represents the best possible process that can be achieved. To maximize the efficiency of a turbomachine, the irreversible entropy production $\Delta S_{irrev}$ must be minimized, and this is a primary objective of any design.

Several important expressions can be obtained using the preceding definition of entropy. For a system of mass $m$ undergoing a reversible process $dQ = dQ_R = mT ds$ and $dW = dW_R = mp dv$. In the absence of motion, gravity, and other effects the first law of thermodynamics, Eq. (1.10b) becomes

$$Tds = du + pdv \quad (1.26a)$$

With $h = u + pv$, then $dh = du + pdv + vdp$, and Eq. (1.26a) then gives

$$Tds = dh + vdp \quad (1.26b)$$
Equations (1.26a) and (1.26b) are extremely useful forms of the second law of thermodynamics because the equations are written only in terms of properties of the system (there are no terms involving $Q$ or $W$). These equations can therefore be applied to a system undergoing any process.

Entropy is a particularly useful property for the analysis of turbomachinery problems. Any increase of entropy in the flow path of a machine can be equated to a certain amount of “lost work” and thus a loss in efficiency. The value of entropy is the same in both the absolute and relative frames of reference (see Figure 1.9) and this means it can be used to track the sources of inefficiency through all the rotating and stationary parts of a machine. The application of entropy to account for lost performance is very powerful and will be demonstrated in later chapters.

### 1.8 Bernoulli’s equation

Consider the steady flow energy equation, Eq. (1.11). For adiabatic flow, with no work transfer,

\[
(h_2 - h_1) + \frac{1}{2}(c_2^2 - c_1^2) + g(z_2 - z_1) = 0
\]  

(1.27)

If this is applied to a control volume whose thickness is infinitesimal in the stream direction (Figure 1.7), the following differential form is derived:

\[
dh + c\,dc + g\,dz = 0
\]  

(1.28)

If there are no shear forces acting on the flow (no mixing or friction), then the flow will be isentropic and, from Eq. (1.26b), $dh = v\,dp = dp/\rho$, giving

\[
\frac{1}{\rho} dp + c\,dc + g\,dz = 0
\]  

(1.29a)

**FIGURE 1.7**

Control volume in a streaming fluid.
Equation (1.29a) is often referred to as the *one-dimensional form* of Euler’s equation of motion. Integrating this equation in the stream direction we obtain

\[ \int_1^2 \frac{1}{\rho} dp + \frac{1}{2} (c_2^2 - c_1^2) + g(z_2 - z_1) = 0 \] (1.29b)

which is Bernoulli’s equation. For an incompressible fluid, \( \rho \) is constant and Eq. (1.29b) becomes

\[ \frac{1}{\rho} (p_{02} - p_{01}) + g(z_2 - z_1) = 0 \] (1.29c)

where the stagnation pressure for an incompressible fluid is \( p_0 = p + (1/2) \rho c^2 \).

When dealing with hydraulic turbomachines, the term *head*, \( H \), occurs frequently and describes the quantity \( z + p_0/\rho g \). Thus, Eq. (1.29c) becomes

\[ H_2 - H_1 = 0 \] (1.29d)

If the fluid is a gas or vapor, the change in gravitational potential is generally negligible and Eq. (1.29b) is then

\[ \int_1^2 \frac{1}{\rho} dp + \frac{1}{2} (c_2^2 - c_1^2) = 0 \] (1.29e)

Now, if the gas or vapor is subject to only small pressure changes the fluid density is sensibly constant and integration of Eq. (1.29e) gives

\[ p_{02} = p_{01} = p_0 \] (1.29f)

i.e., the stagnation pressure is constant (it is shown later that this is also true for a *compressible isentropic process*).

### 1.9 The thermodynamic properties of fluids

The three most familiar fluid properties are the pressure \( p \), the temperature \( T \) and the density \( \rho \). We also need to consider how other associated thermodynamic properties such as the internal energy \( u \), the enthalpy \( h \), the entropy \( s \), and the specific heats \( C_p \) and \( C_v \) change during a flow process.

It is known from studies of statistical thermodynamics that in all fluid processes involving a change in pressure, an enormous number of molecular collisions take place in an extremely short interval which means that the fluid pressure rapidly adjusts to an equilibrium state. We can thus safely assume that all the properties listed above will follow the laws and state relations of classical equilibrium thermodynamics. We will also restrict ourselves to the following pure and homogenous substances: ideal gases, perfect gases, and steam.

**Ideal gases**

Air is a mixture of gases but, in the temperature range 160–2100 K, it can be regarded as a pure substance. Within this temperature range air obeys the ideal gas relationship:

\[ p = \rho RT \quad \text{or} \quad pv = RT \] (1.30)

where \( R = C_p - C_v \) is the gas constant.
The value of the gas constant $R$ for any ideal gas is equal to a Universal Gas Constant $R_0 = 8314 \text{ J/kmol}$ divided by the molecular weight of the gas. In this book many of the problems concern air so it is useful to evaluate a value for this gas mixture which has a molecular weight $M = 28.97 \text{ kg/kmol}$.

$$R_{\text{air}} = \frac{8314}{28.97} = 287 \text{ J/kg K}$$

For air under standard sea-level conditions, the pressure $p_a = 1.01 \text{ bar}$ and the temperature $T_a = 288 \text{ K}$. Thus, the density of air under standardized sea-level conditions is

$$\rho_a = \frac{p_a}{RT_a} = \frac{1.01 \times 10^5}{287 \times 288} = 1.222 \text{ kg/m}^3$$

All gases at high temperatures and at relatively low pressures conform to the ideal gas law. An ideal gas can be either a semi-perfect gas or a perfect gas.

In a semi-perfect gas, the specific heat capacities are functions of temperature only:

$$C_p = \left(\frac{\partial h}{\partial T}\right)_p = \frac{\partial h}{\partial T} = C_p(T) \text{ and } C_v = \left(\frac{\partial u}{\partial T}\right)_p = \frac{\partial u}{\partial T} = C_v(T)$$

Over large temperature differences, air and many other common gases should be treated as semi-perfect gases. The variation in the values of $C_p$ and $\gamma$ for air are shown in Figure 1.8. Note that $\gamma = C_p/C_v$ is the ratio of the specific heats, which is a particularly important parameter in compressible flow analysis (see Section 1.10).
Perfect gases

A perfect gas is an ideal gas for which $C_p$, $C_v$, and $\gamma$, are constants. Many real gases can be treated as perfect gases over a limited range of temperature and pressure. In the calculation of expansion or compression processes in turbomachines the normal practice is to use weighted mean values for $C_p$ and $\gamma$ according to the mean temperature of the process. Accordingly, in the problems in this book values have been selected for $C_p$ and $\gamma$ appropriate to the gas and the temperature range. For example, in air flow at temperatures close to ambient the value of $\gamma$ is taken to be 1.4.

Note that the entropy change for a perfect gas undergoing any process can be calculated from the properties at the start and end of the process. Substituting $dh = C_p dT$ and $pv = RT$ into Eq. (1.26b) gives:

$$T ds = C_p dT - RT dp/p$$

This equation can be integrated between the start state (1) and end state (2) of a process:

$$\int_1^2 ds = C_p \int_1^2 \frac{dT}{T} - R \int_1^2 \frac{dp}{p}$$

$$\therefore s_2 - s_1 = C_p \ln \frac{T_2}{T_1} - R \ln \frac{p_2}{p_1}$$

(1.31)

**EXAMPLE 1.2**

a. A quantity of carbon dioxide undergoes an *isentropic* process. Initially the pressure $p_1 = 120$ kPa and the temperature $T_1 = 120^\circ$C. Finally, at the end of the process, the pressure $p_2 = 100$ kPa. Determine the final temperature $T_2$.

b. Heat is now supplied to the gas at constant volume and the temperature rises to $200^\circ$C. Determine how much heat is supplied per unit mass of the gas, the final pressure, and the specific entropy increase of the gas due to the heat transfer.

Consider CO$_2$ to be a perfect gas with $R = 189$ J/kg K and $\gamma = 1.30$.

**Solution**

a. From Eq. (1.31), with $s_2 = s_1$

$$C_p \ln \frac{T_2}{T_1} = R \ln \frac{p_2}{p_1}$$

from which you can find:

$$T_2 = T_1 \left( \frac{p_2}{p_1} \right)^{(\gamma-1)/\gamma} = 393 \times 0.9588 = 376.8 \text{ K}$$

b. Applying the first law of thermodynamics to a system, Eq. (1.10b):

$$Q = \Delta U = C_v \Delta T, \quad T_3 = 473 \text{ K} \therefore Q = C_v(T_3 - T_2) = \frac{R}{\gamma - 1}(T_3 - T_2)$$
At constant volume, from

\[ p v = R T, \quad \frac{p_3}{p_2} = \frac{T_3}{T_2}, \quad \therefore p_3 = 100 \times \frac{473}{376.8} = 125.5 \text{ kPa} \]

The increase in entropy, from Eq. (1.31) is given by:

\[ \Delta s = C_P \ln \left( \frac{T_3}{T_2} \right) - R \ln \left( \frac{p_3}{p_2} \right) = \frac{\gamma R}{\gamma - 1} \ln \left( \frac{T_3}{T_2} \right) - R \ln \left( \frac{p_3}{p_2} \right) \]

\[ \therefore \Delta s = \frac{1.3 \times 189}{0.3} \ln \left( \frac{473}{376.8} \right) - 189 \ln \left( \frac{125.5}{100} \right) = 142.9 \text{ J/kg K} \]

**Steam**

Steam is the gaseous phase of water formed when pure water is boiled. When steam is in the two-phase region, where liquid and gaseous water coexist, it is known as *wet steam*. Steam turbines use the expansion of high-pressure steam to generate power. They typically operate close to or within the two-phase region, where the ideal gas law is highly inaccurate. No simple formulae apply and it is necessary to use tabulations of property values obtained by experiment and compiled as steam tables or steam charts to determine the effects of a change of state.

The thermodynamic properties of steam were the subject of many *difficult investigations* by groups of scientists and engineers over many years. An interesting summary of the methods used and the difficulties encountered are given in a paper by Harvey and Levelt Sengers (2001). The latest state-of-the-art account of the thermodynamic properties of water was adopted by the *International Association for the Properties of Water and Steam* (IAPWS) (Wagner and Pruss (2002)). The properties calculated from the current IAPWS standards for general and scientific use are distributed in a computer program by the National Institute of Standards and Technology (NIST) Standard Reference Data Program (Harvey, Peskin and Klein (2000)). These properties are also available via a free online calculator and in tabulated form (National Institute of Standards and Technology (2012)).

As well as steam tables the most immediate aid for performing calculations (although less accurate) is the Mollier diagram. This shows the enthalpy \( h \) (kJ/kg) plotted against entropy \( s \) (kJ/kg K) for various values of pressure \( p \) (MPa). A small, single-page Mollier chart is shown in Appendix E, but poster size charts can be obtained which, of course, enable greater accuracy.

**Commonly used thermodynamic terms relevant to steam tables**

i. **Saturation curve**

This is the boundary between the different phases on a property diagram. *Saturated liquid* refers to a state where all the water is in the liquid phase and *saturated vapor* refers to a state where all the water is in the gaseous phase. The two-phase region lies between the liquid and vapor saturation curves. Note that within the two-phase region temperature and pressure are no
longer independent properties. For example, at 1 bar pressure, when water is boiling, all the liquid and gas is at 100°C.

ii. Quality or dryness fraction

This applies within the two-phase region and is the ratio of the vapor mass to the total mass of liquid and vapor. The value of any intensive property within the two-phase region is the mass weighted average of the values on the liquid and vapor saturation curves at the same pressure and temperature. Hence, the quality or dryness fraction can be used to specify the thermodynamic state of the steam.

For example, consider a quantity of wet steam at a state with dryness fraction $x$. The specific enthalpy of the steam at this state will be given by:

$$ h = (1 - x)h_f + xh_g $$

where $h_f$ is the enthalpy on the liquid saturation curve, and $h_g$ is the enthalpy on the vapor saturation curve, both at the same temperature and pressure of the wet steam. The above approach can be used for other intensive properties, such as $u$, $v$, $s$.

iii. Degree of superheat of steam.

When steam is heated at constant pressure in the gaseous phase it will be at a higher temperature than the corresponding saturation temperature. The temperature difference between the steam temperature and the saturation temperature at the same pressure is the degree of superheat.

iv. The Triple Point and the Critical Point.

The triple point for water is the unique temperature and pressure where all three phases coexist: ice, liquid water, and steam. The critical point is the state where the liquid and vapor saturation curves meet at the highest temperature and pressure possible in the two-phase region.

1.10 Compressible flow relations for perfect gases

The Mach number of a flow is defined as the velocity divided by the local speed of sound. For a perfect gas, such as air over a limited temperature range, the Mach number can be written as

$$ M = \frac{c}{a} = \frac{c}{\sqrt{\gamma RT}} $$

Whenever the Mach number in a flow exceeds about 0.3, the flow becomes compressible, and the fluid density can no longer be considered as constant. High power turbomachines require high flow rates and high blade speeds and this inevitably leads to compressible flow. The static and stagnation quantities in the flow can be related using functions of the local Mach number and these are derived later.

Starting with the definition of stagnation enthalpy, $h_0 = h + (1/2)c^2$, this can be rewritten for a perfect gas as

$$ C_p T_0 = C_p T + \frac{c^2}{2} = C_p T + \frac{M^2 \gamma RT}{2} $$

(1.34a)
Given that $\gamma R = (\gamma - 1)C_p$, Eq. (1.34a) can be simplified to

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2 \tag{1.34b}$$

The stagnation pressure in a flow is the static pressure that is measured if the flow is brought isentropically to rest. From Eq. (1.26b), for an isentropic process $dh = dp/\rho$. If this is combined with the equation of state for a perfect gas, $p = \rho RT$, the following equation is obtained:

$$\frac{dp}{p} = \frac{C_p}{R} \frac{dT}{T} = \frac{dT}{T} \frac{\gamma}{\gamma - 1} \tag{1.35}$$

This can be integrated between the static and stagnation conditions to give the following compressible flow relation between the stagnation and static pressure:

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\gamma/(\gamma - 1)} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\gamma/(\gamma - 1)} \tag{1.36}$$

Equation (1.35) can also be integrated along a streamline between any two arbitrary points 1 and 2 within an isentropic flow. In this case, the stagnation temperatures and pressures are related:

$$\frac{p_{02}}{p_{01}} = \left(\frac{T_{02}}{T_{01}}\right)^{\gamma/(\gamma - 1)} \tag{1.37}$$

If there is no heat or work transfer to the flow, $T_0 = \text{constant}$. Hence, Eq. (1.37) shows that, in isentropic flow with no work transfer, $p_{02} = p_{01} = \text{constant}$, which was shown to be the case for incompressible flow in Eq. (1.29f).

Combining the equation of state, $p = \rho RT$ with Eqs. (1.34b), (1.36), and (1.38) the corresponding relationship for the stagnation density is obtained:

$$\frac{\rho_0}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{1/(\gamma - 1)} \tag{1.38}$$

Arguably the most important compressible flow relationship for turbomachinery is the one for nondimensional mass flow rate, sometimes referred to as capacity. It is obtained by combining Eqs. (1.34b), (1.36), and (1.38) with continuity, Eq. (1.8):

$$\frac{\dot{m}\sqrt{C_p T_0}}{A_n p_0} = \frac{\gamma}{\sqrt{\gamma - 1}} M \left(1 + \frac{\gamma - 1}{2} M^2\right)^{-\frac{1}{2}} \left[M^2 \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}\right]^{-\frac{1}{2}} \tag{1.39}$$

This result is important since it can be used to relate the flow properties at different points within a compressible flow turbomachine. The application of Eq. (1.39) is demonstrated in Chapter 3.

Note that the compressible flow relations given previously can be applied in the relative frame of reference for flow within rotating blade rows. In this case relative stagnation properties and relative Mach numbers are used:

$$\frac{p_{0,rel}}{p}, \frac{T_{0,rel}}{T}, \frac{\rho_{0,rel}}{\rho}, \frac{\dot{m}\sqrt{C_p T_{0,rel}}}{A p_{0,rel}} = f(M_{rel})$$
Figure 1.9 shows the relationship between stagnation and static conditions on a temperature—entropy diagram, in which the temperature differences have been exaggerated for clarity. This shows the relative stagnation properties as well as the absolute properties for a single point in a flow. Note that all of the conditions have the same entropy because the stagnation states are defined using an isentropic process. The pressures and temperatures are related using Eq. (1.36).

**EXAMPLE 1.3**

Air flows adiabatically and at high subsonic speed through a duct. At a station which we will call A, flow measurements indicate that the velocity \( c_A \) is 250 m/s, the static temperature \( T_A \) is 315 K and the static pressure \( p_A \) is 180 kPa. Determine the values of the stagnation temperature \( T_{0A} \), the Mach number \( M_A \) the stagnation pressure \( p_{0A} \) and the stagnation density \( \rho_A \). If the duct cross-sectional area is 0.1 m\(^2\), calculate the air mass flow rate. For air take \( R = 287 \text{ J/kg K} \) and \( \gamma = 1.4 \).

**Solution**

From Eq. (1.34a)

\[
T_{0A} = T_A + \frac{c_A^2}{2C_p} = 346 \text{ K}
\]

From Eq. (1.33)

\[
M_A = \frac{c_A}{\sqrt{\gamma RT_A}} = 0.703
\]

From Eq. (1.36)

\[
p_{0A} = p_A \left( 1 + \frac{\gamma - 1}{2} M_A^2 \right)^{\frac{\gamma}{\gamma - 1}} = 250 \text{ kPa}
\]
From Eq. (1.38)

\[ \rho_{0A} = \rho_A \left( 1 + \frac{\gamma - 1}{2} \frac{M_A^2}{\gamma} \right)^{\frac{1}{\gamma}} \]

where \( \rho_A = \frac{p_A}{RT_A} = 1.991 \text{ kg/m}^3 \)

\[ \therefore \rho_{0A} = 2.52 \text{ kg/m}^3 \]

Here, it will be obvious that the stagnation density can be evaluated more directly using the gas law:

\[ \rho_{0A} = \frac{p_{0A}}{RT_{0A}} = 2.52 \text{ kg/m}^3 \]

There are also two ways to evaluate the air mass flow rate. Using Eq. (1.8)

\[ \dot{m} = \rho_A A_c = 1.99 \times 0.1 \times 250 = 49.8 \text{ kg/s} \]

Alternatively, from Eq. (1.39) or Table C.1,

\[ \frac{\dot{m}\sqrt{CpT_{0A}}}{p_{0A}A_A} = f(0.703) = 1.1728 \]

\[ \therefore \dot{m} = 1.1728 \times \frac{p_{0A}A_A}{\sqrt{CpT_{0A}}} = 49.7 \text{ kg/s} \]

Note that Appendix C includes tabulated results for Eqs. (1.34), (1.36), (1.38), and (1.39).

**Choked flow**

For subsonic flow, as flow speed and Mach number increase, the mass flow per unit area increases. This is because, from Eq. (1.8), the mass flow per unit area is \( \dot{m}/A = \rho c \) and as Mach number rises, the flow speed \( c \) increases more rapidly than the density \( \rho \) reduces. However, this is not true for supersonic flow and, above \( M = 1 \), as flow speed and Mach number increase, the mass flow per unit area decreases. There is, therefore, a maximum mass flow per unit area which occurs at sonic conditions (\( M = 1 \)). This maximum can be readily observed by plotting out the nondimensional mass flow function given in Eq. (1.39) for a Mach number range from 0 to 2 using a fixed value of \( \gamma \).

An important consequence of this is that the mass flow through any turbomachinery component reaches a maximum once \( M = 1 \) across the section of minimum flow area. The flow is said to be *choked* and it is not possible to increase the mass flow further (without changing the inlet stagnation conditions). The section of minimum flow area is known as the *throat* and the size of the throat is a critical design parameter since it determines the maximum mass flow that can pass through a transonic turbomachine. Under choked conditions, because pressure waves in the flow travel at \( M = 1 \), changes to the flow downstream of the throat cannot have any effect on the flow upstream of the throat.

Choking is considered in further detail for compressor and turbine blade rows within Sections 3.5 and 3.6, respectively.
1.11 Definitions of efficiency

A large number of efficiency definitions are included in the literature of turbomachines and most workers in this field would agree there are too many. In this book only those considered to be important and useful are included.

Efficiency of turbines

Turbines are designed to convert the available energy in a flowing fluid into useful mechanical work delivered at the coupling of the output shaft. The efficiency of this process, the overall efficiency \( \eta_0 \), is a performance factor of considerable interest to both designer and user of the turbine. Thus,

\[
\eta_0 = \frac{\text{mechanical energy available at coupling of output shaft in unit time}}{\text{maximum energy difference possible for the fluid in unit time}}
\]

Mechanical energy losses occur between the turbine rotor and the output shaft coupling as a result of the work done against friction at the bearings, glands, etc. The magnitude of this loss as a fraction of the total energy transferred to the rotor is difficult to estimate as it varies with the size and individual design of turbomachine. For small machines (several kilowatts) it may amount to 5% or more, but for medium and large machines this loss ratio may become as little as 1%. A detailed consideration of the mechanical losses in turbomachines is beyond the scope of this book and is not pursued further.

The isentropic efficiency \( \eta_t \) or hydraulic efficiency \( \eta_h \) for a turbine is, in broad terms,

\[
\eta_t (\text{or } \eta_h) = \frac{\text{mechanical energy supplied to the rotor in unit time}}{\text{maximum energy difference possible for the fluid in unit time}}
\]

Comparing these definitions it is easily deduced that the mechanical efficiency \( \eta_m \), which is simply the ratio of shaft power to rotor power, is

\[
\eta_m = \frac{\eta_0}{\eta_t (\text{or } \eta_h)} \quad (1.40)
\]

The preceding isentropic efficiency definition can be concisely expressed in terms of the work done by the fluid passing through the turbine:

\[
\eta_t (\text{or } \eta_h) = \frac{\text{actual work}}{\text{ideal (maximum) work}} = \frac{\Delta W_x}{\Delta W_{\text{max}}} \quad (1.41)
\]

The actual work is unambiguous and straightforward to determine from the steady flow energy equation, Eq. (1.11). For an adiabatic turbine, using the definition of stagnation enthalpy,

\[
\Delta W_x = \dot{W}_x / \dot{m} = (h_{01} - h_{02}) + g(z_1 - z_2)
\]

The ideal work is slightly more complicated as it depends on how the ideal process is defined. The process that gives maximum work will always be an isentropic expansion, but the question is one of how to define the exit state of the ideal process relative to the actual process. In the following paragraphs the different definitions are discussed in terms of to what type of turbine they are applied.
Steam and gas turbines

Figure 1.10(a) shows a simplified Mollier diagram representing the expansion process through an adiabatic turbine. Line $1\to C_0$ represents the actual expansion and line $1\to C_0 s$ the ideal or reversible expansion. The fluid velocities at entry to and exit from a turbine may be quite high and the corresponding kinetic energies significant. On the other hand, for a compressible fluid the potential energy terms are usually negligible. Hence, the actual turbine rotor specific work is

$$\Delta W_x = \dot{W}_x/m = h_{01} - h_{02} = (h_1 - h_2) + \frac{1}{2}(c_1^2 - c_2^2)$$

There are two main ways of expressing the isentropic efficiency, the choice of definition depending largely upon whether the exit kinetic energy is usefully employed or is wasted. If the exhaust kinetic energy is useful, then the ideal expansion is to the same stagnation (or total) pressure as the actual process. The ideal work output is, therefore, that obtained between state points 01 and 02s,

$$\Delta W_{\text{max}} = \dot{W}_{\text{max}}/m = h_{01} - h_{02s} = (h_1 - h_{2s}) + \frac{1}{2}(c_1^2 - c_{2s}^2)$$

The relevant adiabatic efficiency, $\eta$, is called the total-to-total efficiency and it is given by

$$\eta_{\text{tt}} = \Delta W_x/\Delta W_{\text{max}} = (h_{01} - h_{02})/(h_{01} - h_{02s})$$  \hspace{1cm} (1.42a)

If the difference between the inlet and outlet kinetic energies is small, i.e., $(1/2)c_1^2 \approx (1/2)c_2^2$, then

$$\eta_{\text{tt}} = (h_1 - h_2)/(h_1 - h_{2s})$$  \hspace{1cm} (1.42b)
An example where the exhaust kinetic energy is not wasted is from the last stage of an aircraft gas turbine where it contributes to the jet propulsive thrust. Likewise, the exit kinetic energy from one stage of a multistage turbine where it can be used in the following stage provides another example.

If, instead, the exhaust kinetic energy cannot be usefully employed and is entirely wasted, the ideal expansion is to the same static pressure as the actual process with zero exit kinetic energy. The ideal work output in this case is that obtained between state points 01 and 2s:

$$\Delta W_{\text{max}} = \dot{m} (h_{01} - h_{2s}) = (h_{1} - h_{2s}) + \frac{1}{2} c_{1}^{2}$$

The relevant adiabatic efficiency is called the total-to-static efficiency $\eta_{ts}$ and is given by

$$\eta_{ts} = \frac{\Delta W_{s}}{\Delta W_{\text{max}}} = \frac{(h_{01} - h_{02})}{(h_{01} - h_{2s})}$$  (1.43a)

If the difference between inlet and outlet kinetic energies is small, Eq. (1.43a) becomes

$$\eta_{ts} = \frac{(h_{1} - h_{2})}{(h_{1} - h_{2s} + \frac{1}{2} c_{1}^{2})}$$  (1.43b)

A situation where the outlet kinetic energy is wasted is a turbine exhausting directly to the surroundings rather than through a diffuser. For example, auxiliary turbines used in rockets often have no exhaust diffusers because the disadvantages of increased mass and space utilization are greater than the extra propellant required as a result of reduced turbine efficiency.

By comparing Eqs. (1.42) and (1.43) it is clear that the total-to-static efficiency will always be lower than the total-to-total efficiency. The total-to-total efficiency relates to the internal losses (entropy creation) within the turbine, whereas the total-to-static efficiency relates to the internal losses plus the wasted kinetic energy.

### EXAMPLE 1.4

A steam turbine receives 10 kg/s of superheated steam at 20 bar and 350°C which then expands through the turbine to a pressure of 0.3 bar and a dryness fraction of 0.95. Neglecting any changes in kinetic energy, determine

a. the change in enthalpy of the steam in its passage through the turbine
b. the increase in entropy of the steam
c. the total-to-total efficiency of the turbine.
d. the power output of the turbine

**Solution**

A small Mollier diagram for steam is shown in Appendix E. This can be used to verify the enthalpy and entropy values for the expansion given below.

<table>
<thead>
<tr>
<th></th>
<th>$T$ °C</th>
<th>$h$ kJ/kg</th>
<th>$s$ kJ/kg K</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inlet Steam at 20 bar</td>
<td>350</td>
<td>3140</td>
<td>6.96</td>
</tr>
<tr>
<td>Saturated Liquid at 0.3 bar</td>
<td>69.1</td>
<td>289.3</td>
<td>0.944</td>
</tr>
<tr>
<td>Saturated Vapor at 0.3 bar</td>
<td>69.1</td>
<td>2624.5</td>
<td>7.767</td>
</tr>
</tbody>
</table>
a. First determine the specific enthalpy and entropy at exit from the steam turbine (state 2). Using Eq. (1.32) for a dryness fraction of 0.95:

\[ h_2 = 0.95h_g + 0.05h_f = 0.95 \times 2624.5 + 0.05 \times 289.3 = 2510 \text{ kJ/kg} \]

\[ s_2 = 0.95s_g + 0.05s_f = 0.95 \times 7.767 + 0.05 \times 0.944 = 7.43 \text{ kJ/kg K} \]

\[ \Delta h_0 = 630 \text{ kJ/kg} \]

\[ \Delta s = 0.47 \text{ kJ/kg K} \]

b. The efficiency of the turbine expansion process is

\[ \eta_{tt} = \frac{h_{01} - h_{02}}{h_{01} - h_{02s}} = \frac{630}{790} = 0.797 \]

Note that \( h_{02s} = 2350 \text{ kJ/kg} \) is the enthalpy where \( p = 0.3 \text{ bar} \) and \( s = 6.96 \text{ kJ/kg K} \).

d. The power output is \( W = \dot{m}(h_{01} - h_{02}) = 10 \times 630 = 6.3 \text{ MW} \)

**Hydraulic turbines**

The turbine hydraulic efficiency is a form of the total-to-total efficiency expressed previously. The steady flow energy equation (Eq. 1.11) can be written in differential form for an adiabatic turbine as

\[ dW_x = \dot{m} \left[ dh + \frac{1}{2} d(c^2) + gdz \right] \]

For an isentropic process, \( Tds = 0 = dh - dp/\rho \). The maximum work output for an expansion to the same exit static pressure, kinetic energy, and height as the actual process is, therefore,

\[ W_{\text{max}} = \dot{m} \left[ \int_{1}^{2} \frac{1}{\rho} dp + \frac{1}{2} (c_1^2 - c_2^2) + g(z_1 - z_2) \right] \]

For an incompressible fluid, the maximum work output from a hydraulic turbine (ignoring frictional losses) can be written

\[ W_{\text{max}} = \dot{m} \left[ \frac{1}{\rho} (p_1 - p_2) + \frac{1}{2} (c_1^2 - c_2^2) + g(z_1 - z_2) \right] = \dot{m}g(H_1 - H_2) \]

where \( gH = p/\rho + (1/2)c^2 + gz \) and \( \dot{m} = \rho Q \).

The turbine hydraulic efficiency, \( \eta_h \), is the work supplied by the rotor divided by the hydrodynamic energy difference of the fluid, i.e.,

\[ \eta_h = \frac{W_x}{W_{\text{max}}} = \frac{\Delta W_x}{g[H_1 - H_2]} \]  

(1.44)
Efficiency of compressors and pumps

The isentropic efficiency, $\eta_c$, of a compressor or the hydraulic efficiency of a pump, $\eta_h$, is broadly defined as

$$\eta_c (or \ \eta_h) = \frac{\text{useful (hydrodynamic) energy input to fluid in unit time}}{\text{power input to rotor}}$$

The power input to the rotor (or impeller) is always less than the power supplied at the coupling because of external energy losses in the bearings, glands, etc. Thus, the overall efficiency of the compressor or pump is

$$\eta_o = \frac{\text{useful (hydrodynamic) energy input to fluid in unit time}}{\text{power input to coupling of shaft}}$$

Hence, the mechanical efficiency is

$$\eta_m = \eta_o / \eta_c (or \ \eta_o / \eta_h) \quad (1.45)$$

For a complete adiabatic compression process going from state 1 to state 2, the specific work input is

$$\Delta W_c = (h_{02} - h_{01}) + g(z_2 - z_1)$$

Figure 1.10(b) shows a Mollier diagram on which the actual compression process is represented by the state change 1–2 and the corresponding ideal process by 1–2s. For an adiabatic compressor in which potential energy changes are negligible, the most meaningful efficiency is the total-to-total efficiency, which can be written as

$$\eta_c = \frac{\text{ideal (minimum) work input}}{\text{actual work input}} = \frac{h_{02s} - h_{01}}{h_{02} - h_{01}} \quad (1.46a)$$

If the difference between inlet and outlet kinetic energies is small, $(1/2)c_1^2 \approx (1/2)c_2^2$ then

$$\eta_c = \frac{h_{2s} - h_1}{h_2 - h_1} \quad (1.46b)$$

For incompressible flow, the minimum work input is given by

$$\Delta W_{\text{min}} = \dot{W}_{\text{min}}/\dot{m} = \left[ (p_2 - p_1)/p + \frac{1}{2}(c_2^2 - c_1^2) + g(z_2 - z_1) \right] = g[H_2 - H_1]$$

For a pump the hydraulic efficiency is therefore defined as

$$\eta_h = \frac{\dot{W}_{\text{min}}}{\dot{W}_c} = \frac{g[H_2 - H_1]}{\Delta W_c} \quad (1.47)$$
EXAMPLE 1.5
A hydraulic pump delivers 0.4 m$^3$/s of water against a head of 6.0 m. If the efficiency of the pump is known to be 85%, how much power is needed to drive the pump?

Solution
From Eq. (1.47)

$$\eta_h = \frac{g\Delta H}{\Delta W_c} \implies \Delta W_c = g\Delta H/\eta_h = \frac{9.81 \times 6}{0.85} = 69.25 \text{ J/kg}$$

$$\therefore P = \rho Q\Delta W_c = 10^3 \times 0.4 \times 69.25 = 27.7 \text{ kW}$$

1.12 Small stage or polytropic efficiency

The isentropic efficiency described in the preceding section, although fundamentally valid, can be misleading if used for comparing the efficiencies of turbomachines of differing pressure ratios. Now, any turbomachine may be regarded as being composed of a large number of very small stages, irrespective of the actual number of stages in the machine. If each small stage has the same efficiency, then the isentropic efficiency of the whole machine will be different from the small stage efficiency, the difference depending upon the pressure ratio of the machine. This perhaps rather surprising result is a manifestation of a simple thermodynamic effect concealed in the expression for isentropic efficiency and is made apparent in the following argument.

Compression process

Figure 1.11 shows an enthalpy—entropy diagram on which adiabatic compression between pressures $p_1$ and $p_2$ is represented by the change of state between points 1 and 2. The corresponding reversible process is represented by the isentropic line 1 to 2s. It is assumed that the compression process may be divided into a large number of small stages of equal efficiency $\eta_p$. For each small stage the actual work input is $\delta W$ and the corresponding ideal work in the isentropic process is $\delta W_{\text{min}}$. With the notation of Figure 1.11,

$$\eta_p = \frac{\delta W_{\text{min}}}{\delta W} = \frac{h_{xs} - h_1}{h_x - h_1} = \frac{h_{ys} - h_x}{h_y - h_x} = \ldots$$

Since each small stage has the same efficiency, then $\eta_p = (\Sigma \delta W_{\text{min}}/\Sigma \delta W)$ is also true.

From the relation $Tds = dh - vdp$, for a constant pressure process, $(\partial h/\partial s)_{p1} = T$. This means that the higher the fluid temperature, the greater is the slope of the constant pressure lines on the Mollier diagram. For a gas where $h$ is a function of $T$, constant pressure lines diverge and the slope of the line $p_2$ is greater than the slope of line $p_1$ at the same value of entropy. At equal values of $T$, constant pressure lines are of equal slope as indicated in Figure 1.11. For the special case of a
perfect gas (where $C_p$ is constant), $C_p(dT/ds) = T$ for a constant pressure process. Integrating this expression results in the equation for a constant pressure line, $s = C_p \log T + \text{constant}$.

Returning now to the more general case, since

$$\sum dW = \{(h_x - h_1) + (h_y - h_2) + \cdots\} = (h_2 - h_1)$$

then

$$\eta_p = [(h_{xs} - h_1) + (h_{ys} - h_3) + \cdots]/(h_2 - h_1)$$

The adiabatic efficiency of the whole compression process is

$$\eta_c = (h_{2s} - h_1)/(h_2 - h_1)$$

Due to the divergence of the constant pressure lines

$$\{(h_{xs} - h_1) + (h_{ys} - h_2) + \cdots\} > (h_{2s} - h_1)$$

i.e.,

$$\sum \delta W_{\text{min}} > W_{\text{min}}$$

Therefore,

$$\eta_p > \eta_c$$

FIGURE 1.11
Compression process by small stages.
Thus, for a compression process the isentropic efficiency of the machine is *less* than the small stage efficiency, the difference being dependent upon the divergence of the constant pressure lines. Although the foregoing discussion has been in terms of static states it also applies to stagnation states since these are related to the static states via isentropic processes.

**Small stage efficiency for a perfect gas**

An explicit relation can be readily derived for a perfect gas between small stage efficiency, the overall isentropic efficiency and the pressure ratio. The analysis is for the limiting case of an infinitesimal compressor stage in which the incremental change in pressure is $dp$ as indicated in Figure 1.12. For the actual process the incremental enthalpy rise is $dh$ and the corresponding ideal enthalpy rise is $dh_{is}$.

The polytropic efficiency for the small stage is

$$\eta_P = \frac{dh_{is}}{dh} = \frac{vdp}{C_p dT}$$

since for an isentropic process $Tds = 0 = dh_{is} - vdp$. Substituting $v = RT/p$ into Eq. (1.48) and using $C_p = \gamma R/(\gamma - 1)$ gives

$$\frac{dT}{T} = \frac{(\gamma - 1) dp}{\gamma \eta_P \ p}$$

Integrating Eq. (1.49) across the whole compressor and taking equal efficiency for each infinitesimal stage gives

$$\frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right)^{(\gamma - 1)/\eta_P \gamma}$$

(1.50)
Now the isentropic efficiency for the whole compression process is

\[ \eta_c = \frac{(T_{2s} - T_1)}{(T_2 - T_1)} \quad (1.51) \]

if it is assumed that the velocities at inlet and outlet are equal.

For the *ideal* compression process put \( \eta_p = 1 \) in Eq. (1.50) and so obtain

\[ \frac{T_{2s}}{T_1} = \left( \frac{p_2}{p_1} \right)^{(\gamma - 1)/\gamma} \quad (1.52) \]

which is equivalent to Eq. (1.37). Substituting Eqs. (1.50) and (1.52) into Eq. (1.51) results in the expression

\[ \eta_c = \left[ \left( \frac{p_2}{p_1} \right)^{(\gamma - 1)/\gamma} - 1 \right] \left/ \left[ \left( \frac{p_2}{p_1} \right)^{(\gamma - 1)/\eta_p \gamma} - 1 \right] \right. \quad (1.53) \]

Values of “overall” isentropic efficiency have been calculated using Eq. (1.53) for a range of pressure ratio and different values of \( \eta_p \); these are plotted in Figure 1.13. This figure amplifies the observation made earlier that the isentropic efficiency of a finite compression process is *less* than the efficiency of the small stages. Comparison of the isentropic efficiency of two machines of different pressure ratios is not a valid procedure since, for equal polytropic efficiency, the compressor with the higher pressure ratio is penalized by the *hidden* thermodynamic effect.

**EXAMPLE 1.6**

An axial flow air compressor is designed to provide an overall total-to-total pressure ratio of 8 to 1. At inlet and outlet the stagnation temperatures are 300 and 586.4 K, respectively.

Determine the overall total-to-total efficiency and the polytropic efficiency for the compressor. Assume that \( \gamma \) for air is 1.4.

**Solution**

From Eq. (1.46), substituting \( h = C_p T \), the efficiency can be written as

\[ \eta_c = \frac{T_{02s} - T_{01}}{T_{02} - T_{01}} = \frac{(p_{02}/p_{01})^{(\gamma - 1)/\gamma} - 1}{T_{02}/T_{01} - 1} = \frac{8^{1/3.5} - 1}{586.4/300 - 1} = 0.85 \]

From Eq. (1.50), taking logs of both sides and rearranging, we get

\[ \eta_p = \frac{\gamma - 1}{\gamma} \ln(p_{02}/p_{01}) = \frac{1}{3.5} \times \left( \frac{\ln 8}{\ln 1.9547} \right) = 0.8865 \]
Turbine polytropic efficiency

A similar analysis to the compression process can be applied to a perfect gas expanding through an adiabatic turbine. For the turbine the appropriate expressions for an expansion, from a state 1 to a state 2, are

\[
\frac{T_2}{T_1} = \left( \frac{p_2}{p_1} \right)^{\eta_p (\gamma - 1)/\gamma}
\]

(1.54)

\[
\eta_t = \left[ 1 - \left( \frac{p_2}{p_1} \right)^{\eta_p (\gamma - 1)/\gamma} \right] \left/ \left[ 1 - \left( \frac{p_2}{p_1} \right)^{(\gamma - 1)/\gamma} \right] \right.
\]

(1.55)

The derivation of these expressions is left as an exercise for the student. “Overall” isentropic efficiencies have been calculated for a range of pressure ratios and polytropic efficiencies, and these are shown in Figure 1.14. The most notable feature of these results is that, in contrast with a compression process, for an expansion, isentropic efficiency exceeds small stage efficiency.

Reheat factor

The foregoing relations cannot be applied to steam turbines as vapors do not obey the perfect gas laws. It is customary in steam turbine practice to use a reheat factor \( R_H \) as a measure of the inefficiency of the complete expansion. Referring to Figure 1.15, the expansion process through an
FIGURE 1.14
Turbine isentropic efficiency against pressure ratio for various polytropic efficiencies ($\gamma = 1.4$).

FIGURE 1.15
Mollier diagram showing expansion process through a turbine split up into a number of small stages.
adiabatic turbine from state 1 to state 2 is shown on a Mollier diagram, split into a number of small stages. The reheat factor is defined as

$$R_H = \frac{[(h_1 - h_{x1}) + (h_x - h_{ys}) + \cdots]/(h_1 - h_{2s}) = (\Sigma \Delta h_{is})/(h_1 - h_{2s})}{1/\sqrt{C_138}}$$

Due to the gradual divergence of the constant pressure lines on a Mollier chart, $R_H$ is always greater than unity. The actual value of $R_H$ for a large number of stages will depend upon the position of the expansion line on the Mollier chart and the overall pressure ratio of the expansion. In normal steam turbine practice the value of $R_H$ is usually between 1.03 and 1.08.

Now, since the isentropic efficiency of the turbine is

$$\eta_t = \frac{h_1 - h_2}{h_1 - h_{2s}} = \frac{h_1 - h_2}{\Sigma \Delta h_{is}} \times \frac{\Sigma \Delta h_{is}}{h_1 - h_{2s}}$$

then

$$\eta_t = \eta_p R_H$$

which establishes the connection between polytropic efficiency, reheat factor and turbine isentropic efficiency.

1.13 The inherent unsteadiness of the flow within turbomachines

It is a less well-known fact often ignored by designers of turbomachinery that turbomachines can only work the way they do because of flow unsteadiness. This subject was discussed by Dean (1959), Horlock and Daneshyar (1970), and Greitzer (1986). Here, only a brief introduction to an extensive subject is given.

In the absence of viscosity, the equation for the stagnation enthalpy change of a fluid particle moving through a turbomachine is

$$\frac{Dh_0}{Dt} = \frac{1}{\rho} \frac{\partial p}{\partial t}$$

where $D/Dt$ is the rate of change following the fluid particle. Eq. (1.57) shows us that any change in stagnation enthalpy of the fluid is a result of unsteady variations in static pressure. In fact, without unsteadiness, no change in stagnation enthalpy is possible and thus no work can be done by the fluid. This is the so-called “Unsteadiness Paradox.” Steady approaches can be used to determine the work transfer in a turbomachine, yet the underlying mechanism is fundamentally unsteady.

A physical situation considered by Greitzer (1986) is the axial compressor rotor as depicted in Figure 1.16a. The pressure field associated with the blades is such that the pressure increases from the suction surface ($S$) to the pressure surface ($P$). This pressure field moves with the blades and is therefore steady in the relative frame of reference. However, for an observer situated at the point* (in the absolute frame of reference), a pressure that varies with time would be recorded, as shown
in Figure 1.16b. This unsteady pressure variation is directly related to the blade pressure field via the rotational speed of the blades,

\[
\frac{\partial p}{\partial t} = \Omega \frac{\partial p}{\partial \theta} = U \frac{\partial p}{r \partial \theta}
\]  

(1.58)

Thus, the fluid particles passing through the rotor experience a positive pressure increase with time (i.e., \(\partial p/\partial t > 0\)) enthalpies are increased.

**PROBLEMS**

1. **a.** Air flows adiabatically through a long straight horizontal duct, 0.25 m diameter, at a measured mass flow rate of 40 kg/s. At a particular section along the duct the measured values of static temperature \(T = 150^\circ\text{C}\) and static pressure \(p = 550\) kPa. Determine the average velocity of the airflow and its stagnation temperature.

   **b.** At another station further along the duct, measurements reveal that the static temperature has dropped to \(147^\circ\text{C}\) as a consequence of wall friction. Determine the average velocity and the static pressure of the airflow at this station.

   Also determine the change in entropy per unit of mass flow between the two stations.

   For air assume that \(R = 287\) J/(kg K) and \(\gamma = 1.4\).
2. Nitrogen gas at a stagnation temperature of 300 K and a static pressure of 2 bar flows adiabatically through a pipe duct of 0.3 m diameter. At a particular station along the duct length the Mach number is 0.6. Assuming the flow is frictionless, determine
a. the static temperature and stagnation pressure of the flow;
b. the mass flow of gas if the duct diameter is 0.3 m.
For nitrogen gas take \( R = 297 \text{ J/(kg K)} \) and \( \gamma = 1.4 \).

3. Air flows adiabatically through a horizontal duct and at a section numbered (1) the static pressure \( p_1 = 150 \text{ kPa} \), the static temperature \( T_1 = 200^\circ \text{C} \) and the velocity \( c_1 = 100 \text{ m/s} \). At a station further downstream the static pressure \( p_2 = 50 \text{ kPa} \) and the static temperature \( T_2 = 150^\circ \text{C} \). Determine the velocity \( c_2 \) and the change in entropy per unit mass of air. For air take \( R = 287 \text{ J/(kg K)} \) and \( \gamma = 1.4 \).

4. For the adiabatic expansion of a perfect gas through a turbine, show that the overall efficiency \( \eta_t \) and small stage efficiency \( \eta_p \) are related by
\[
\eta_t = \frac{1 - \varepsilon \eta_p}{1 - \varepsilon}
\]
where \( \varepsilon = r^{(1-\gamma)/\gamma} \), and \( r \) is the expansion pressure ratio, \( \gamma \) is the ratio of specific heats. An axial flow turbine has a small stage efficiency of 86%, an overall pressure ratio of 4.5 to 1 and a mean value of \( \gamma \) equal to 1.333. Calculate the overall turbine efficiency.

5. Air is expanded in a multistage axial flow turbine, the pressure drop across each stage being very small. Assuming that air behaves as a perfect gas with ratio of specific heats \( \gamma \), derive pressure—temperature relationships for the following processes:
a. reversible adiabatic expansion;
b. irreversible adiabatic expansion, with small stage efficiency \( \eta_p \);
c. reversible expansion in which the heat loss in each stage is a constant fraction \( k \) of the enthalpy drop in that stage;
d. reversible expansion in which the heat loss is proportional to the absolute temperature \( T \).
Sketch the first three processes on a \( T, s \) diagram. If the entry temperature is 1100 K and the pressure ratio across the turbine is 6 to 1, calculate the exhaust temperatures in each of the first three cases. Assume that \( \gamma = 1.333 \), that \( \eta_p = 0.85 \), and that \( k = 0.1 \).

6. Steam at a pressure of 80 bar and a temperature of 500\(^\circ\)C is admitted to a turbine where it expands to a pressure of 0.15 bar. The expansion through the turbine takes place adiabatically with an isentropic efficiency of 0.9 and the power output from the turbine is 40 MW. Using a Mollier chart and/or steam tables determine the enthalpy of the steam at exit from the turbine and the flow rate of the steam.

7. A multistage high-pressure steam turbine is supplied with steam at a stagnation pressure of 7 MPa and a stagnation temperature of 500\(^\circ\)C. The corresponding specific enthalpy is 3410 \( \text{kJ/kg} \). The steam exhausts from the turbine at a stagnation pressure of 0.7 MPa, the steam having been in a superheated condition throughout the expansion. It can be assumed that the steam behaves like a perfect gas over the range of the expansion and that \( \gamma = 1.3 \). Given that the turbine flow process has a small-stage efficiency of 0.82, determine
a. the temperature and specific volume at the end of the expansion;
b. the reheat factor.

The specific volume of superheated steam is represented by \( pv = 0.231(h - 1943) \), where \( p \) is in kPa, \( v \) is in m\(^3\)/kg, and \( h \) is in kJ/kg.

**8.** A 20 MW back-pressure turbine receives steam at 4 MPa and 300°C, exhausting from the last stage at 0.35 MPa. The stage efficiency is 0.85, reheat factor 1.04, and external losses 2% of the actual isentropic enthalpy drop. Determine the rate of steam flow. At the exit from the first stage nozzles, the steam velocity is 244 m/s, specific volume 68.6 dm\(^3\)/kg, mean diameter 762 mm, and steam exit angle 76° measured from the axial direction. Determine the nozzle exit height of this stage.

**9.** Steam is supplied to the first stage of a five-stage pressure-compounded steam turbine at a stagnation pressure of 1.5 MPa and a stagnation temperature of 350°C. The steam leaves the last stage at a stagnation pressure of 7.0 kPa with a corresponding dryness fraction of 0.95. By using a Mollier chart for steam and assuming that the stagnation state point locus is a straight line joining the initial and final states, determine

a. the stagnation conditions between each stage assuming that each stage does the same amount of work;
b. the total-to-total efficiency of each stage;
c. the overall total-to-total efficiency and total-to-static efficiency assuming the steam enters the condenser with a velocity of 200 m/s;
d. the reheat factor based upon stagnation conditions.

**10.** Carbon dioxide gas (CO\(_2\)) flows adiabatically along a duct. At station 1 the static pressure \( p_1 = 120 \) kPa and the static temperature \( T_1 = 120^\circ C \). At station 2 further along the duct the static pressure \( p_2 = 75 \) kPa and the velocity \( c_2 = 150 \) m/s. Determine

a. the Mach number \( M_2 \);
b. the stagnation pressure \( p_{02} \);
c. stagnation temperature \( T_{02} \);
d. the Mach number \( M_1 \).

For CO\(_2\) take \( R = 188 \) J/(kg K) and \( \gamma = 1.30 \).

**11.** Air enters the first stage of an axial flow compressor at a stagnation temperature of 20°C and at a stagnation pressure of 1.05 bar and leaves the compressor at a stagnation pressure of 11 bar. The total-to-total efficiency of the compressor is 83%. Determine, the exit stagnation temperature of the air and the polytropic efficiency of the compressor. Assume for air that \( \gamma = 1.4 \).

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**References**


2

Dimensional Analysis: Similitude

If you have known one you have known all.
Terence, Phormio

2.1 Dimensional analysis and performance laws

The widest comprehension of the general behavior of all turbomachines is, without doubt, obtained from dimensional analysis. This is the formal procedure whereby the group of variables representing some physical situation is reduced to a smaller number of dimensionless groups. When the number of independent variables is not too great, dimensional analysis enables experimental relations between variables to be found with the greatest economy of effort. Dimensional analysis applied to turbomachines has two further important uses: (a) prediction of a prototype’s performance from tests conducted on a scale model (similitude), and (b) determination of the most suitable type of machine, on the basis of maximum efficiency, for a specified range of head, speed, and flow rate. Several methods of constructing nondimensional groups have been described by Douglas, Gasiorek, and Swaffield (1995) and Shames (1992), among other authors. The subject of dimensional analysis was made simple and much more interesting by Taylor (1974) in his comprehensive account of the subject and this approach is the one adopted in this book.

Adopting the simple approach of elementary thermodynamics, a control surface of fixed shape, position, and orientation is drawn around the turbomachine (Figure 2.1). Across this boundary, fluid flows steadily, entering at station 1 and leaving at station 2. As well as the flow of fluid, there is a flow of work across the control surface, transmitted by the shaft either to or from the machine. All details of the flow within the machine can be ignored and only externally observed features such as shaft speed, flow rate, torque, and change in fluid properties across the machine need be considered. To be specific, let the turbomachine be a pump (although the analysis could apply to other classes of turbomachine) driven by an electric motor. The speed of rotation \( \Omega \) can be adjusted by altering the current to the motor; the volume flow rate \( Q \) can be independently adjusted by means of a throttle valve. For fixed values of the set \( Q \) and \( \Omega \), all other variables, such as torque, \( \tau \), and head, \( H \), are thereby established. The choice of \( Q \) and \( \Omega \) as control variables is clearly arbitrary and any other pair of independent variables such as \( \tau \) and \( H \) could equally well have been chosen. The important point to recognize is that there are, for this pump, two control variables.

If the fluid flowing is changed for another of different density, \( \rho \) and viscosity, \( \mu \), the performance of the machine will be affected. Note also that, for a turbomachine handling compressible fluids, other fluid properties are important and these are discussed later.
So far we have considered only one particular turbomachine, namely a pump of a given size. To extend the range of this discussion, the effect of the geometric variables on the performance must now be included. The size of machine is characterized by the impeller diameter, $D$, the shape can be expressed by a number of length ratios, $l_1/D$, $l_2/D$, etc., and the surface finish can be characterized by a representative roughness length, $e$.

### 2.2 Incompressible fluid analysis

The performance of a turbomachine can be expressed in terms of the control variables, geometric variables, and fluid properties. Take as an example a hydraulic pump. It is convenient to regard the net energy transfer, $gH$, the efficiency, $\eta$, and the power supplied, $P$, as dependent variables and to write the three functional relationships as

$$gH = f_1 \left( Q, \Omega, D, \rho, \mu, e, \frac{l_1}{D}, \frac{l_2}{D}, \ldots \right)$$

(2.1a)

$$\eta = f_2 \left( Q, \Omega, D, \rho, \mu, e, \frac{l_1}{D}, \frac{l_2}{D}, \ldots \right)$$

(2.1b)

$$P = f_3 \left( Q, \Omega, D, \rho, \mu, e, \frac{l_1}{D}, \frac{l_2}{D}, \ldots \right)$$

(2.1c)

For a family of geometrically similar machines, the shape parameters, $l_1/D$ and $l_2/D$ are constant and may be omitted. Dimensional analysis\(^1\) can then be applied to determine the dimensionless

\(^1\)This is the approach used to reduce the experimental variables in a fluid mechanical problem (and in other areas, as well) to the minimum number of nondimensional parameters. It is explained at some length in elementary textbooks such as Franzini and Finnemore (1997) and White (2011).
groups that are needed to describe dynamic similarity. The number of dimensionless groups required can be found using Buckingham’s π-theorem (Buckingham, 1914). This theorem states that for \( M \) independent variables and \( N \) dimensions, there must be at least \( M - N \) nondimensional groups. In this case, for 6 variables \((Q, \Omega, D, \rho, \mu, e)\) and 3 dimensions (mass, length, time), there must be \( 6 - 3 = 3 \) independent nondimensional groups. However, the form of the nondimensional groups required is not obvious and consideration of the physics is necessary. For a pump, the selection of \( \rho \), \( \Omega \), and \( D \) as common factors avoids the appearance of special fluid terms (e.g., \( \mu, Q \)) in more than one group and allows \( gH, \eta, \) and \( P \) to be made explicit. Hence, the three relationships in Eqs. (2.1a–c) can be reduced to the following easily verified forms:

Energy transfer coefficient, sometimes called head coefficient:

\[
\psi = \frac{gH}{(\Omega D)^2} = f_4 \left( \frac{Q}{\Omega D^3}, \frac{\rho \Omega D^2}{\mu}, \frac{e}{D} \right)
\]  

(2.2a)

Efficiency, which is already nondimensional:

\[
\eta = f_5 \left( \frac{Q}{\Omega D^3}, \frac{\rho \Omega D^2}{\mu}, \frac{e}{D} \right)
\]  

(2.2b)

Power coefficient:

\[
\hat{P} = \frac{P}{\rho \Omega^2 D^5} = f_4 \left( \frac{Q}{\Omega D^3}, \frac{\rho \Omega D^2}{\mu}, \frac{e}{D} \right)
\]  

(2.2c)

The nondimensional group \( Q/(\Omega D^3) \) is a volumetric flow coefficient. In nonhydraulic flow turbomachines, an alternative to \( Q/(\Omega D^3) \) that is frequently used is the velocity (or flow) coefficient \( \Phi = c_m/U \), where \( U \) is the mean blade speed and \( c_m \) the average meridional velocity. Since

\[
Q = c_m \times \text{flow area} \propto c_m D^2 \quad \text{and} \quad U \propto \Omega D
\]

then

\[
\frac{Q}{\Omega D^3} \propto \frac{c_m}{U} = \Phi
\]

Both of these nondimensional groups are usually referred to as a flow coefficient, \( \Phi \).

The nondimensional group \( \rho \Omega D^2/\mu \) is a form of Reynolds number, denoted \( Re \). Physically, Reynolds number represents the ratio between the inertial forces and the viscous forces within a fluid flow. For low viscosity fluid moving at high velocity, the Reynolds number is high; conversely for slow moving fluid with high viscosity, the Reynolds number is low. It is found experimentally that provided \( Re > 2 \times 10^4 \), the effects of \( Re \) on the performance of turbomachines is small. This is true because at high \( Re \), the viscous boundary layers on the blades of a turbomachine are generally turbulent and very thin. They, therefore, have little impact on the global flow field. Efficiency is the variable that can be most affected by Reynolds number and typically \( \eta \) will rise up to a few per cent as \( Re \) increases an order of magnitude. Note that for turbomachines handling water, the kinematic viscosity, \( \nu = \mu/\rho \), is very small and, therefore, the corresponding Reynolds number is always high and its effects may be ignored to a first approximation.
The effects of surface finish are captured by the nondimensional group, $e/D$, called the roughness ratio or relative roughness. At high Reynolds numbers, greater surface roughness tends to increase skin friction losses and thus reduce the efficiency. The effects at lower Reynolds numbers are more complex as the boundary layers may be laminar or undergoing transition to turbulence. If it is assumed that both the surface finish effects are small and that the Reynolds numbers are high, the functional relationships for geometrically similar hydraulic turbomachines are:

$$\psi = f_4\left(\frac{Q}{\Omega D^3}\right)$$  \hspace{1cm} (2.3a)

$$\eta = f_5\left(\frac{Q}{\Omega D^3}\right)$$  \hspace{1cm} (2.3b)

$$\hat{P} = f_6\left(\frac{Q}{\Omega D^3}\right)$$  \hspace{1cm} (2.3c)

This is as far as the reasoning with dimensional analysis alone can be taken; the actual form of the functions $f_4$, $f_5$, and $f_6$ must be ascertained by experiment.

One relation between $\psi$, $\Phi$, $\eta$, and $\hat{P}$ may be immediately stated. For a pump, the net hydraulic power, $P_N$, equals $\rho QgH$, which is the minimum shaft power required in the absence of all losses. As shown in Chapter 1, we define pump efficiency as $\eta = P_N/P = \rho QgH/P$, where $P$ is the actual power to drive the pump. Therefore,

$$P = \frac{1}{\eta} \left(\frac{Q}{\Omega D^3}\right) \frac{gH}{\Omega^2 D^2} \rho \Omega^3 D^5$$  \hspace{1cm} (2.4)

Thus, $f_6$ may be derived from $f_4$ and $f_5$ since $\hat{P} = \Phi \psi / \eta$. For a turbine, the net hydraulic power supplied to it, $P_N$, is clearly greater than the actual power output from the machine and the efficiency $\eta = P/P_N$. By reasoning similar to that provided for the pump, we can see that for a turbine $\hat{P} = \Phi \psi \eta$.

### 2.3 Performance characteristics for low-speed machines

The operating condition of a turbomachine will be dynamically similar at two different rotational speeds if all fluid velocities at corresponding points within the machine are in the same direction and proportional to the blade speed. In other words, the flow is dynamically similar if the streamline patterns relative to the blades are geometrically similar. When two flow fields are dynamically similar, then all the dimensionless groups are the same. As shown by Eqs. (2.3a–c), for an incompressible flow machine (one in which $M < 0.3$ everywhere) operating at high Reynolds number, dynamic similarity is achieved once the flow coefficient is the same. Thus, the nondimensional presentation of performance data has the important practical advantage of collapsing results into a single curve that would otherwise require a multiplicity of curves if plotted dimensionally.

Evidence in support of the foregoing assertion is provided in Figure 2.2, which shows experimental results obtained by one author (at the University of Liverpool) on a simple centrifugal laboratory pump. Within the normal operating range of this pump, $0.03 < Q/\left(\Omega D^3\right) < 0.06$, very little systematic scatter is apparent, which might be associated with a Reynolds number effect, for the range of speeds $2500 \leq \Omega \leq 5000$ rpm. For smaller flows, $Q/\left(\Omega D^3\right) < 0.025$, the flow became unsteady and the manometer readings of uncertain accuracy, but nevertheless, dynamically similar
conditions still appear to hold true. At high flow rates there is a systematic deviation away from the “single-curve” law at higher rotational speeds. This effect is due to cavitation, a high-speed phenomenon of hydraulic machines caused by the release of vapor bubbles at low pressures, which is discussed later in this chapter. It will be clear at this stage that under cavitating flow conditions, dynamical similarity is not possible.

The nondimensional results shown in Figure 2.2 have, of course, been obtained for a particular pump. They would also be approximately valid for a range of different pump sizes so long as all these pumps are geometrically similar and cavitation is absent. Thus, neglecting any change in performance due to change in Reynolds number, the dynamically similar results in Figure 2.2 can be applied to predicting the dimensional performance of a given pump for a series of required speeds. Figure 2.3 shows such a dimensional presentation. It will be clear from the preceding discussion that the locus of dynamically similar points in the $H-Q$ field lies on a parabola since $H$ varies as $\Omega^2$ and $Q$ varies as $\Omega$.

**EXAMPLE 2.1**

A model centrifugal pump with an efficiency of 88% is tested at a rotational speed of 3000 rpm and delivers 0.12 m$^3$/s of water against a head of 30 m. Using the similarity rules given above, determine the rotational speed, volume flow rate, and power requirement of a geometrically similar prototype at eight times the scale of the model and working against a head of 50 m.

**Solution**

From the similarity laws, for the same head coefficient,

$$H_p/(\Omega_p^2 D_p^2) = H_m/(\Omega_m^2 D_m^2)$$
where subscript \( m \) is for the model and \( p \) for the prototype. Hence,

\[
\Omega_p = \Omega_m \left( \frac{D_m}{D_p} \right) \left( \frac{H_p}{H_m} \right)^{\frac{1}{3}} = 3000 \times \frac{1}{8} \times \left( \frac{50}{30} \right)^{\frac{1}{3}} = 484.1 \text{ rpm}
\]

Operating at the same volumetric flow coefficient,

\[
\frac{Q_p}{\Omega_p D_p^3} = \frac{Q_m}{\Omega_m D_m^3}
\]

\[
Q_p = Q_m \left( \frac{D_p}{D_m} \right)^3 = 0.12 \times \frac{484.1}{3000} \times 8^3 = 9.914 \text{ m}^3/\text{s}
\]

Finally, the power for the prototype can be determined assuming the efficiency is the same as the model:

\[
P_p = \frac{\rho g Q_p H_p}{\eta_p} = \left( 10^3 \times 9.81 \times 9.914 \times 50 \right)/0.88 = 5.526 \times 10^6 = 5.536 \text{ MW}
\]

### 2.4 Compressible flow analysis

The application of dimensional analysis to compressible flow increases, not unexpectedly, the complexity of the functional relationships obtained in comparison with those already found for
incompressible fluids. Even if the fluid is regarded as a perfect gas, in addition to the previously used fluid properties, two further characteristics are required; these are $a_{01}$, the stagnation speed of sound at entry to the machine, and $\gamma$, the ratio of specific heats $C_p/C_v$. In the following analysis, the compressible fluids under discussion are either perfect gases or dry vapors approximating in behavior to a perfect gas.

Another choice of variables is preferred when appreciable density changes occur across the machine. Instead of volume flow rate $Q$, the mass flow rate $\dot{m}$ is used; likewise for the head change $H$, the isentropic stagnation enthalpy change $\Delta h_{0s}$ is employed. The choice of this last variable is a significant one for, in an ideal and adiabatic process, $\Delta h_{0s}$ is equal to the work done per unit mass of fluid. Since heat transfer from the casings of turbomachines is, in general, of negligible magnitude compared with the flux of energy through the machine, temperature on its own may be safely excluded as a fluid variable. However, temperature is an easily observable characteristic and, for a perfect gas, can be easily introduced by means of the equation of state, $p/\rho = RT$.

The performance parameters $\Delta h_{0s}$, $\eta$, and $P$, for a turbomachine handling a compressible flow, can be expressed functionally as

$$\Delta h_{0s}, \eta, P = f(\mu, \Omega, D, \dot{m}, \rho_{01}, a_{01}, \gamma)$$  \hspace{1cm} (2.5)

Because $\rho_0$ and $a_0$ change through a turbomachine, the values of these fluid variables are selected at inlet, denoted by subscript 1. Equation (2.5) expresses three separate functional relationships, each of which consists of eight variables. Again, selecting $\rho_{01}$, $\Omega$, and $D$ as common factors, each of these three relationships may be reduced to five dimensionless groups:

$$\frac{\Delta h_{0s}}{\Omega^2D^2}, \eta, \frac{P}{\rho_{01}\Omega^3D^5} = f\left(\frac{\dot{m}}{\rho_{01}D^3}, \frac{\rho_{01}\Omega D^2}{\mu}, \frac{\Omega D}{a_{01}}, \gamma\right)$$  \hspace{1cm} (2.6a)

The group $\Omega D/a_{01}$ can be regarded as a blade Mach number because $\Omega D$ is proportional to blade speed. Since this appears as an independent variable on the right-hand side of the equation, it can be used to rewrite the preceding relationships in terms of the inlet stagnation speed of sound $a_{01}$:

$$\frac{\Delta h_{0s}}{a_{01}^2}, \eta, \frac{P}{\rho_{01}a_{01}D^2} = f\left(\frac{\dot{m}}{\rho_{01}a_{01}D^2}, \frac{\rho_{01}a_{01}D}{\mu}, \frac{\Omega D}{a_{01}}, \gamma\right)$$  \hspace{1cm} (2.6b)

For a machine handling a perfect gas, a different set of functional relationships is often more useful. These may be found either by selecting the appropriate variables for a perfect gas and working through again from first principles or, by means of some rather straightforward transformations, rewriting Eq. (2.6b) to give more suitable groups. The latter procedure is preferred here as it provides a useful exercise. As an example, consider an adiabatic compressor handling a perfect gas. The isentropic stagnation enthalpy rise can be written as $C_p(T_{02s} - T_{01})$ for a perfect gas. As shown in Chapter 1, the isentropic relationship between temperature and pressure is given by

$$\frac{T_{02s}}{T_{01}} = \left(\frac{p_{02}}{p_{01}}\right)^{(\gamma - 1)/\gamma}$$

The isentropic stagnation enthalpy rise can therefore be written as

$$\Delta h_{0s} = C_p T_{01} \left[\left(\frac{p_{02}}{p_{01}}\right)^{(\gamma - 1)/\gamma} - 1\right]$$  \hspace{1cm} (2.7)
Since \( C_p = \gamma R (\gamma - 1) \) and \( a_{01}^2 = \gamma R T_{01} \), then \( a_{01}^2 = (\gamma - 1) C_p T_{01} \) and thus,

\[
\frac{\Delta h_{0s}}{a_{01}^2} = \frac{\Delta h_{0s}}{(\gamma - 1) C_p T_{01}} = \frac{1}{(\gamma - 1)} \left[ \left( \frac{p_{02}}{p_{01}} \right)^{(\gamma - 1)/\gamma} - 1 \right] = f(p_{02}/p_{01}, \gamma)
\]

Using the equation of state, \( p/\rho = RT \), the nondimensional mass flow can be more conveniently expressed as

\[
\dot{m} = \frac{m}{\rho_0 a_{01} D^2} = \frac{m R T_{01}}{p_0 \sqrt{\gamma R T_{01} D^2}} = \frac{m \sqrt{\gamma R T_{01}}}{D^2 p_0 \gamma}
\]

The power coefficient can also be rewritten as

\[
\hat{p} = \frac{P}{\rho_0 a_{01} D^2} = \frac{m C_p \Delta T_0}{(\rho_0 a_{01} D^2) a_{01}^2} = \frac{m C_p \Delta T_0}{a_{01}^2} = \frac{\dot{m}}{(\gamma - 1) T_{01}} \Delta T_0
\]

Collecting together these newly formed nondimensional groups and inserting them in Eq. (2.6b) leads to a simpler and more useful functional relationship:

\[
\frac{p_{02}}{p_{01}}, \eta, \frac{\Delta T_0}{T_{01}} = f\left( \frac{m \sqrt{\gamma R T_{01}}}{D^2 p_0 \gamma}, \frac{\Omega D}{\sqrt{\gamma R T_{01}}}, Re, \gamma \right)
\]

A key advantage of Eq. (2.8) over Eq. (2.6b) is that the nondimensional groups are in terms of inlet and exit stagnation temperatures and pressures, which are parameters that are readily measured for a turbomachine. For a machine handling a single gas, \( \gamma \) can be dropped as an independent variable. If, in addition, the machine operates only at high Reynolds numbers (or over a small speed range), \( Re \) can also be dropped. Equation (2.8) can then be written with just two nondimensional groups on the right-hand side:

\[
\frac{p_{02}}{p_{01}}, \eta, \frac{\Delta T_0}{T_{01}} = f\left( \frac{m \sqrt{C_p T_{01}}}{D^2 p_0}, \frac{\Omega D}{\sqrt{\gamma R T_{01}}} \right)
\]

(2.9a)

In this equation, the nondimensional group, \( m \sqrt{C_p T_{01}}/D^2 p_0 \) is often referred to as the flow capacity, introduced in Section 1.10 of Chapter 1. This is the most widely used form of nondimensional mass flow, although the forms in Eqs (2.6b) and (2.8) are also valid. For machines of a known size and fixed working fluid, it has become customary, in industry at least, to delete \( \gamma, R, C_p, \) and \( D \) from Eq. (2.9a) and similar expressions. Under these conditions, Eq. (2.9a) becomes

\[
\frac{p_{02}}{p_{01}}, \eta, \frac{\Delta T_0}{T_{01}} = f\left( \frac{m \sqrt{T_{01}}}{p_0}, \frac{\Omega}{\sqrt{T_{01}}} \right)
\]

(2.9b)

Note that by omitting the diameter \( D \) and gas constant \( R \), the independent variables on the right-hand side of Eq. (2.9b) are no longer dimensionless.

For a given turbomachine, Eq. (2.9b) is sometimes expressed in terms of corrected flow and corrected speed. These are the mass flow and speed that would be measured if the machine was operating at standard sea-level atmospheric pressure and temperature, \( p_a \) and \( T_a \).
The corrected mass flow and corrected speed are defined as

\[ \frac{\dot{m}\sqrt{\theta}}{\delta} \quad \text{and} \quad \frac{\Omega}{\sqrt{\theta}} \]

where

\[ \theta = \frac{T_{01}}{T_a} \quad \text{and} \quad \delta = \frac{p_{01}}{p_a} \]

The functional relationships in Eq. (2.9b) can then be rewritten as

\[ \frac{p_{02}}{p_{01}}, \eta, \frac{\Delta T_0}{T_{01}} = f \left( \frac{\dot{m}\sqrt{\theta}}{\delta}, \frac{\Omega}{\sqrt{\theta}} \right) \] (2.9c)

Note that the parameters on the right-hand side are no longer nondimensional. The units of the first parameter are kg/s and that of the second are rad/s. To nondimensionalize these parameters, they can be normalized by their values at the design point.

Equations (2.9a–c) show that two variables are required to fix the operating point of a compressible flow machine. This compares to the one variable needed to fix the operating point of an incompressible flow machine, Eqs. (2.3a–c). In all cases, for dynamic similarity, the streamline pattern relative to the blades must be geometrically similar. In an incompressible flow machine, it is enough just to fix the relative inlet angle to the blades (via the flow coefficient). In a compressible flow machine, the streamline pattern within the blade rows also depends on the variation of density through the blade passages. Therefore, a second parameter is needed to fix the flow Mach numbers and thus fix the variation of density.

Similarly to the incompressible case, the performance parameters, \( \frac{p_{02}}{p_{01}}, \eta, \frac{\Delta T_0}{T_{01}} \) are not entirely independent and it is straightforward to write an equation relating the three. For a compressor, the isentropic efficiency is defined in Chapter 1 and can be written as

\[ \eta_c = \frac{\Delta h_{0s}}{\Delta h_0} = \frac{(p_{02}/p_{01})^{\gamma/(\gamma-1)} - 1}{\Delta T_0/T_{01}} \] (2.10a)

The corresponding isentropic efficiency for a turbine is

\[ \eta_t = \frac{\Delta h_0}{\Delta h_{0s}} = \frac{\Delta T_0/T_{01}}{[(p_{01}/p_{02})^{(\gamma-1)/\gamma} - 1]} \] (2.10b)

where \( p_{01}/p_{02} \) is the overall total pressure ratio of the turbine.

### Flow coefficient and stage loading

In compressible flow machines, the flow coefficient, \( \Phi \), is an important parameter for design and analysis. It is defined in the same way as given earlier for incompressible machines, i.e., \( \Phi = c_m/U \), where \( U \) is the mean blade speed and \( c_m \) the average meridional velocity. However, in the compressible case, the flow coefficient alone cannot be used to fix the operating condition of a
This is because the flow coefficient is also a function of the nondimensional parameters given in Eq. (2.9a). It is straightforward to show this via the following algebraic manipulation:

$$\varphi = \frac{c_m}{U} = \frac{\dot{m}}{\rho_0 A_1 U} = \frac{\dot{m} R T_{01}}{\rho_0 A_1 U} \times \frac{\dot{m} \sqrt{(C_p T_{01})}}{D^2 p_0} \times \frac{\sqrt{(C_p T_{01})}}{U} = f \left( \frac{\dot{m} \sqrt{C_p T_{01}}}{D^2 p_0}, \frac{\Omega D}{\sqrt{\gamma R T_{01}}} \right)$$

Note that the nondimensional mass flow, $\dot{m} \sqrt{C_p T_{01}}/D^2 p_0$ is distinct from a flow coefficient because it does not involve the blade speed.

The stage loading, $\psi$, is another key design parameter for all nonhydraulic turbomachines. It is defined as

$$\psi = \frac{\Delta h_0}{U^2}$$

(2.11)

This parameter is similar in form to the head coefficient $\psi$ used in hydraulic machines (Eq. (2.2a)), but there are subtle differences. Most importantly, stage loading is a nondimensional form of the actual specific stagnation enthalpy change, whereas the head coefficient is a nondimensional measure of the maximum, or isentropic, work that a hydraulic machine can achieve. Note that the stage loading can be related to the nondimensional parameters in Eq. (2.9a) as follows:

$$\psi = \frac{\Delta h_0}{U^2} = \frac{C_p \Delta T_0}{C_p T_{01} U^2} \times \frac{C_p T_{01}}{U^2} = \Delta T_0 \left( \frac{U}{\sqrt{(C_p T_{01})}} \right)^2 = f \left\{ \frac{\dot{m} \sqrt{C_p T_{01}}}{D^2 p_0}, \frac{\Omega D}{\sqrt{\gamma R T_{01}}} \right\}$$

Thus, the stage loading is also fixed once both the nondimensional mass flow and the nondimensional blade speed (or blade Mach number) are fixed. In many cases, the stage loading is used in place of the power coefficient $\Delta T_0/T_0$ given in Eq. (2.9a).

### 2.5 Performance characteristics for high-speed machines

#### Compressors

The performance (or characteristic) map of a high-speed compressor is essentially a graphical representation of the functional relationships given in Eq. (2.9b). Figure 2.4 shows a performance map for a transonic fan and Figure 2.5 shows a performance map for a high-speed multistage axial compressor. In both cases, the pressure ratio across the machine is plotted as a function of $\dot{m} \sqrt{T_{01}}/p_0$ for several fixed values of $\Omega/\sqrt{T_{01}}$, which is the usual method of presentation. Figures 2.4 and 2.5 also show contours of compressor isentropic efficiency on the same axes.

Each of the constant speed curves on the compressor characteristic terminate at the instability line (often referred to as the surge or stall line). Beyond this point, the operation is unstable. A discussion of the phenomena of surge and stall is included in Chapter 5. At high speeds and low pressure ratios, the constant speed curves become vertical. In these regions of the characteristic, no further increase in $\dot{m} \sqrt{T_{01}}/p_0$ is possible since the Mach number across a section of the machine has reached unity and the flow is choked.

A compressor is able to operate anywhere below and to the right of the surge line. However, it is usually constrained to a single operating line, which is set by the flow area downstream of the compressor. A single operating line is shown in Figure 2.4. The design operating line is usually
specified so that it passes as close as possible to the point of peak compressor efficiency. However, its exact position is a matter of judgment for the compressor designer. The term \textit{stall margin} is often used to describe the relative position of the operating line and the surge line. There are several ways of defining the surge margin ($SM$) and a fairly simple one often used is

$$SM = \frac{(pr)_s - (pr)_o}{(pr)_o}$$  \hspace{1cm} (2.12)

where $(pr)_o$ is a pressure ratio at a point on the operating line at a certain corrected speed $\Omega/\sqrt{T_{01}}$ and $(pr)_s$ is the corresponding pressure ratio on the surge line at the same corrected speed. With this definition a surge margin of 20% would be typical for a compressor used within a turbojet engine. Several other definitions of \textit{stall margin} and their merits are discussed by Cumpsty (1989).

\section*{Turbines}

Figure 2.6 shows a typical high-speed axial turbine characteristic. The behavior of turbines is very different to that of compressors and this is reflected in the way the characteristic has been presented. Turbines are able to operate with a high-pressure ratio across each stage because the
boundary layers on the surfaces of the turbine blades are accelerating and therefore stable. The high-pressure ratios soon lead to choking in the turbine stator blades and therefore a fixed nondimensional mass flow through the machine. Once the turbine stators are fully choked, the operating point is independent of $\Omega/\sqrt{T_{01}}$ because the rotation of the blades has virtually no influence on either the turbine pressure ratio or the nondimensional mass flow rate.

As shown by Figure 2.6, it is more revealing to plot the flow capacity and turbine efficiency as a function of the turbine pressure ratio rather than the other way around, since it is usually the pressure ratio across a turbine that is specified and, for a high-speed case, there is limited variation in $m\sqrt{T_{01}/p_0}$ for different values of $\Omega/\sqrt{T_{01}}$.

**EXAMPLE 2.2**

The compressor with the performance map shown in Figure 2.5 is tested at sea level on a stationary test bed on a day when the atmospheric temperature and pressure is 298 K and 101 kPa, respectively. When running at its design operating point, the mass flow rate through the compressor is measured as 15 kg/s and the rotational speed is 6200 rpm. Determine the mass flow rate and rotational speed when the compressor is operating at the design operating point during high altitude cruise with an inlet stagnation temperature of 236 K and an inlet stagnation pressure of 10.2 kPa.

**FIGURE 2.5**

Performance map of a 10-stage high-speed axial compressor. *(Adapted from Cline et al., 1983)*
The design pressure ratio of the compressor is 22. Using the compressor characteristic in Figure 2.5, determine the compressor isentropic and polytropic efficiency at the design point. Hence calculate the required power input at the cruise condition. Assume throughout for air that $\gamma = 1.4$ and $C_p = 1005 \text{ J/kg/K}$.

**Solution**

At cruise and during the test the compressor is operating at its design nondimensional operating point. Therefore, all the nondimensional performance parameters of the compressor will be the same at both conditions.

**FIGURE 2.6**

Overall characteristic of a two-stage high-speed axial turbine.
The nondimensional mass flow is

\[ \frac{\dot{m}\sqrt{\gamma RT_1}}{D^2 p_0} \text{cruise} = \frac{\dot{m}\sqrt{\gamma RT_1}}{D^2 p_0} \text{test} \]

Since there is no change in the dimensions of the compressor or in the gas properties of the working fluid, this reduces to

\[ \frac{\dot{m}\sqrt{T_1}}{p_0} \text{cruise} = \frac{\dot{m}\sqrt{T_1}}{p_0} \text{test} \]

During the test, the compressor is stationary and therefore the inlet air stagnation temperature and pressure are equal to the atmospheric static temperature and pressure. The mass flow at cruise is thus

\[ \dot{m}_\text{cruise} = \frac{p_0}{\sqrt{T_1}} \text{cruise} \times \left[ \frac{\dot{m}\sqrt{T_1}}{p_0} \right] \text{test} = 10.2 \times \frac{15 \times \sqrt{298}}{101} = 1.70 \text{ kg/s} \]

Similarly for the nondimensional speed,

\[ \left[ \frac{\Omega}{\sqrt{T_1}} \right] \text{cruise} = \left[ \frac{\Omega}{\sqrt{T_1}} \right] \text{test} \]

and thus,

\[ \Omega_{\text{cruise}} = \sqrt{T_{1,\text{cruise}}} \times \left[ \frac{\Omega}{\sqrt{T_1}} \right] \text{test} = \sqrt{236} \times \left[ \frac{6200}{\sqrt{298}} \right] = 5520 \text{ rpm} \]

From Figure 2.5, at 100% speed and a pressure ratio of 22, \( \eta_c = 0.81 \).

\[ \frac{T_{02}}{T_{01}} = \frac{(p_{02}/p_{01})^{(\gamma - 1)/\gamma} - 1}{\eta_c} + 1 = \frac{221^{3.5} - 1}{0.81} + 1 = 2.751 \]

From Eq. (1.50), the polytropic efficiency is given by

\[ \eta_p = \frac{\gamma - 1}{\gamma} \ln\left(\frac{p_{02}}{p_{01}}\right) = \frac{1}{3.5 \ln(2.751)} = 0.873 \]

As expected, the polytropic efficiency is significantly higher than the isentropic efficiency at this pressure ratio. The input power to the compressor at the cruise condition can be found using the fact that the nondimensional power coefficient \( \Delta T_0/T_0 \) is unchanged between the two conditions:

\[ \frac{\Delta T_0}{T_0} = \frac{T_{02}}{T_{01}} - 1 = 1.751 \]

\[ P_{\text{cruise}} = [\dot{m}C_p \Delta T_0]_{\text{cruise}} = [\dot{m}C_p T_{01}]_{\text{cruise}} \frac{\Delta T_0}{T_{01}} = 1.70 \times 1005 \times 236 \times 1.751 = 706 \text{ kW} \]
2.6 Specific speed and specific diameter

The turbomachine designer is often faced with the basic problem of deciding what type of machine will be the best choice for a given duty. At the outset of the design process, some overall requirements of the machine will usually be known. For a hydraulic pump, these would include the head required, $H$, the volume flow rate, $Q$, and the rotational speed, $\Omega$. In contrast, if a high-speed gas turbine was being considered, the initial specification would probably cover the mass flow rate, $\dot{m}$, the specific work, $\Delta h_0$, and the preferred rotational speed, $\Omega$.

Two nondimensional parameters called the specific speed, $\Omega_s$, and specific diameter, $D_s$, are often used to decide upon the choice of the most appropriate machine (see Balje (1981)). The specific speed is derived from the nondimensional groups defined in Eqs. (2.3a–c) in such a way that the characteristic diameter $D$ of the turbomachine is eliminated. The value of $\Omega_s$ gives the designer a guide to the type of machine that will provide the normal requirement of high efficiency at the design condition. Similarly, the specific diameter is derived from these groups by eliminating the speed, $\Omega$.

Consider a hydraulic turbomachine with fixed geometry. As shown by Eq. (2.3b), there will be a unique relationship between efficiency and flow coefficient if Reynolds number effects are negligible and cavitation absent. If the maximum efficiency $\eta = \eta_{\text{max}}$ occurs at a unique value of flow coefficient $\Phi = \Phi_1$ and corresponding unique values of $\psi = \psi_1$ and $\hat{P} = \hat{P}_1$, it is possible to write

\[
\frac{Q}{\Omega D^3} = \Phi_1 = \text{constant} \quad (2.13a)
\]

\[
\frac{gH}{\Omega^2 D^2} = \psi_1 = \text{constant} \quad (2.13b)
\]

\[
\frac{P}{\rho \Omega^3 D^5} = \hat{P}_1 = \text{constant} \quad (2.13c)
\]

It is a simple matter to combine any pair of these expressions in such a way as to eliminate the diameter. For a pump, the customary way of eliminating $D$ is to divide $\Phi_1^{1/2}$ by $\psi_1^{3/4}$. Thus, at the operating point giving maximum efficiency,

\[
\Omega_s = \frac{\Phi_1^{1/2}}{\psi_1^{3/4}} = \frac{\Omega Q^{1/2}}{(gH)^{3/4}} \quad (2.14)
\]

where $\Omega_s$ is called the specific speed. The term specific speed is justified only to the extent that $\Omega_s$ is directly proportional to $\Omega$. It is sometimes referred to as a shape factor since its value characterizes the shape of the machine required.

In the case of a hydraulic turbine, the power specific speed $\Omega_{sp}$ is often used and it is defined by

\[
\Omega_{sp} = \frac{\hat{P}_1^{1/2}}{\psi_1^{3/4}} = \frac{\Omega (P/\rho)^{1/2}}{(gH)^{5/4}} \quad (2.15)
\]

There is a simple connection between $\Omega_s$ and $\Omega_{sp}$. By dividing Eq. (2.15) by Eq. (2.14), we obtain, for a hydraulic turbine,

\[
\frac{\Omega_{sp}}{\Omega_s} = \frac{\Omega (P/\rho)^{1/2}}{(gH)^{5/4}} \frac{(gH)^{3/4}}{\Omega Q^{1/2}} = \left( \frac{P}{\rho gQH} \right)^{1/2} = \sqrt{\eta} \quad (2.16)
\]
Similarly to specific speed, to form the specific diameter, any pair of expressions in Eqs. (2.13a–c) can be used to eliminate the speed, \( \Omega \). In the case of a pump, we divide \( \psi^{1/4} \) by \( \Phi^{1/2} \). Thus,

\[
D_s = \frac{\psi^{1/4}}{\Phi^{1/2}} = \frac{D(gH)^{1/4}}{Q^{1/2}}
\]

Equations (2.14), (2.15), and (2.17) are dimensionless. It is always safer and less confusing to calculate specific speed and specific diameter in one or another of these forms rather than dropping the factors \( g \) and \( \rho \), which would make the equations dimensional and any values of specific speed or specific diameter obtained using them would then depend upon the choice of the units employed. The dimensionless forms of \( \Omega_s \) (and \( \Omega_{sp} \)) and \( D_s \) are the only ones used in this book. Another point arises from the fact that the rotational speed, \( \Omega \), can be expressed in rad/s, rev/s or rpm, and therefore, although \( \Omega_s \) is dimensionless, numerical values of specific speed are sometimes specified in rev/s rather than rad/s. In this book, unless otherwise stated, the speed of rotation is taken to be in rad/s.

The concept of specific speed just described is illustrated in Figure 2.7. This shows contours of \( \Omega_s \) plotted as a function of flow coefficient, \( \Phi \), and head coefficient, \( \psi \), using Eq. (2.14). Also plotted on the same axes are typical characteristics of three types of hydraulic pumps. This plot demonstrates how for a given type of machine, one value of \( \Omega_s \) passes through the operating point of peak efficiency. In other words, once the specific speed is known, the machine type giving peak efficiency can be determined. Figure 2.7 also shows how a low specific speed suits radial machines, since these tend to give a high pressure change to a low mass flow rate. In contrast, axial flow stages with widely spaced blades are suited to high specific speed applications because they impart a small pressure change to a large mass flow rate.

**FIGURE 2.7**

Contours of specific speed showing characteristics of various pump types.
Given that specific speed is defined at the point of maximum efficiency of a turbomachine, it becomes a parameter of great importance in selecting the type of machine required for a given duty. The maximum efficiency condition replaces the condition of geometric similarity, so that any alteration in specific speed implies that the machine design changes. Broadly speaking, each different class of machine has its optimum efficiency within its own fairly narrow range of specific speed. Figure 2.8 shows the ranges of specific speed appropriate to different types of turbomachine. Once the specific speed at the design condition is found, a well-designed machine selected using Figure 2.8 should give the maximum possible design efficiency.

**EXAMPLE 2.3**

a. A hydraulic turbine with a runner outside diameter of 4.31 m operates with an effective head, $H$, of 543 m at a volume flow rate of 71.5 m$^3$/s and produces 350 MW of shaft power at a rotational speed of 333 rpm. Determine the specific speed, the specific diameter, and efficiency of this turbine.
b. Another geometrically and dynamically similar turbine with a runner 6.0 m diameter is to be built to operate with an effective head of 500 m. Determine the required flow rate, the expected power output, and the rotational speed of the turbine.

**Solution**

a. Note: All speeds are first converted to rad/s; therefore, \( \Omega = 333 \times \pi/30 = 34.87 \text{ rad/s} \).

Using Eq. (2.14), the specific speed is

\[
\Omega_s = \Omega Q^{1/2}/(gH)^{3/4} = \frac{34.87 \times 71.5^{0.5}}{(9.81 \times 543)^{0.75}} = 0.473 \text{ rad} \]

Using Eq. (2.17), the specific diameter is

\[
D_s = \frac{D(gH)^{1/4}}{Q^{1/2}} = \frac{4.31 \times (9.81 \times 543)^{1/4}}{71.4^{1/2}} = 4.354 \]

For the turbine, the net hydraulic power is

\[
P_n = \rho gQH = 9810 \times 71.5 \times 543 = 380.9 \times 10^6 = 380.9 \text{ MW} \]

The turbine efficiency is

\[ \eta = 350/380.9 = 0.919 \]

b. Transposing Eq. (2.17), we can find the volume flow rate:

\[
Q = (D/D_s)^2(gH)^{1/2} = (6/4.354)^2(9.81 \times 500)^{1/2} = 133 \text{ m}^3/\text{s} \]

and the power output is

\[
P = \eta \rho gQH = 0.919 \times 9810 \times 133 \times 500 = 599.5 \text{ MW} \]

We can determine the rotational speed in rpm from Eq. (2.14) as

\[
\Omega = \Omega_s(gH)^{3/4}/Q^{1/2} = 0.473 \times \frac{30}{\pi} \times (9.81 \times 500)^{3/4}/133^{1/2} = 229.6 \text{ rpm} \]

---

**The Cordier diagram**

A rough but useful guide to the selection of the most appropriate type and size of compressor, pump, or fan for a given duty and optimum efficiency is obtained by means of the Cordier diagram, Figure 2.9. Although the method was originally devised by Cordier (1953), further details are more readily accessed from the work of Csanady (1964) and, with some added elaboration, by Lewis (1996). Figure 2.9 shows, on the right-hand side, the recommended ranges for various types of turbomachines for which the method applies. It must be mentioned that the line presented is, in fact, a mean curve based upon results obtained from a large number of machines, so it represents a fairly broad spread of results on either side of the line. For many designs, it would be possible to diverge from the line and still obtain high-performance pumps, fans, or compressors.
Following Lewis, an interesting and useful alternative presentation of the Cordier diagram can be made with ordinates $\Phi$ and $\psi$ from the relationships already given. From Eqs (2.14) and (2.17), we can derive the flow coefficient, $\Phi$, and stage loading coefficient, $\psi$, as

$$\Phi = 1/(\Omega_s D_s^3)$$

(2.18)

$$\psi = 1/(\Omega_s^2 D_s^2)$$

(2.19)

By introducing the Cordier line data into these last two equations and reploting this information, a new and more definite shape of the optimum machine curves results, shown in Figure 2.10. The new curve is clearly divided into two main parts with centrifugal pumps operating at a fairly constant head coefficient at roughly $\psi = 0.1$ over a flow coefficient range of $0.001 \leq \Phi \leq 0.04$ and axial machines operating with a wide range of stage loading coefficients, $0.005 \leq \psi \leq 0.05$ and also a wide range of $\Phi$. Casey, Zwyssig, and Robinson (2010) show that the shape of the Cordier line and the two distinct parts of the curve in Figure 2.10 are caused by the variation in centrifugal effects in the different compressor types: In radial machines, almost all the pressure change is due to the centrifugal effects generated by a change in flow radius, whereas these effects are absent in axial machines (see Chapter 7).
Mixed-flow machines are stuck in between axials and radials with quite a narrow range of both $\psi$ and $\Phi$. However, in some cases, mixed-flow machines are the crucial choice. Lewis (1996) points out that applications that require a high mass flow at a high pressure ratio, such as gas cooled nuclear reactors and hovercraft lift fans, are ideally suited for mixed-flow fans rather than a single-stage axial compressor. Recently, mixed-flow turbomachinery has found application in specialist domestic appliances. Figure 2.11 shows a mixed-flow fan used for air movement.

**EXAMPLE 2.4**

The mixed-flow fan shown in Figure 2.11 is designed to provide a pressure rise of 450 Pa to air at a volume flow rate of 27 L/s. The impeller design rotational speed is 8300 rpm and its tip diameter is 90 mm.

Calculate the specific speed and specific diameter of the fan and mark the location of the design on the Cordier line in Figure 2.9. Also determine the design flow coefficient and head coefficient. Assuming that the required flow rate and pressure rise cannot be changed, estimate the rotational speed that would be needed for an axial flow fan to be suitable for the design.

Take air density to be 1.21 kg/m$^3$. 

![Chart of $\psi$ versus $\Phi$ for various pumps and fans.](image)
Solution

The specific speed can be calculated from the design specification, as follows:

\[
\Omega_s = \frac{\Omega Q^{1/2}}{(gH)^{3/4}} = \frac{\Omega Q^{1/2}}{(\Delta p/\rho)^{3/4}} = \frac{8300 \times \pi/30 \times \sqrt{27 \times 10^{-3}}}{(450/1.21)^{0.75}} \approx 1.69 \quad \text{rad}
\]

Similarly, the specific diameter can be calculated:

\[
D_s = \frac{D(gH)^{1/4}}{Q^{1/2}} = \frac{D(\Delta p/\rho)^{1/4}}{Q^{1/2}} = \frac{0.09 \times (450/1.21)^{25}}{\sqrt{27 \times 10^{-3}}} \approx 2.41
\]

Marking these values on Figure 2.9, it is clear that the design lies close to the Cordier line and that a mixed-flow device is most suitable. The design flow coefficient and head coefficient, using Eqs (2.18) and (2.19), are

\[
\Phi = \frac{Q}{\Omega D^3} = \frac{1}{(\Omega_s D_s^3)} = \frac{1}{1.69 \times 2.41^3} = 0.042
\]

\[
\psi = \frac{\Delta p/\rho}{\Omega^2 D^2} = \frac{1}{(\Omega_s^2 D_s^2)} = \frac{1}{1.69^2 \times 2.41^2} = 0.060
\]

For an axial machine to be suitable, Figures 2.8 and 2.9 suggest that the specific speed must be increased to a value of around 3 or higher. With a fixed flow and pressure rise, the specific speed is proportional to the rotational speed. Therefore, a specific speed greater than 3 requires a rotational speed:

\[
\Omega_2 \geq \Omega_1 \frac{\Omega_{s2}}{\Omega_{s1}} = 8300 \times \frac{3}{1.69} = 14700 \quad \text{rpm}
\]
Compressible specific speed

Specific speed as defined in Eq. (2.14) has mostly been applied to the design and selection of low-speed and hydraulic turbomachines. However, the notion of specific speed can equally be applied to a compressible flow machine, and it is particularly useful for determining whether an axial or a radial flow machine is best for a particular requirement. As described in Baskharone (2006), the application of the important concept of specific speed to compressible turbomachines has to be modified because of the large variation in the values of volume flow rate, $Q$, as well as the particular meaning of the head, $H$. The specific speed when applied to high-speed turbomachines is therefore expressed in terms of parameters appropriate to compressible flow:

$$\Omega_s = \Omega \left( \frac{\dot{m} \rho_e}{\dot{m} \rho_e} \right)^{1/2} (\Delta h_{0s})^{-3/4}$$  \hspace{1cm} (2.20)

Note that in Eq. (2.20), the isentropic specific work, $\Delta h_{0s}$, is used rather than the actual specific work, $\Delta h_0$. In the case of a compressor, this makes sense since the isentropic specific work can be determined from the required pressure ratio $p_{02}/p_{01}$ using Eq. (2.7). The required pressure ratio is likely to be known at the outset of the design process, whereas the actual specific work input depends on the compressor efficiency, which in general will not be known. In the case of a turbine, the actual specific work is more likely to be a known requirement. The efficiency can be estimated or the isentropic work approximated to be equal to the actual work required.

Equation (2.20) also requires the density of the working fluid at exit $\rho_e$. This can be estimated from $\rho_e = p_e/RT_e$, with $p_e$ and $T_e$ taken as the isentropic static pressure and temperature at exit from the machine. Other definitions are sometimes used, but this is the simplest and any extra uncertainty introduced is likely to be small and will have no effect on the preferred type of machine selected.

EXAMPLE 2.5

An air turbine is required for a dentist’s drill. For the drill bit to effectively abrade tooth enamel, the turbine must rotate at high speed, around 300,000 rpm. The turbine must also be very small so that it can be used to access all parts of a patient’s mouth and an exit air flow rate in the region of 10 L/min is required for this. The turbine is to be driven by supply air at a pressure of 3 bar and a temperature of 300 K.

Calculate the specific speed of the turbine and use this to determine the type of machine required. Also estimate the power consumption of the turbine and account for how this power is used.

**Solution**

Putting the quantities into standard SI units,

- the rotational speed, $\Omega = 300,000 \times \pi/30 = 10,000\pi$ rad/s
- the exit volume flow rate, $\dot{m}/\rho_e = Q_e = 10/(1000 \times 60) = 0.000167$ m$^3$/s
The isentropic specific work can be estimated assuming an isentropic expansion through the turbine. Treating air as a perfect gas with $\gamma = 1.4$ and $C_p = 1005 \text{ J/kg/K}$,

$$\Delta h_{0s} = C_p T_0 \left[ 1 - (p_{02}/p_{01})^{(\gamma - 1)/\gamma} \right] = 1005 \times 300 \times \left[ 1 - \left( \frac{1}{3} \right)^{0.4/1.4} \right] = 81.29 \text{ kJ/kg}$$

The specific speed can now be calculated from the information provided using Eq. (2.20):

$$\Omega_s = \frac{\Omega Q^{1/2}}{(gH)^{3/4}} = \Omega \left( \frac{\dot{m}}{\rho_e} \right)^{1/2} (\Delta h_{0s})^{-3/4} = \frac{10,000 \times \pi \times 0.000167^{1/2}}{(81,290)^{3/4}} \approx 0.084 \text{ rad}$$

Using the plot of machine type versus specific speed presented in Figure 2.8, it is immediately apparent that the only kind of turbine suitable for this very low specific speed is a Pelton wheel. In fact, all modern high-speed dentist drills use Pelton wheels and a photograph of a typical impeller from one is shown in Figure 2.12.

The power used by the turbine can be approximated from the mass flow rate and the specific isentropic work output. Using a typical value for the exit air density, this gives

$$P = \dot{m} \Delta h_{0s} = \rho_e Q_e \Delta h_{0s} \approx 1.16 \times 0.000167 \times 81,290 = 15.7 \text{W}$$

The majority of this power will be dissipated as heat through friction in the bearings, losses in the Pelton wheel, and friction with the tooth. This heat dissipation is the reason why an appreciable amount of cooling water is required for modern high-speed dentist drills!

**FIGURE 2.12**

Pelton Wheel Turbine Impeller from a High Speed Dental Drill, Tip Diameter 10 mm.

(With kind permission of Sirona Dental)

### 2.7 Cavitation

Cavitation is the boiling of a liquid at normal temperature when the static pressure is made sufficiently low. It may occur at the entry to pumps or at the exit from hydraulic turbines in the vicinity of the moving blades. The dynamic action of the rotor blades causes the static pressure to reduce
locally in a region that is already normally below atmospheric pressure and cavitation can commence. The phenomenon is accentuated by the presence of dissolved gases that are released with a reduction in pressure.

For the purpose of illustration, consider a centrifugal pump operating at constant speed and capacity. By steadily reducing the inlet pressure head, a point is reached when streams of small vapor bubbles appear within the liquid and close to solid surfaces. This is called *cavitation inception* and commences in the regions of lowest pressure. These bubbles are swept into regions of higher pressure where they collapse. This condensation occurs suddenly, the liquid surrounding the bubbles either hitting the walls or adjacent liquid. The pressure wave produced by bubble collapse (with a magnitude on the order of 400 MPa) momentarily raises the pressure level in the vicinity and the action ceases. The cycle then repeats itself and the frequency may be as high as 25 kHz (Shepherd, 1956). The repeated action of bubbles collapsing near solid surfaces leads to the well-known cavitation erosion.

The collapse of vapor cavities generates noise over a wide range of frequencies—up to 1 MHz has been measured (Pearsall, 1972), i.e., so-called white noise. Apparently the collapsing smaller bubbles cause the higher frequency noise, and the larger cavities the lower frequency noise. Noise measurement can be used as a means of detecting cavitation (Pearsall, 1967). Pearsall and McNulty (1968) have shown experimentally that there is a relationship between cavitation noise levels and erosion damage on cylinders and conclude that a technique could be developed for predicting the occurrence of erosion.

Up to this point, no detectable deterioration in performance occurs. However, with further reduction in inlet pressure, the bubbles increase both in size and number, coalescing into pockets of vapor that affects the whole field of flow. This growth of vapor cavities is usually accompanied by a sharp drop in pump performance as shown conclusively in Figure 2.2 (for the 5000 rpm test data). It may seem surprising to learn that, with this large change in bubble size, the solid surfaces are much less likely to be damaged than at inception of cavitation. The avoidance of cavitation inception in conventionally designed machines can be regarded as one of the essential tasks of both pump and turbine designers. However, in certain recent specialized applications, pumps have been designed to operate under *supercavitating* conditions. Under these conditions, large size vapor bubbles are formed, but bubble collapse takes place downstream of the impeller blades. An example of the specialized application of a supercavitating pump is the fuel pumps of rocket engines for space vehicles, where size and mass must be kept low at all costs. Pearsall (1973) has shown that the supercavitating principle is most suitable for axial flow pumps of high specific speed and has suggested a design technique using methods similar to those employed for conventional pumps.

Pearsall (1973) was one of the first to show that operating in the supercavitating regime was practicable for axial flow pumps, and he proposed a design technique to enable this mode of operation to be used. A detailed description was published in Pearsall (1972), and the cavitation performance was claimed to be much better than that of conventional pumps. Some further details are given in Chapter 7.

**Cavitation limits**

In theory, cavitation commences in a liquid when the static pressure is reduced to the vapor pressure corresponding to the liquid’s temperature. However, in practice, the physical state of the liquid will determine the pressure at which cavitation starts (Pearsall, 1972). Dissolved gases come out of
solution as the pressure is reduced, forming gas cavities at pressures in excess of the vapor pressure. Vapor cavitation requires the presence of nuclei—submicroscopic gas bubbles or solid non-wetted particles—in sufficient numbers. It is an interesting fact that in the absence of such nuclei, a liquid can withstand negative pressures (i.e., tensile stresses)! Perhaps the earliest demonstration of this phenomenon was that performed by Reynolds (1882) before a learned society. He showed how a column of mercury more than twice the height of the barometer could be (and was) supported by the internal cohesion (stress) of the liquid. More recently Ryley (1980) devised a simple centrifugal apparatus for students to test the tensile strength of both plain, untreated tap water in comparison with water that had been filtered and then deaerated by boiling. Young (1989) gives an extensive literature list covering many aspects of cavitation including the tensile strength of liquids. At room temperature, the theoretical tensile strength of water is quoted as being as high as 1000 atm (100 MPa)! Special pretreatment (i.e., rigorous filtration and pre-pressurization) of the liquid is required to obtain this state. In general, the liquids flowing through turbomachines will contain some dust and dissolved gases and under these conditions negative pressure does not arise.

A useful parameter is the available suction head at entry to a pump or at exit from a turbine. This is usually referred to as the net positive suction head, NPSH, defined as

$$H_s = (p_o - p_v)/(\rho g)$$ (2.21)

where \(p_o\) and \(p_v\) are the absolute stagnation and vapor pressures, respectively, at pump inlet or at turbine outlet.

To take into account the effects of cavitation, the performance laws of a hydraulic turbomachine should include the additional independent variable \(H_s\). Ignoring the effects of Reynolds number, the performance laws of a constant geometry hydraulic turbomachine are then dependent on two groups of variable. Thus, the efficiency,

$$\eta = f(\varphi, \Omega_{ss})$$ (2.22)

where the suction specific speed \(\Omega_{ss} = \Omega Q^{1/2}/(gH_s)^{3/4}\), determines the effect of cavitation, and \(\Phi = Q/(\Omega D^3)\), as before.

It is known from experiments made by Wislicenus (1965) that cavitation inception occurs for an almost constant value of \(\Omega_{ss}\) for all pumps (and, separately, for all turbines) designed to resist cavitation. This is because the blade sections at the inlet to these pumps are broadly similar (likewise, the exit blade sections of turbines are similar) and the shape of the low-pressure passages influences the onset of cavitation.

Using the alternative definition of suction specific speed \(\Omega_{ss} = \Omega Q^{1/2}/(gH_s)^{3/4}\), where \(\Omega\) is the rotational speed in rad/s, \(Q\) is the volume flow in m\(^3\)/s, and \(gH_s\) is in m\(^2\)/s\(^2\). Wislicenus showed that

\[\Omega_{ss} = 3.0 \text{ (rad)}\] (2.23a)

for pumps, and

\[\Omega_{ss} = 4.0 \text{ (rad)}\] (2.23b)

for turbines.

Pearsall (1967) describes a supercavitating pump with a cavitation performance much better than that of conventional pumps. For this pump, suction specific speeds \(\Omega_{ss}\) up to 9.0 were readily obtained and, it was claimed, even better values might be possible but at the cost of reduced head
and efficiency. It is likely that supercavitating pumps will be increasingly used in the search for higher speeds, smaller sizes, and lower costs.

PROBLEMS

1. A fan operating at 1750 rpm at a volume flow rate of 4.25 m$^3$/s develops a head of 153 mm measured on a water-filled U-tube manometer. It is required to build a larger, geometrically similar fan that will deliver the same head at the same efficiency as the existing fan but at a speed of 1440 rpm. Calculate the volume flow rate of the larger fan.

2. An axial flow fan 1.83 m diameter is designed to run at a speed of 1400 rpm with an average axial air velocity of 12.2 m/s. A quarter scale model has been built to obtain a check on the design and the rotational speed of the model fan is 4200 rpm. Determine the axial air velocity of the model so that dynamical similarity with the full-scale fan is preserved. The effects of Reynolds number change may be neglected. A sufficiently large pressure vessel becomes available in which the complete model can be placed and tested under conditions of complete similarity. The viscosity of the air is independent of pressure and the temperature is maintained constant. At what pressure must the model be tested?

3. The water pump used to generate the plot shown in Figure 2.2 has an impeller diameter of 56 mm. When tested at a speed of 4500 rpm, the head–volume flow rate characteristic produced can be approximated by the equation

$$H = 8.6 - 5.6Q^2$$

where $H$ is in meters and $Q$ in dm$^3$/s. Show that, provided viscous and cavitation effects are negligible, the characteristic of all geometrically similar pumps may be written in dimensionless form as

$$\psi = 0.121(1 - 4460\Phi^2)$$

where $\psi$ is the dimensionless head coefficient, $gH/\Omega^2D^2$, $\Phi$ is the flow coefficient, $Q/\Omega D^3$, and $\Omega$ is expressed in rad/s. Show that this result is consistent with Figure 2.2, where $\Omega$ is expressed in rev/s.

4. A water turbine is to be designed to produce 27 MW when running at 93.7 rpm under a head of 16.5 m. A model turbine with an output of 37.5 kW is to be tested under dynamically similar conditions with a head of 4.9 m. Calculate the model speed and scale ratio. Assuming a model efficiency of 88%, estimate the volume flow rate through the model. It is estimated that the force on the thrust bearing of the full-size machine will be 7.0 GN. For what thrust must the model bearing be designed?

5. Derive the nondimensional groups that are normally used in the testing of gas turbines and compressors. A compressor has been designed for normal atmospheric conditions (101.3 kPa and 15°C). To economize on the power required, it is being tested with a throttle in the entry duct to reduce the entry pressure. The characteristic curve for its normal design speed of
4000 rpm is being obtained on a day when the ambient temperature is 20°C. At what speed should the compressor be run? At the point on the characteristic curve at which the mass flow would normally be 58 kg/s, the entry pressure is 55 kPa. Calculate the mass flow rate during the test.

6. Describe, with the aid of sketches, the relationship between geometry and specific speed for pumps.
   a. A model centrifugal pump with an impeller diameter of 20 cm is designed to rotate at 1450 rpm and to deliver 20 dm³/s of freshwater against a pressure of 150 kPa. Determine the specific speed and diameter of the pump. How much power is needed to drive the pump if its efficiency is 82%?
   b. A prototype pump with an impeller diameter of 0.8 m is to be tested at 725 rpm under dynamically similar conditions as the model. Determine the head of water the pump must overcome, the volume flow rate, and the power needed to drive the pump.

7. A hydraulic turbine is to be installed where the net head is 120 m and the normal available flow rate is 1.5 m³/s. A 48 pole synchronous generator is available (to operate with a 60 Hz electrical system) and has an adequate power capacity matching the turbine. Determine
   a. the rotational speed and the electrical power that can be delivered if the system efficiency (turbine and generator) is 85%;
   b. the power specific speed of the turbine;
   What type of turbine is being used in this application?

8. A hydraulic turbine running at 160 rpm, discharges 11 m³/s and develops 2400 kW at a net head of 25 m. Determine
   a. the efficiency of the turbine;
   b. the speed, flow rate, and power output of this turbine when running under a net head of 40 m assuming homologous conditions and the same efficiency.

9. A hydraulics engineer is planning to utilize the water flowing in a stream, normally able to provide water at a flow rate of 2.7 m³/s, and a head of 13 m for power generation. The engineer is planning to use a 2.0 m diameter turbine operating at a rotational speed of 360 rpm and at a hoped for efficiency of 88%.
   a. Determine the likely power developed by the turbine, the specific speed and specific diameter, and the most suitable type of turbine for this duty.
   b. The engineer then decides, first of all, to test a geometrically similar model turbine with a diameter of 0.5 m (operating at the same specific speed and specific diameter as the prototype) and with a head of 4.0 m. Determine, for the model, the volume flow rate, the rotational speed, and the power.

10. A single-stage axial flow gas turbine is to be tested in a “cold rig” so as to simulate the design-point operation. The two sets of operating conditions are:
   1. Design—point operation of turbine
      Stage—inlet total pressure, $p_{01} = 11$ bar
      Stage—inlet total temperature, $T_{01} = 1400$ K
      Stage—exit total pressure, $p_{02} = 5.0$ bar
Speed of rotation, \( N = 55,000 \text{ rpm} \)
Stage efficiency, \( \eta_s = 87\% \)
Mass flow rate \( \dot{m} = 3.5 \text{ kg/s} \)

2. Cold—rig operation
Stage—inlet total pressure, \( p_{01(\text{cr})} = 2.5 \text{ bar} \)
Stage—inlet total temperature, \( T_{01(\text{cr})} = 365 \text{ K} \)

For both sets of conditions, assume that the axial velocity across the stage remains constant.

Determine

a. the stage—exit total temperature \( T_{02(\text{cr})} \);

b. the power output in the cold rig.

Assume that the average specific heat ratio for both operating conditions is given by \( \gamma = 1.36 \).

References