

# Effect of plate skew angle and crack parameters on buckling of cracked CNT-reinforced composite skew plates

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## ABSTRACT

The present study is the first to investigate the effect of plate skew angle and crack parameters on the buckling of cracked CNT-reinforced composite skew plates. A MATLAB code, developed within the extended finite element method (XFEM), analyzes the buckling stability of various cracked CNT-reinforced composite skew plate models under different in-plane loading and boundary conditions. The analysis considers varying plate skew angles, crack lengths, and crack locations. The effect of single-walled carbon nanotube (SWCNT) volume fraction and dispersion, modeled using the Eshelby-Mori-Tanaka homogenization scheme, on critical buckling is also examined. Results indicate that the buckling capacity of cracked CNT-reinforced composite skew plates is significantly affected by not only the skew angle and boundary conditions of the plate but also by the length and location of the crack. Moreover, the agglomeration of CNTs reduces the effective stiffness and buckling capacity.

## 1. Introduction

Skew plates are used in various engineering applications, particularly in aerospace structures such as rockets, satellites, and aircraft components, as well as in shipbuilding, automotive, and the construction of panels for skew bridges. Consequently, understanding their static and dynamic behavior is crucial for engineers. While analytical and numerical methods exist for analyzing skew plate buckling, accurate design and modeling necessitate advanced techniques. Various analytical and numerical approaches exist to solve the equations governing the buckling of skew plates. Regarding skew plates without cracks, studies under in-plane compressive or shear stresses using methods such as the Galerkin method [1], Rayleigh-Ritz method [2–8], classical plate theory [9,10], general theory of plates [11], extended Kantorovich method [12], differential quadrature method [13,14], discrete singular convolution [15], and FEM method [16–18] have been reported. A complicating factor in the analysis of skew plates is the non-orthogonal coordinate system used in deriving the governing equations and solving the resulting coupled equations. Conventional plate-bending finite elements can be applied to the analysis of skew plates. However, they involve formulating the transformation matrix between global and local degrees of freedom for the nodes on the skew edges. This transformation makes it easier to impose boundary conditions normal and tangential to

the edges [17].

Generally, thin plates are highly susceptible to defects, with cracking being the most common type. Cracks can initiate under various conditions, such as initial material weaknesses, fatigue, yield loading, and production imperfections. On the other hand, buckling is a phenomenon that can most likely occur in thin-walled plates, especially in cases with defects. In engineering design, the presence of buckling cracks can lead to catastrophic failure. For this reason, crack analysis is very crucial in the design of such structures. Accordingly, several research efforts have been made on the buckling of rectangular plates with central and edge cracks under different loads using numerical [19–21] and experimental [22–24] methods. These studies have investigated the effects of parameters such as crack length and angle, Poisson's ratio, and boundary conditions on the buckling load of cracked plates. The results indicate that these parameters significantly affect the buckling load. Cracks can form at any location on a plate, making it essential to evaluate the impact of crack location on the stability behavior of cracked plates. Researchers have investigated the eccentricity of cracks in rectangular plates under compressive [25–28], tensile [29], and shear [30] loads. They found that the effect of crack eccentricity largely depends on the type of plate support conditions [28]. In addition, the buckling capacity of a cracked plate is highly dependent on crack parameters, such as crack length and location, and the critical buckling mode shape is affected by

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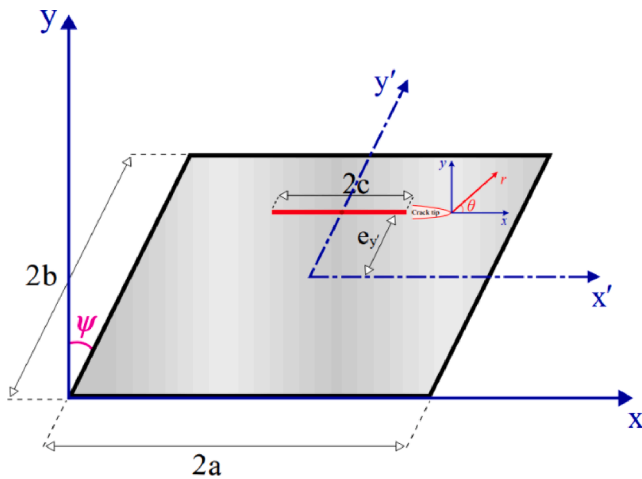
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**Fig. 1.** Schematic illustration of the CNT-reinforced composite skew plate with an off-center crack showing global  $(x, y)$  and local coordinate  $(x', y')$  systems. In this figure,  $2a$ ,  $2b$ , and  $\psi$  are the length, width, and angle of skew of the CNTRC skew plate, respectively. The aspect ratio of the cracked skew plate is assumed to be one, and its thickness is 1 mm. In addition,  $2c$  denotes the crack length. Parameter  $c^*$  is the relative crack length ( $c^* = c/a$ ). The crack location is expressed by the eccentricity ratio parameter  $e_y^*$  ( $e_y^* = e_y/b$ ), where  $e_y$  is the distance from the center of the crack to the skew plate centerline. Here, the stresses are displayed on an infinitesimal element around the crack tip located at a point with local polar coordinates  $(r, \theta)$ . The polar coordinates of the local polar coordinate system are defined with an origin at the tip of the crack and with basis vectors defined by the unit vector's tangent and normal to the crack at the crack tip. The skew plate is made of epoxy reinforced by SWCNTs, in which the effect of agglomeration and volume fraction of CNTs is considered.

the interaction effects of various cracking parameters [30].

In response to new engineering demands, composites reinforced with carbon nanotubes [31–33] play a crucial role due to CNTs' unique mechanical properties, including high strength, hardness, a high aspect ratio, and low density [34–40]. Studies show that even small additions of CNTs can significantly improve the electrical, thermal, and mechanical properties of polymer composites [41–51], provided that good surface bonding between CNTs and the polymer, as well as proper dispersion of carbon nanotubes in the polymer matrix, are achieved [52–55]. CNT-reinforced composites are suitable for use as structural beams, plates, or shells in diverse engineering applications. An important problem to consider when analyzing the buckling of polymer composite plates is the effect of cracks, if any, on the buckling behavior of the plate. Polymer composites are prone to premature failure due to the creation of cracks and microcracks during operation. The evolution of cracks and microcracks can cause catastrophic material failure [56]. Considering the increasing use of polymer materials and carbon nanotubes in aeronautics and astronautics and the need for advanced structures, it is essential to analyze the buckling behavior of carbon nanotube-reinforced cracked composite skew plates.

In the literature on skew plates, the buckling solutions obtained by different researchers for various boundary conditions differ somewhat from each other. Some researchers attribute this difference to the boundary conditions at the corners of the sheet, which significantly affect the results [17]. When dealing with complex geometries and boundary conditions, a method should be chosen that has flexibility. The finite element method (FEM) is one of the most widely used methods in structural mechanics. Since FEM can be considered a local Ritz method and does not account for bending moment singularity in the assumed displacement functions, it may experience a lower convergence rate when analyzing a thin inclined plate with large skew angles [13]. Therefore, a very fine mesh must be used to obtain accurate buckling solutions. The standard FEM has been successful in common fracture mechanics simulations, such as modeling inclined plates without cracks.

However, in the case of media with cracks, due to complications related to stress and oscillatory displacement fields near the crack tip lead to low performance and accuracy. Numerical methods more suitable for solving discontinuity problems are element-free methods based on enrichment techniques. Consequently, the Extended Finite Element Method (XFEM) offers significant advantages over conventional methods like FEM, particularly in the buckling analysis of cracked CNT-reinforced composite skew plates. In terms of computational accuracy, XFEM inherently provides more reliable results by precisely modeling stress and displacement fields around the crack tip without requiring remeshing in the vicinity of the crack. This capability substantially enhances computational efficiency, as XFEM converges with fewer elements compared to FEM, which often necessitates extremely fine and complex meshes when dealing with discontinuities. This approach effectively overcomes the limitations of FEM, such as its reduced convergence rate in analyzing thin skew plates with large skew angles and its diminished accuracy in the presence of cracks. Overall, XFEM presents a more robust and accurate solution for simulating the complex mechanical behavior of these structures.

The study of plate skew angles and cracks in skew plates is crucial for meeting modern engineering demands and leveraging advanced materials. Existing research has mainly focused on the buckling of cracked rectangular plates and uncracked skew plates. Therefore, this study, for the first time, investigates the effect of plate skew angle and crack parameters on the buckling of CNTRC cracked skew plates using a MATLAB-based XFEM code. To achieve this goal, initially, the equivalent mechanical properties of the nanocomposite material are estimated using the Eshelby-Mori-Tanaka homogenization scheme, considering the random orientation of carbon nanotubes within the polymer matrix and accounting for three agglomeration modes (null, partial, and full). Subsequently, a computational code is developed in MATLAB to generate numerical models of cracked CNTRC skew plates for buckling analysis. The analysis encompasses the effect of several parameters, including the volume fraction and agglomeration of CNTs, crack length and location, loading and boundary conditions, and the skew angle of the CNTRC skew plates.

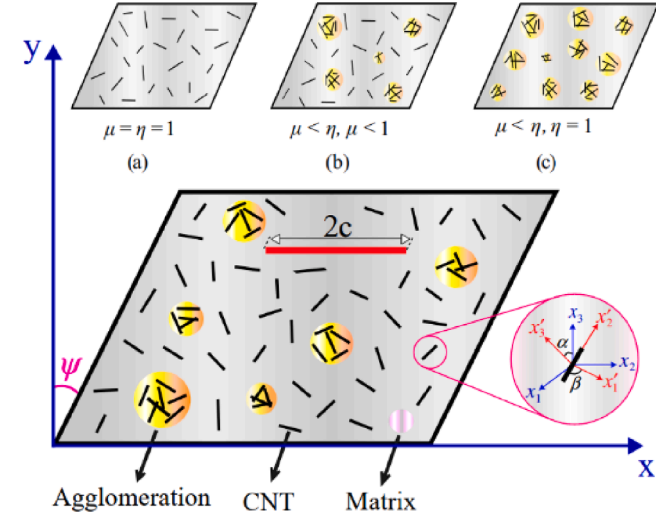
## 2. Geometric and mechanical characterization

In this work, we considered a cracked CNT-reinforced composite skew plate with the associated coordinate system and geometric parameters (See Fig. 1). The location of the crack is characterized by the eccentricity ratio parameter  $e_y^*$  ( $e_y^* = e_y/b$ ), where  $e_y$  is the distance from the center of the crack to the plate centerlines. For a center crack, the value of  $e_y^*$  is zero, and for off-center cracks, the value of  $e_y^*$  ranges between zero and one. As shown in Fig. 1, around the crack tip, the stresses are on an infinitesimal element located at a point with polar coordinates  $(r, \theta)$ . The XFEM method is based on the concept of partition of unity for enriching the FEM approximation to include the effects of singular or discontinuous fields around a crack. Nanocomposite skew plates were considered from a mixture of SWCNTs and an isotropic matrix with certain Young's modulus ( $E_m$ ) and Poisson's ratio ( $\nu_m$ ). Polymer composites are widely used in space technologies and space structures due to their strength, lightweight, and good thermal and electrical insulation properties. The widest application of epoxy resins is perhaps utilizing them as a matrix for polymer-based composite materials. Nevertheless, epoxy resins are mostly brittle materials, and cracking is a common problem for epoxy resins and composites that serve under a combination of mechanical stress and chemically aggressive environments. Therefore, many attempts have been made to improve their physical properties using various toughening or strengthening agents. CNTs are the most well-known nanoparticles used as reinforcements in polymeric materials.

CNTs have low bending stiffness and a high aspect ratio, making them prone to agglomeration in a polymer matrix [57,58]. The effective mechanical properties of nanocomposites, in which the straight CNTs

**Table 1**  
Material properties of cracked CNTRC skew plates [30].

| SWCNT (10,10)               |      |                              |      | Matrix      |                           |
|-----------------------------|------|------------------------------|------|-------------|---------------------------|
| Hill's elastic moduli (GPa) |      | Density (kg/m <sup>3</sup> ) |      | Epoxy resin |                           |
| $k_{CNT}$                   | 271  | $\rho_{CNT}$                 | 1400 | $E_m$       | 2.1 (GPa)                 |
| $l_{CNT}$                   | 88   |                              |      | $\rho_m$    | 1150 (kg/m <sup>3</sup> ) |
| $m_{CNT}$                   | 17   |                              |      | $\nu_m$     | 0.34                      |
| $n_{CNT}$                   | 1089 |                              |      |             |                           |
| $p_{CNT}$                   | 442  |                              |      |             |                           |



**Fig. 2.** Agglomeration model of CNTs for the cracked CNT-reinforced composite skew plate and orientation of a straight CNT by two Euler angles  $\alpha$  and  $\beta$ : (a) null agglomeration; (b) partial agglomeration; (c) complete agglomeration. The variable  $\mu$  denotes the volume fraction of clusters relative to the total RVE volume, while  $\eta$  refers to the volume ratio of CNTs in the clusters compared to the total CNTs in the RVE. In general, three limit cases of the agglomeration ratio can be defined depending on the values of  $\mu$  and  $\eta$ . The case  $\mu = \eta = 1$ , where all nanotubes are uniformly dispersed in the matrix, represents a null level of agglomeration. It is assumed that the nanotubes are randomly dispersed in the matrix (Fig. 2a). For  $\mu < \eta$  and  $\mu < 1$ , it is assumed that some CNTs are uniformly distributed throughout the matrix, while others appear in cluster form due to agglomeration (Fig. 2b). Finally, the complete agglomeration of CNTs is defined by  $\mu < \eta$  and  $\eta = 1$  (Fig. 2c). In other words, all the CNTs are located within the spherical clusters. The direction of a straight CNT is determined by two Euler angles,  $\alpha$  and  $\beta$ .

are randomly oriented and aggregated in some clusters, are obtained based on the Eshelby–Mori–Tanaka approach [59,60]. In this regard, a version of the EMT approach introduced by Shi et al. [61] was employed to estimate the mechanical constants of the nanocomposite layer. The reinforcing phase is represented by SWCNTs as linear elastic and homogeneous cylindrical solids with transversely isotropic features. Five independent parameters (Hill's elastic modulus) are needed to characterize CNTs, which are presented in Table 1 for a single-walled CNT with a chiral index equal to 10. According to the rule of mixtures, the density of a CNT-reinforced composite can be calculated as follows:

$$\rho_{CNTRC} = \rho_m V_m + \rho_{CNT} V_{CNT} \quad (1)$$

The volume fraction of the polymer matrix ( $V_m$ ) and the volume fraction of CNTs ( $V_{CNT}$ ) are interrelated as follows:

$$V_{CNT} + V_m = 1 \quad (2)$$

The volume fraction of CNTs ( $V_{CNT}$ ) can be expressed as a function of the mass fraction ( $w_r$ ) as described:

$$V_{CNT} = \left( \frac{\rho_{CNT}}{w_{CNT}\rho_m} - \frac{\rho_{CNT}}{\rho_m} + 1 \right)^{-1} \quad (3)$$

in which  $w_{CNT}$  represents the mass fraction of CNTs, while  $\rho_{CNT}$  and  $\rho_m$  denote the densities of the CNT and the matrix, respectively. The total volume of the CNTs in the representative volume element (RVE) can be divided into the following two parts:

$$V_{CNT} = V_{CNT}^{cluster} + V_{CNT}^m \quad (4)$$

where  $V_{CNT}^{cluster}$  and  $V_{CNT}^m$  represent the total volume of CNTs inside a cluster and inside the matrix outside the clusters, respectively. Following Shi et al. [61], a representative volume element is divided into two parts: one reinforced with aggregated CNT and the other with fully dispersed CNT. The two parameters ( $\mu$ ,  $\eta$ ) used to describe agglomeration are defined as:

$$\mu = \frac{V_{cluster}}{V}, \eta = \frac{V_{CNT}^{cluster}}{V_{CNT}} \quad 0 \leq \mu, \eta \leq 1 \quad (5)$$

where  $V$ ,  $V_{cluster}$ , and  $V_{CNT}^{cluster}$  represent the RVE volume, total cluster volume, and total CNTs volume inside a cluster, respectively. The variable  $\mu$  denotes the volume fraction of clusters relative to the total RVE volume  $V$ , while  $\eta$  refers to the volume ratio of CNTs in the clusters compared to the total CNTs in the RVE. Thus,  $\mu = \eta = 1$  indicates null agglomeration,  $\mu < \eta$  and  $\mu < 1$  signifies partial agglomeration, and  $\mu < \eta$  with  $\eta = 1$  reflects complete agglomeration (Fig. 2). The orientation of a straight CNT is defined by two Euler angles,  $\alpha$  and  $\beta$ , as depicted in Fig. 2. The average volume fraction of CNTs in the composite is  $c_{CNT}$ .

$$c_{CNT} = \frac{V_{CNT}}{V} \quad (6)$$

By combining Eqs. (7–9) the volume fractions of CNTs in the clusters and the matrix are expressed, respectively, as

$$\frac{V_{CNT}^{cluster}}{V_{cluster}} = \frac{c_{CNT}\eta}{\mu}, \quad \frac{V_{CNT}^m}{V - V_{cluster}} = \frac{c_{CNT}(1 - \eta)}{(1 - \mu)} \quad (7)$$

The orientation of a straight CNT is characterized by two Euler angles  $\alpha$  and  $\beta$ , as shown in Fig. 2. The base vectors  $\mathbf{e}_i$  and  $\mathbf{e}'_i$  of the global ( $o-x_1 \times x_2 \times x_3$ ) and the local coordinate systems ( $o-x'_1 \times x'_2 \times x'_3$ ) are related via the transformation matrix  $\mathbf{g}$ ,

$$\mathbf{e}_i = g_{ij} \mathbf{e}'_j \quad (8)$$

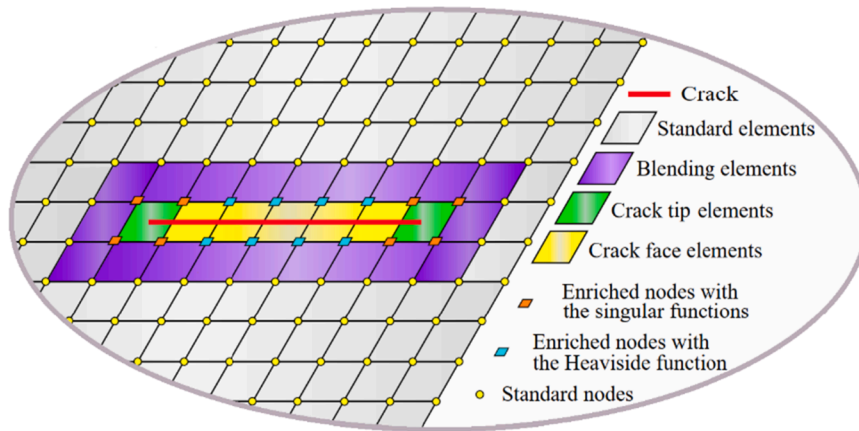
where  $\mathbf{g}$  is defined as

$$\mathbf{g} = \begin{bmatrix} \cos \beta & -\cos \alpha \sin \beta & \sin \alpha \sin \beta \\ \sin \beta & \cos \alpha \cos \beta & -\sin \alpha \cos \beta \\ 0 & \sin \alpha & \cos \alpha \end{bmatrix} \quad (9)$$

The orientation distribution of CNTs in a nanocomposite is represented by a probability density function  $p(\alpha, \beta)$  satisfying the below normalization condition

$$\int_0^{2\pi} \int_0^{\pi/2} p(\alpha, \beta) \sin \alpha d\alpha d\beta = 1 \quad (10)$$

If CNTs are randomly oriented, the density function is defined as  $p(\alpha, \beta) = 1/2\pi$ . Here a carbon nanotube-reinforced composite is considered a system consisting of spherical clusters in a matrix. The effective elastic stiffness of the clusters and the matrix is first estimated separately, and then the overall properties of the composite system as a whole are calculated. It is assumed that the CNTs are randomly oriented in the clusters, the cluster materials are considered isotropic. The effective values of the bulk modulus ( $K$ ) and shear modulus ( $G$ ) for both the inside and outside of the clusters are estimated using the EMT approach as follows [61]:



**Fig. 3.** Different element types and enriched nodes in XFEM modeling for a crack. In the XFEM method for crack modeling, both the Heaviside function and singular functions are employed to represent cracks. To address the incompatibility between enriched and non-enriched domains, a transition zone is introduced. Consequently, four types of subdomains are defined to model a general crack problem: Elements intersected by a crack and enriched by the Heaviside function, crack tip enrichment domain, Blending region, and Standard FEM domain. In addition, nodes surrounding the crack are enriched using the Heaviside function, while the four nodes around the crack tip are enriched using singularity functions.

$$K_{in} = K_m + \frac{V_{CNT}\eta(\delta_{CNT} - 3K_m\alpha_{CNT})}{3(\mu - V_{CNT}\eta + V_{CNT}\eta\alpha_{CNT})} \quad (11)$$

$$G_{in} = G_m + \frac{V_{CNT}\eta(\eta_{CNT} - 2G_m\beta_{CNT})}{2(\mu - V_{CNT}\eta + V_{CNT}\eta\beta_{CNT})} \quad (12)$$

$$K_{out} = K_m + \frac{V_{CNT}(1-\eta)(\delta_{CNT} - 3K_m\alpha_{CNT})}{3[1-\mu - V_{CNT}(1-\eta) + V_{CNT}(1-\eta)\alpha_{CNT}]} \quad (13)$$

$$G_{out} = G_m + \frac{V_{CNT}(1-\eta)(\eta_{CNT} - 2G_m\beta_{CNT})}{2[1-\mu - V_{CNT}(1-\eta) + V_{CNT}(1-\eta)\beta_{CNT}]} \quad (14)$$

$$K^* = K_{out} \left[ 1 + \frac{\mu \left( \frac{K_{in}}{K_{out}} - 1 \right)}{1 + \alpha(1-\mu) \left( \frac{K_{in}}{K_{out}} - 1 \right)} \right] \quad (15)$$

$$G^* = G_{out} \left[ 1 + \frac{\mu \left( \frac{G_{in}}{G_{out}} - 1 \right)}{1 + \beta(1-\mu) \left( \frac{G_{in}}{G_{out}} - 1 \right)} \right] \quad (16)$$

$G_m$  and  $K_m$  specify the shear and bulk moduli of the isotropic matrix, respectively, and can be calculated as,

$$G_m = \frac{E_m}{2(1+\nu_m)}, \quad K_m = \frac{E_m}{3(1-2\nu_m)} \quad (17)$$

where  $E_m$  and  $\nu_m$  represent the elastic modulus and Poisson's ratio of the isotropic matrix, respectively. In addition,  $\alpha_{CNT}$ ,  $\beta_{CNT}$ ,  $\delta_{CNT}$ , and  $\eta_{CNT}$  are defined as follows:

$$\alpha_{CNT} = \frac{3(K_m + G_m) + k_{CNT} - l_{CNT}}{3(G_m + k_{CNT})} \quad (18)$$

$$\delta_{CNT} = \frac{1}{3} \left[ n_{CNT} + 2l_{CNT} + \frac{(2k_{CNT} + l_{CNT})(3K_m + 2G_m - l_{CNT})}{G_m + k_{CNT}} \right] \quad (20)$$

$$\eta_{CNT} = \frac{1}{5} \left[ \frac{2}{3} (n_{CNT} - l_{CNT}) + \frac{8G_m p_{CNT}}{G_m + p_{CNT}} + \frac{2(k_{CNT} - l_{CNT})(2G_m + l_{CNT})}{3(G_m + k_{CNT})} + \frac{8m_{CNT}G_m(3K_m + 4G_m)}{3K_m(m_{CNT} + G_m) + G_m(7m_{CNT} + G_m)} \right] \quad (21)$$

where  $k_{CNT}$ ,  $l_{CNT}$ ,  $m_{CNT}$ ,  $n_{CNT}$ , and  $p_{CNT}$  are Hill's elastic moduli of CNTs as specified in Table 1; and

$$\alpha = \frac{(1 + \nu_{out})}{3(1 - \nu_{out})} \quad (22)$$

$$\beta = \frac{2(4 - 5\nu_{out})}{15(1 - \nu_{out})} \quad (23)$$

$$\nu_{out} = \frac{(3K_{out} - 2G_{out})}{(6K_{out} + 2G_{out})} \quad (24)$$

Finally, the effective Young's modulus ( $E^*$ ) and Poisson's ratio ( $\nu^*$ ) of the CNTRC skew plate in the bulk modulus and shear modulus are defined as:

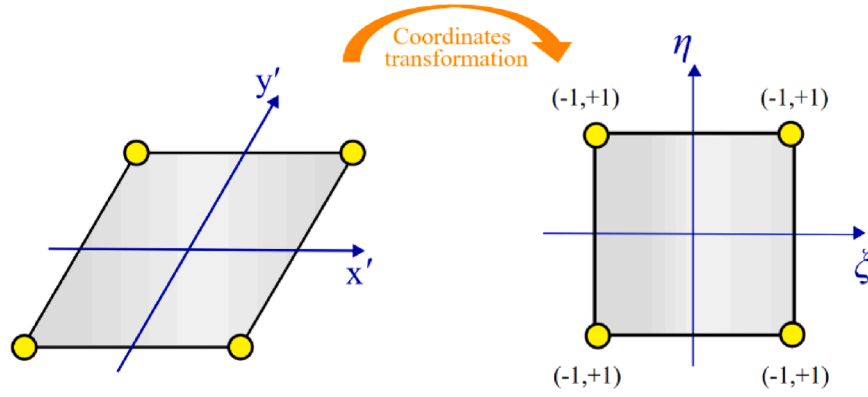
$$E^* = \frac{9K^*G^*}{3K^* + G^*} \quad (25)$$

$$\nu^* = \frac{3K^* - 2G^*}{6K^* + 2G^*} \quad (26)$$

### 3. Numerical approach

This section details the computational methodology employed. Specifically, Mindlin's first-order shear deformation theory (FSDT) was implemented within the extended finite element method (XFEM) framework to analyze in-plane equations and eigen-buckling behavior. A MATLAB code was developed to perform buckling analysis of cracked

$$\beta_{CNT} = \frac{1}{5} \left\{ \frac{4G_m + 2k_{CNT} + l_{CNT}}{3(G_m + k_{CNT})} + \frac{4G_m}{G_m + p_{CNT}} + \frac{2[G_m(3K_m + G_m) + G_m(3K_m + 7G_m)]}{G_m(3K_m + G_m) + m_{CNT}(3K_m + 7G_m)} \right\} \quad (19)$$



**Fig. 4.** An isoparametric Q4 cracked CNTRC plate element in an oblique coordinate system (real element) mapped to a local natural coordinate system (reference element). The local coordinate is a Cartesian coordinate system assumed to coincide  $(\xi, \eta, \zeta)$ .  $N_i(\xi, \eta)$  are the conventional finite element shape functions for the 4-noded elements, and  $\zeta$  is perpendicular to the mid-surface in the local coordinate system. The geometry of the shell is defined using the coordinates of the mid-surface of the shell and the normal unit vector, which is perpendicular to the surface of the shell at each node,  $V_i^3$ , where  $e_{xi}$  and  $e_{yi}$  are the orthonormal vectors of normal unit vector at node  $i$ .

skew plates composed of carbon nanotube-reinforced composite (CNTRC) material. The study investigates the influence of carbon nanotube volume fraction on the critical buckling coefficient, considering variations in crack length and location, boundary conditions, and skew angle. Furthermore, the loading conditions applied to the CNTRC cracked skew plates are presented, including three different loading models. Finally, mesh convergence and validation of the buckling results are demonstrated through comparisons with analytical and numerical solutions documented in established literature.

### 3.1. Extended finite element method

The XFEM method is based on the concept of partition of unity, which enriches the classical FEM approximation to account for singular or discontinuous fields around a crack. XFEM models the stress field singularity near crack tips and captures strong discontinuities as jumps in the displacement field across crack faces. The initial studies with XFEM method for crack modeling without remeshing were presented [62–64]. These papers presented a minimal remeshing FEM for crack growth by including the discontinuous enrichment functions to the finite element approximation to account for the presence of the cracks. In XFEM, Heaviside and singular functions are used to enrich element nodes for crack modeling. Consider a mesh of various elements and an independent crack geometry as shown in Fig. 3. In the XFEM framework, the crack is modeled by enriching the standard approximation with discontinuous and near-tip functions. This figure illustrates that nodes around the crack are enriched by the Heaviside function (jump function), while the singular functions (crack tip functions) enrich the four nodes around the crack tip. In addition, Fig. 3 depicts the representation of standard, blending, and enriched elements (tip and crack face). The Heaviside and singular enrichment functions are defined according to Eqs. (27) and (28), respectively. Eq. (28) is defined in a local polar coordinate system  $(r, \theta)$  at the crack tip [65,66]:

$$H(\mathbf{x}) = \begin{cases} +1 & \text{above the crack} \\ -1 & \text{under the crack} \end{cases} \quad (27)$$

$$B(r, \theta) = \{B_1, B_2, B_3, B_4\} \\ = \left\{ \sqrt{r} \sin\left(\frac{\theta}{2}\right), \sqrt{r} \cos\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta), \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta) \right\} \quad (28)$$

### 3.2. Displacement field and XFEM discretization

The XFEM approximation field in modeling of crack interface for an isotropic flat plate can be written by applying some simplifications on the case of general shell element. A 4-noded quadrilateral plate element, illustrated in Fig. 4, was employed to discretize the plate's geometry. The displacement field was derived from Mindlin-Risner plate theory, as shown in Eq. (29) [67].

$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_{i=1}^n N_i(\xi, \eta) \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} + \frac{t}{2} \sum_{i=1}^n \zeta N_i(\xi, \eta) \begin{Bmatrix} V_i^3 x_i \\ V_i^3 y_i \\ V_i^3 z_i \end{Bmatrix} \quad (29)$$

where,  $t$  is the thickness of the shell element;  $N_i(\xi, \eta)$  are the conventional finite element shape functions in the natural coordinate system;  $V_i^3$  is the normal unit vector at node  $i$ ; and  $\zeta$  is perpendicular to the mid-surface in the natural coordinate system, as depicted in Fig. 4. The general displacement field in XFEM for each arbitrary point is obtained according to the Eq. (30):

$$\mathbf{u} = \mathbf{u}_{fem} + \mathbf{u}_{enrich} \quad , \quad \mathbf{u} = \{u, v, w\}^T \quad (30)$$

$\mathbf{u}_{fem}$  and  $\mathbf{u}_{enrich}$  represent the standard FEM and enrichment displacement fields, respectively. In addition,  $u$ ,  $v$ , and  $w$  denote the displacement components along the  $x$ ,  $y$ , and  $z$ .  $\mathbf{u}_{fem}$  can be expressed as:

$$\mathbf{u}_{fem} = \sum_{i=1}^n N_i \mathbf{u}_i - \frac{t}{2} \sum_{i=1}^n N_i \zeta \mathbf{e}_{xi} \alpha_i + \frac{t}{2} \sum_{i=1}^n N_i \zeta \mathbf{e}_{yi} \beta_i \quad (31)$$

In Eq. (31),  $\mathbf{e}_{xi}$  and  $\mathbf{e}_{yi}$  represent the orthonormal unit vectors at node  $i$ , as illustrated in Fig. 4. The parameters  $\alpha_i$  and  $\beta_i$  denote rotations about the  $x$  and  $y$  directions, respectively. The displacement field  $\mathbf{u}_{enrich}$  comprises two components:  $\mathbf{u}_{crack-face}$  and  $\mathbf{u}_{crack-tip}$ , which are enriched by Heaviside and singular functions, respectively, as outlined in Eqs. (32–34) [65,66].

$$\mathbf{u}_{enrich} = \mathbf{u}_{crack-face} + \mathbf{u}_{crack-tip} \quad (32)$$

$$\mathbf{u}_{crack-face} = \sum_{i=1}^n N_i H(\mathbf{x}) \mathbf{a}_i + \frac{t}{2} \sum_{i=1}^n \zeta N_i H(\mathbf{x}) [-\mathbf{e}_{xi} \alpha_i + \mathbf{e}_{yi} \beta_i] \quad (33)$$

$$\mathbf{u}_{crack-tip} = \sum_{i=1}^{nt} N_i \sum_{j=1}^m (F_j(\mathbf{x}) - F_j(\mathbf{x}_i)) \begin{Bmatrix} b_i^u \\ b_i^v \\ 0 \end{Bmatrix} + \sum_{i=1}^{nt} N_i \sum_{j=1}^m (G_j(\mathbf{x}) - G_j(\mathbf{x}_i)) \begin{Bmatrix} 0 \\ 0 \\ b_i^w \end{Bmatrix} + \sum_{i=1}^{nt} N_i \sum_{j=1}^m (R_j(\mathbf{x}) - R_j(\mathbf{x}_i)) [-\mathbf{e}_{xi} b_{ai}^j + \mathbf{e}_{yi} b_{\beta i}^j] \quad (34)$$

In the above equation,  $m$  is the number of crack tip enrichment functions;  $nt$  is the number of nodes enriched by the crack tip functions; and  $b_{ai}^j$  and  $b_{\beta i}^j$  are the additional enrichment degrees of freedom. The vector of nodal displacements and enrichments are defined as Eqs. (35–37):

$$\underline{\mathbf{d}} = \{u, v, w, \alpha, \beta\}^T \quad (35)$$

$$\mathbf{h} = \{a_i^u, a_i^v, a_i^w, a_{ai}, a_{\beta i}\}^T \quad (36)$$

$$\mathbf{t} = \{b_i^u, b_i^v, b_i^w, b_{ai}, b_{\beta i}\}^T \quad (37)$$

The overall nodal displacement vector is defined as follows:

$$\mathbf{D} = \left\{ \underline{\mathbf{d}} \quad \mathbf{h} \quad \mathbf{t} \right\}^T \quad (38)$$

where,  $F$ ,  $G$ , and  $R$  are singular enrichment functions corresponding to in-plane, out-of-plane, and rotational degrees of freedom (tip enrichment functions), respectively, which in the local polar coordinate system  $(r, \theta)$  are expressed as [65,66],

$$F(r, \theta) = \left\{ \sqrt{r} \sin\left(\frac{\theta}{2}\right), \sqrt{r} \cos\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta), \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta) \right\} \quad (39)$$

$$G(r, \theta) = \left\{ \sqrt{r} \sin\left(\frac{\theta}{2}\right), r^{3/2} \sin\left(\frac{\theta}{2}\right), r^{3/2} \cos\left(\frac{\theta}{2}\right), r^{3/2} \sin\left(\frac{3\theta}{2}\right), r^{3/2} \cos\left(\frac{3\theta}{2}\right) \right\} \quad (40)$$

$$R(r, \theta) = \left\{ \sqrt{r} \sin\left(\frac{\theta}{2}\right), \sqrt{r} \cos\left(\frac{\theta}{2}\right), \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta), \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta) \right\} \quad (41)$$

### 3.3. Energy method

The energy method is used to obtain stiffness and geometric matrices. The total potential energy of the whole system ( $\Pi$ ) is defined according to Eq. (42):

$$\Pi = \frac{1}{2} \int_V \boldsymbol{\varepsilon}_L^T \mathbf{C} \boldsymbol{\varepsilon}_L dV + \frac{1}{2} \int_V \boldsymbol{\varepsilon}_{NL}^T \boldsymbol{\sigma}_0 dV \quad (42)$$

where,  $\boldsymbol{\varepsilon}_L$  and  $\boldsymbol{\varepsilon}_{NL}$  are the linear and nonlinear parts of Green-Lagrange strain vectors, respectively, and from their sum, the total strain as given in Eq. (43);  $\mathbf{C}$  denotes the material properties matrix [67];  $\boldsymbol{\sigma}_0$  is the stress state computed from the linear static analysis,

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_L + \boldsymbol{\varepsilon}_{NL} = \left\{ \varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{xz}, \gamma_{yz} \right\}^T \quad (43)$$

and

$$\mathbf{C} = \frac{E^*}{(1 - \nu^{*2})} \begin{bmatrix} 1 & \nu^* & 0 & 0 & 0 & 0 \\ \nu^* & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1 - \nu^*)/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa(1 - \nu^*)/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa(1 - \nu^*)/2 \end{bmatrix} \quad (44)$$

in which,  $E^*$  and  $\nu^*$  represent the modulus of elasticity and Poisson's ratio, respectively, while  $\kappa$ , the shear coefficient and the second part of Eq. (42) can be expressed as Eq. (45):

$$\frac{1}{2} \int_V \boldsymbol{\varepsilon}_{NL}^T \boldsymbol{\sigma}_0 dV = \frac{1}{2} \int_V \boldsymbol{\mu}^T \mathbf{S}_0 \boldsymbol{\mu} dV \quad (45)$$

where  $\boldsymbol{\sigma}_0$  is the stress vector obtained from in-plane linear static analysis and is expressed as follow:

$$\boldsymbol{\sigma}_0 = \{\sigma_{x0}, \sigma_{y0}, \sigma_{z0}, \tau_{xy0}, \tau_{yz0}, \tau_{xz0}\}^T \quad (46)$$

and

$$\boldsymbol{\mu} = \{\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z, \mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z, \mathbf{w}_x, \mathbf{w}_y, \mathbf{w}_z\}^T \quad (47)$$

In Eq. (46),  $\mathbf{S}_0$  is computed from the stresses obtained from the static analysis inside the plane and is expressed according to Eqs. (48), (49) [67]:

$$\mathbf{S}_0 = \begin{bmatrix} \bar{\mathbf{S}}_0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{S}}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{S}}_0 \end{bmatrix} \quad (48)$$

$$\bar{\mathbf{S}}_0 = \begin{bmatrix} \sigma_{x0} & \tau_{xy0} & \tau_{xz0} \\ \tau_{xy0} & \sigma_{y0} & \tau_{yz0} \\ \tau_{xz0} & \tau_{yz0} & \sigma_{z0} \end{bmatrix} \quad (49)$$

The vectors  $\boldsymbol{\varepsilon}_L$  and  $\boldsymbol{\mu}$  are associated with the nodal displacement vector ( $\mathbf{D}$ ) and are stated as follows:

$$\boldsymbol{\varepsilon}_L = \mathbf{B} \mathbf{D} \quad , \quad \boldsymbol{\mu} = \mathbf{B}_G \mathbf{D} \quad (50)$$

The total potential energy is found by substituting Eqs. (46) and (50) into Eq. (42),

$$\Pi = \frac{1}{2} \int_V \mathbf{D}^T \mathbf{B}^T \mathbf{C} \mathbf{B} \mathbf{D} dV + \frac{1}{2} \int_V \mathbf{D}^T \mathbf{B}_G^T \mathbf{S}_0 \mathbf{B}_G \mathbf{D} dV \quad (51)$$

To calculate the critical buckling load value by zeroing the total potential energy variations and, after manipulation, is derived the stability equation as given in Eq. (53):

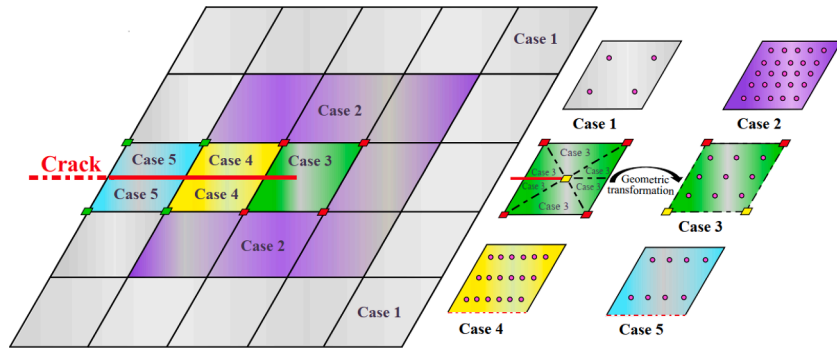
$$\delta \Pi = \delta \mathbf{D}^T \left\{ \int_V \mathbf{B}^T \mathbf{C} \mathbf{B} dV + \int_V \mathbf{B}_G^T \mathbf{S}_0 \mathbf{B}_G dV \right\} \mathbf{D} = 0 \quad (52)$$

$$(\mathbf{K}_S + \lambda \mathbf{K}_G) \{\mathbf{D}\} = \mathbf{0} \quad (53)$$

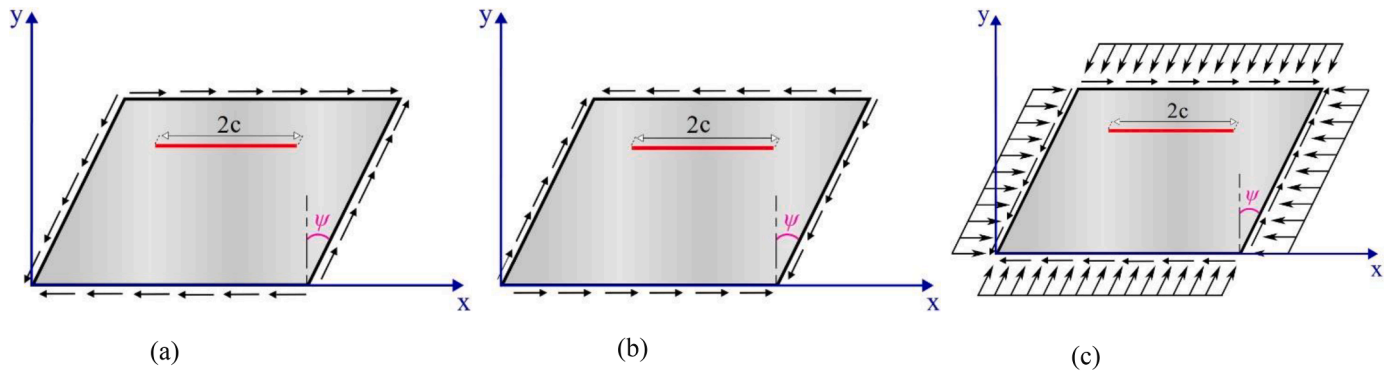
Eq. (53) is an eigenvalue equation where  $\lambda$  denotes the eigenvalues and  $\mathbf{D}$  represents the eigenvectors. The standard stiffness matrix ( $\mathbf{K}_S$ ) and the geometric stiffness matrix ( $\mathbf{K}_G$ ) are defined as follows:

$$\mathbf{K}_S = \int_V \mathbf{B}_S^T \mathbf{C} \mathbf{B}_S dV \quad (54)$$

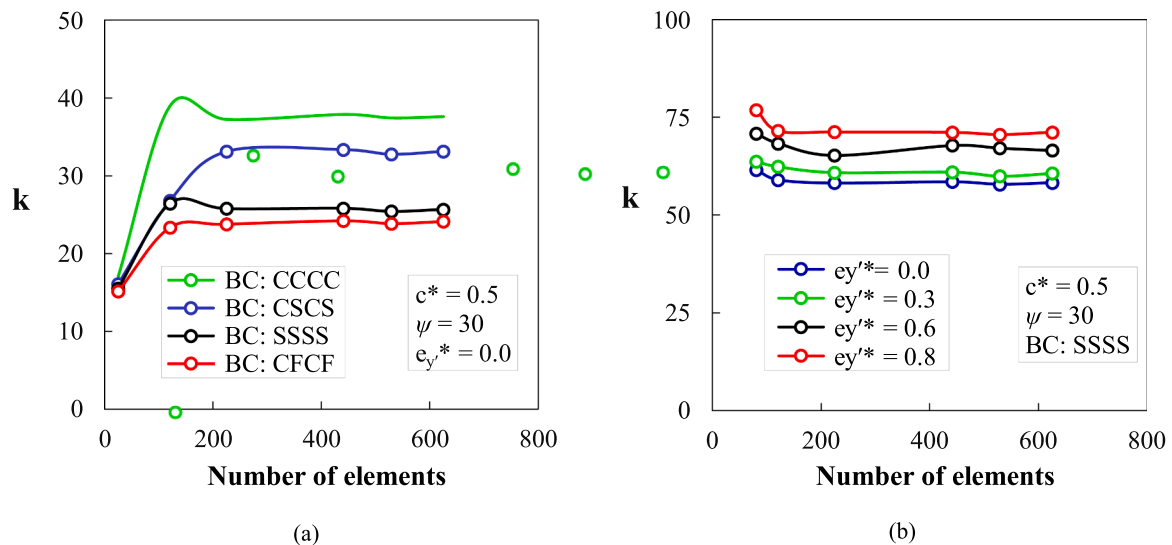
$$\mathbf{K}_G = \int_V \mathbf{B}_G^T \mathbf{S}_0 \mathbf{B}_G dV \quad (55)$$



**Fig. 5.** Positions of Gauss points in different element types. The standard elements (case 1) used  $2 \times 2$  Gauss integration. The elements with crack tip enrichments (case 2) used more sub-domains and more Gauss points due to the singular nature of stress fields in the vicinity of the crack tip; accordingly, these elements were divided into six triangular sub-domains, and each one was mapped to a quadrilateral domain with  $3 \times 3$  Gauss points (Fig. 6a). The elements cut by a crack and enriched by the Heaviside function (case 3) were divided into two quadrilateral sub-domains, and for each one,  $2 \times 4$  Gauss points were used (Fig. 6b). The elements not cut by a crack but partially enriched by crack tip enrichments (case 4) used  $5 \times 5$  Gauss points. Finally, elements cut by a crack and partially enriched by crack tip enrichments (case 5) were divided into two quadrilateral sub-domains with  $3 \times 6$  Gauss points per sub-domain.



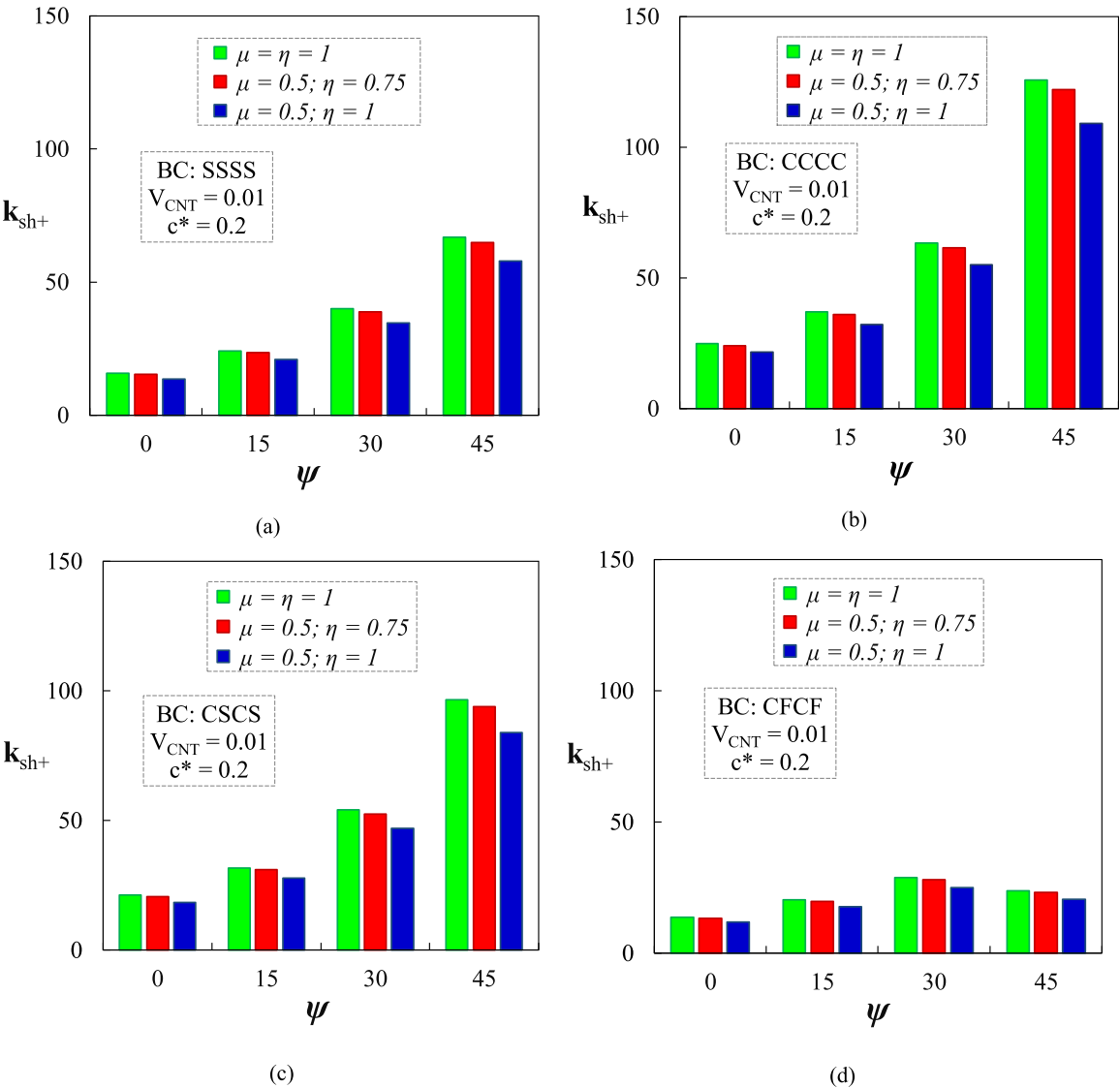
**Fig. 6.** Schematic of the cracked CNTRC skew plate with skew angle  $\psi$  under different in-plane load schemes: (a) positive shear load, (b) negative shear load, (c) Combination of positive shear and compressive load.



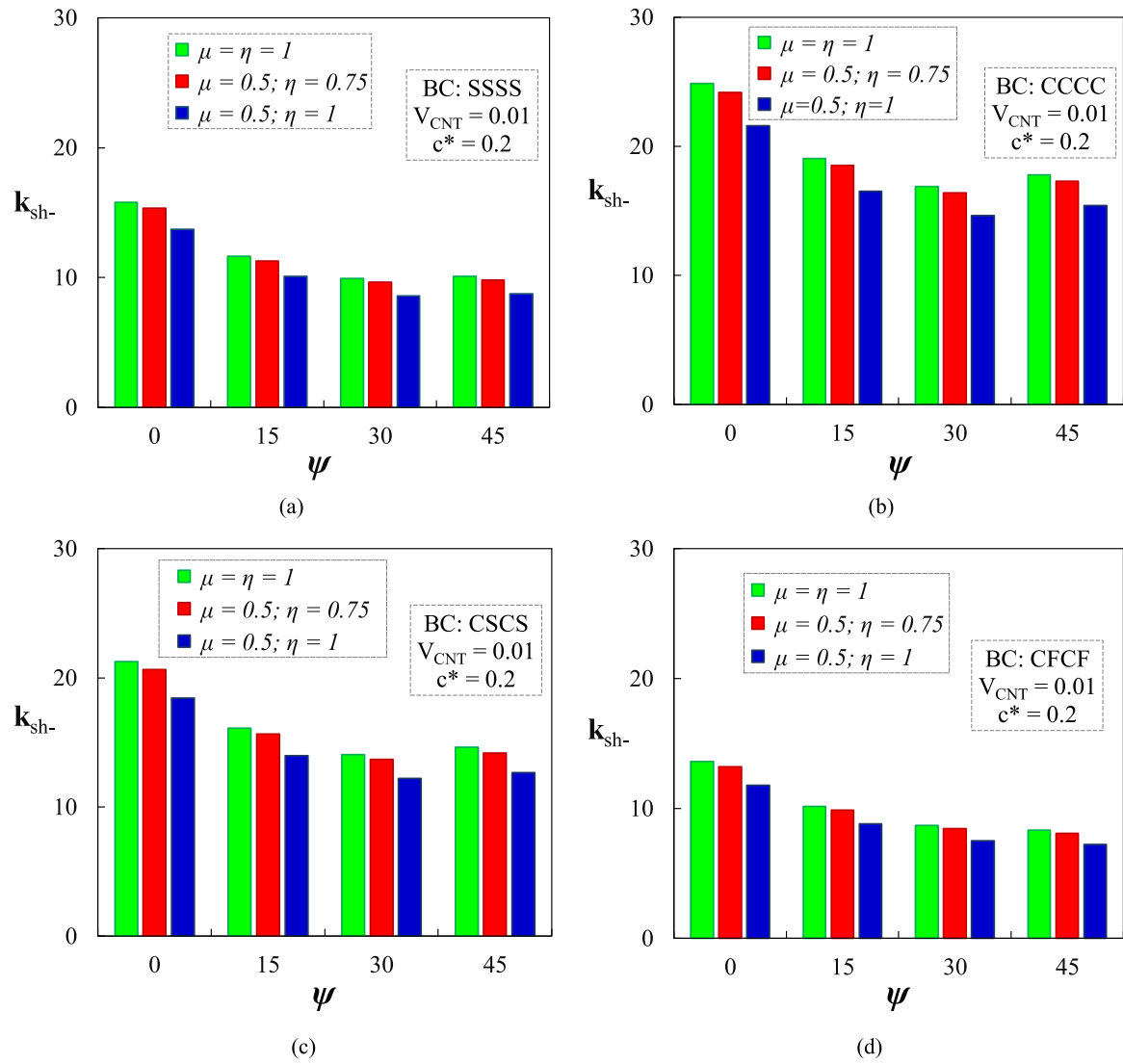
**Fig. 7.** Mesh sensitivity analysis of CNTRC skew plates with: (a) center crack; (b) off-center crack.

**Table 2**  
Comparison of buckling coefficient validation results for uncracked skew plates under shear loading.

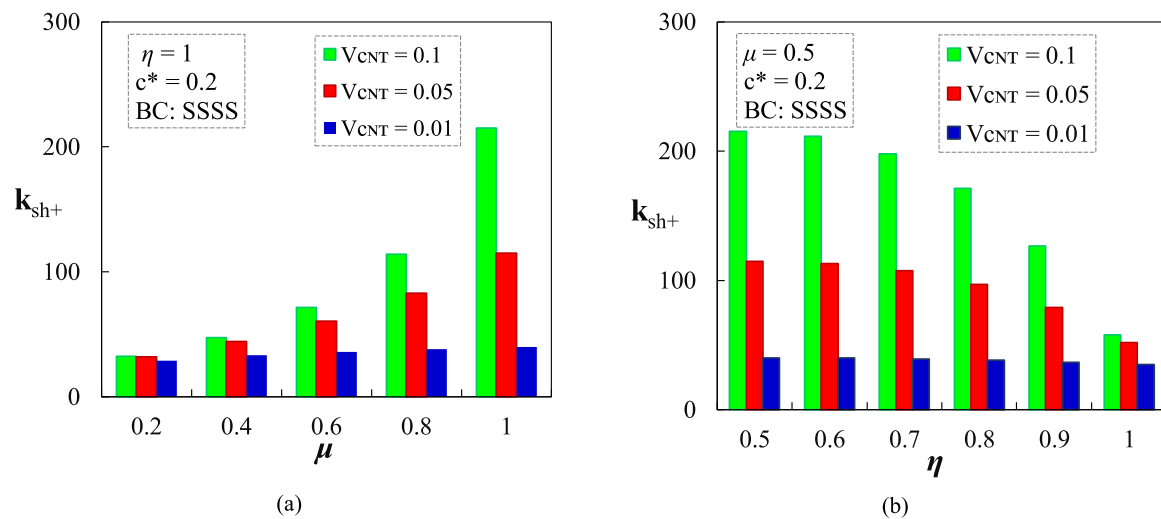
| BC   | $\psi$ | Ref. [12] | Ref. [13] | Present study | % Difference (Ref. [12]) | % Difference (Ref. [13]) |
|------|--------|-----------|-----------|---------------|--------------------------|--------------------------|
| SSSS | 0      | 9.32      | 9.32      | 9.46          | 1.50                     | 1.50                     |
|      | 15     | 7.10      | 7.07      | 6.96          | -1.97                    | -1.56                    |
|      | 30     | 6.64      | 6.61      | 6.05          | -8.89                    | -8.47                    |
|      | 45     | 7.97      | 7.92      | 6.56          | -17.69                   | -17.17                   |
| CCCC | 0      | 14.64     | 14.64     | 14.77         | 0.89                     | 0.89                     |
|      | 15     | 11.40     | 11.40     | 11.24         | -1.40                    | -1.40                    |
|      | 30     | 10.89     | 10.88     | 10.20         | -6.34                    | -6.25                    |
|      | 45     | 13.07     | 13.08     | 11.07         | -15.30                   | -15.37                   |
| CSCS | 0      | 12.55     | 12.56     | 12.86         | 2.47                     | 2.39                     |
|      | 15     | 9.76      | 9.75      | 9.71          | -0.51                    | -0.41                    |
|      | 30     | 9.31      | 9.30      | 8.59          | -7.73                    | -7.63                    |
|      | 45     | 11.25     | 11.18     | 9.23          | -17.96                   | -17.44                   |
| CFCF | 0      | 7.46      | 7.48      | 7.63          | 2.28                     | 2.01                     |
|      | 15     | 5.67      | 5.68      | 5.70          | 0.53                     | 0.35                     |
|      | 30     | 5.13      | 5.13      | 4.90          | -4.48                    | -4.48                    |
|      | 45     | 5.68      | 5.64      | 4.78          | -15.85                   | -15.25                   |



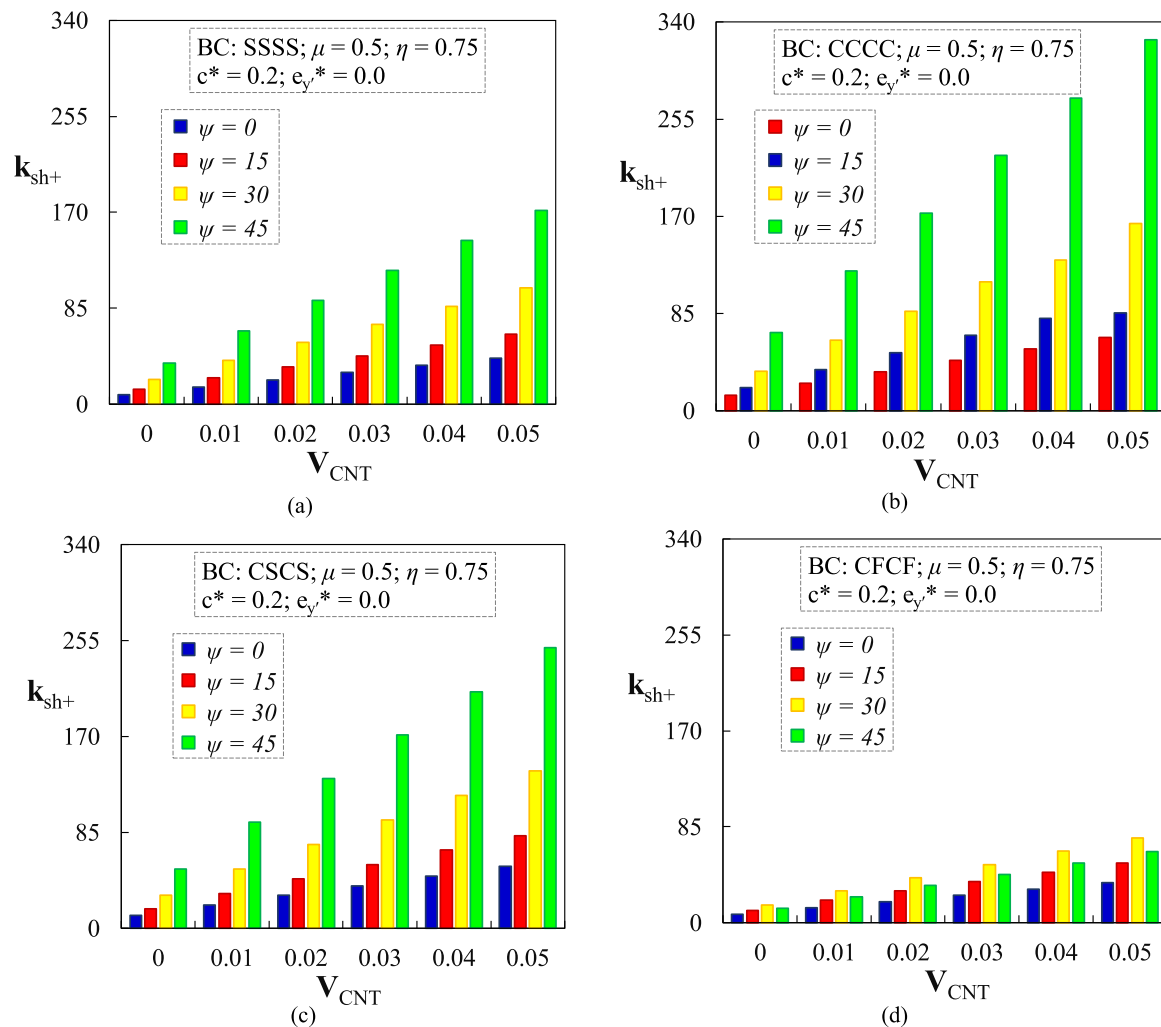
**Fig. 8.** Effect of CNTs agglomeration parameter on critical buckling coefficient for centrally cracked CNTRC skew plates under positive shear load: (a) BC: SSSS; (b) BC: CCCC; (c) BC: CSCS; (d) BC: CFCF.



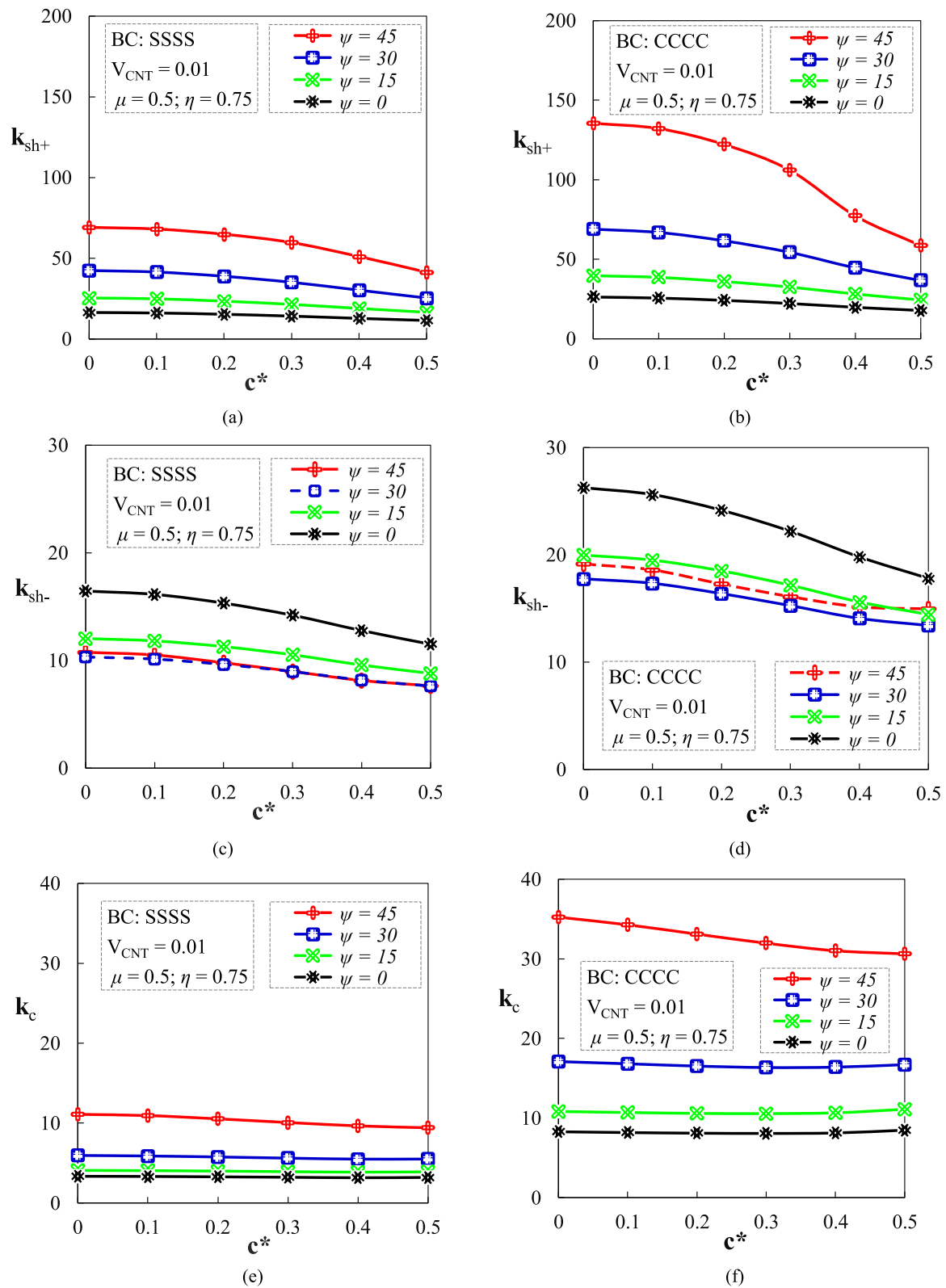
**Fig. 9.** Effect of CNTs agglomeration parameter on critical buckling coefficient for centrally cracked CNTRC skew plates under negative shear load: (a) BC: SSSS; (b) BC: CCCC; (c) BC: CSCS; (d) BC: CFCF.



**Fig. 10.** Effect of agglomeration parameters (a)  $\mu$ , and (b)  $\eta$  on critical buckling coefficient of CNTRC skew plates with center crack under positive shear load.



**Fig. 11.** Effect of the CNT volume fraction on critical buckling coefficient for CNTRC skew plates with center crack under positive shear loading: (a) BC:SSSS; (b) BC:CCCC; (c) BC:CSCS; (d) BC:CFCF.



**Fig. 12.** Critical buckling coefficient vs. relative crack length for centrally cracked CNTRC skew plates with: (a) positive shear load, BC: SSSS; (b) positive shear load, BC: CCCC; (c) negative shear load, BC: SSSS; (d) negative shear load, BC: CCCC; (e) combined load, BC: SSSS; (f) combined load, BC: CCCC.

where,  $B_S$  and  $B_G$  are as below:

$$B_{S_i} = \begin{bmatrix} B_{S_i}^d & B_{S_i}^h & B_{S_i}^t \end{bmatrix} \quad (56)$$

$$B_{G_i} = \begin{bmatrix} B_{G_i}^d & B_{G_i}^h & B_{G_i}^t \end{bmatrix} \quad (57)$$

where,  $B_S^t$  and  $B_G^t$  can be obtained for plate elements [67]

$$B_{S_i}^t = \begin{bmatrix} \frac{\partial \phi}{\partial x} & 0 & 0 & \frac{\partial}{\partial x} \left( -\frac{\zeta t}{2} \mathbf{e}_{x1} \varphi \right) & \frac{\partial}{\partial x} \left( \frac{\zeta t}{2} \mathbf{e}_{y1} \varphi \right) \\ 0 & \frac{\partial \phi}{\partial y} & 0 & \frac{\partial}{\partial y} \left( -\frac{\zeta t}{2} \mathbf{e}_{x2} \varphi \right) & \frac{\partial}{\partial y} \left( \frac{\zeta t}{2} \mathbf{e}_{y2} \varphi \right) \\ 0 & 0 & \frac{\partial \psi}{\partial z} & \frac{\partial}{\partial z} \left( -\frac{\zeta t}{2} \mathbf{e}_{x3} \varphi \right) & \frac{\partial}{\partial z} \left( \frac{\zeta t}{2} \mathbf{e}_{y3} \varphi \right) \\ \frac{\partial \phi}{\partial y} \frac{\partial \phi}{\partial x} & 0 & \frac{\partial}{\partial y} \left( -\frac{\zeta t}{2} \mathbf{e}_{x1} \varphi \right) + \frac{\partial}{\partial x} \left( -\frac{\zeta t}{2} \mathbf{e}_{x2} \varphi \right) & \frac{\partial}{\partial y} \left( \frac{\zeta t}{2} \mathbf{e}_{y1} \varphi \right) + \frac{\partial}{\partial x} \left( \frac{\zeta t}{2} \mathbf{e}_{y2} \varphi \right) \\ 0 & \frac{\partial \phi}{\partial z} \frac{\partial \psi}{\partial y} & \frac{\partial}{\partial z} \left( -\frac{\zeta t}{2} \mathbf{e}_{x2} \varphi \right) + \frac{\partial}{\partial y} \left( -\frac{\zeta t}{2} \mathbf{e}_{x3} \varphi \right) & \frac{\partial}{\partial z} \left( \frac{\zeta t}{2} \mathbf{e}_{y2} \varphi \right) + \frac{\partial}{\partial y} \left( \frac{\zeta t}{2} \mathbf{e}_{y3} \varphi \right) \\ \frac{\partial \phi}{\partial z} & 0 & \frac{\partial \psi}{\partial x} \frac{\partial}{\partial z} \left( -\frac{\zeta t}{2} \mathbf{e}_{x1} \varphi \right) + \frac{\partial}{\partial x} \left( -\frac{\zeta t}{2} \mathbf{e}_{x3} \varphi \right) & \frac{\partial}{\partial z} \left( \frac{\zeta t}{2} \mathbf{e}_{y1} \varphi \right) + \frac{\partial}{\partial x} \left( \frac{\zeta t}{2} \mathbf{e}_{y3} \varphi \right) \end{bmatrix} \quad (58)$$

$$B_{G_i}^t = \begin{bmatrix} \frac{\partial \phi}{\partial x} & 0 & 0 & \frac{\partial}{\partial x} \left( -\frac{\zeta t}{2} \mathbf{e}_{x1} \varphi \right) & \frac{\partial}{\partial x} \left( \frac{\zeta t}{2} \mathbf{e}_{y1} \varphi \right) \\ \frac{\partial \phi}{\partial y} & 0 & 0 & \frac{\partial}{\partial y} \left( -\frac{\zeta t}{2} \mathbf{e}_{x1} \varphi \right) & \frac{\partial}{\partial y} \left( \frac{\zeta t}{2} \mathbf{e}_{y1} \varphi \right) \\ \frac{\partial \phi}{\partial z} & 0 & 0 & \frac{\partial}{\partial z} \left( -\frac{\zeta t}{2} \mathbf{e}_{x1} \varphi \right) & \frac{\partial}{\partial z} \left( \frac{\zeta t}{2} \mathbf{e}_{y1} \varphi \right) \\ 0 & \frac{\partial \phi}{\partial x} & 0 & \frac{\partial}{\partial x} \left( -\frac{\zeta t}{2} \mathbf{e}_{x2} \varphi \right) & \frac{\partial}{\partial x} \left( \frac{\zeta t}{2} \mathbf{e}_{y2} \varphi \right) \\ 0 & \frac{\partial \phi}{\partial y} & 0 & \frac{\partial}{\partial y} \left( -\frac{\zeta t}{2} \mathbf{e}_{x2} \varphi \right) & \frac{\partial}{\partial y} \left( \frac{\zeta t}{2} \mathbf{e}_{y2} \varphi \right) \\ 0 & \frac{\partial \phi}{\partial z} & 0 & \frac{\partial}{\partial z} \left( -\frac{\zeta t}{2} \mathbf{e}_{x2} \varphi \right) & \frac{\partial}{\partial z} \left( \frac{\zeta t}{2} \mathbf{e}_{y2} \varphi \right) \\ 0 & 0 & \frac{\partial \psi}{\partial x} & \frac{\partial}{\partial x} \left( -\frac{\zeta t}{2} \mathbf{e}_{x3} \varphi \right) & \frac{\partial}{\partial x} \left( \frac{\zeta t}{2} \mathbf{e}_{y3} \varphi \right) \\ 0 & 0 & \frac{\partial \psi}{\partial y} & \frac{\partial}{\partial y} \left( -\frac{\zeta t}{2} \mathbf{e}_{x3} \varphi \right) & \frac{\partial}{\partial y} \left( \frac{\zeta t}{2} \mathbf{e}_{y3} \varphi \right) \\ 0 & 0 & \frac{\partial \psi}{\partial z} & \frac{\partial}{\partial z} \left( -\frac{\zeta t}{2} \mathbf{e}_{x3} \varphi \right) & \frac{\partial}{\partial z} \left( \frac{\zeta t}{2} \mathbf{e}_{y3} \varphi \right) \end{bmatrix} \quad (59)$$

where  $\phi$ ,  $\varphi$  and  $\chi$  are defined as follows:

$$\phi = N_i(F_j(\mathbf{x}) - F_j(\mathbf{x}_i)) \quad (60)$$

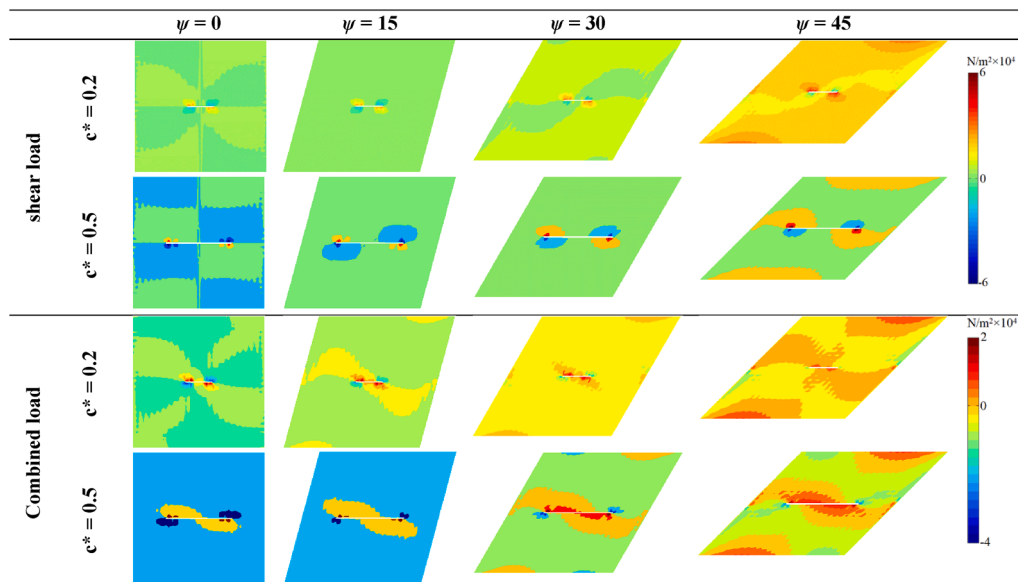
$$\varphi = N_i(R_j(\mathbf{x}) - R_j(\mathbf{x}_i)) \quad (61)$$

$$\chi = N_i(G_j(\mathbf{x}) - G_j(\mathbf{x}_i)) \quad (62)$$

By making some simplifications,  $B_S^d$ ,  $B_S^h$ ,  $B_G^d$  and  $B_G^h$  can be obtained for plate elements [67], such that in Eqs. (58) and (59) if:  $t \rightarrow d \Rightarrow \phi = \varphi = \chi = N_i$  and if:  $t \rightarrow h \Rightarrow \phi = \varphi = \chi = N_i(H(x) - H(x_i))$ .

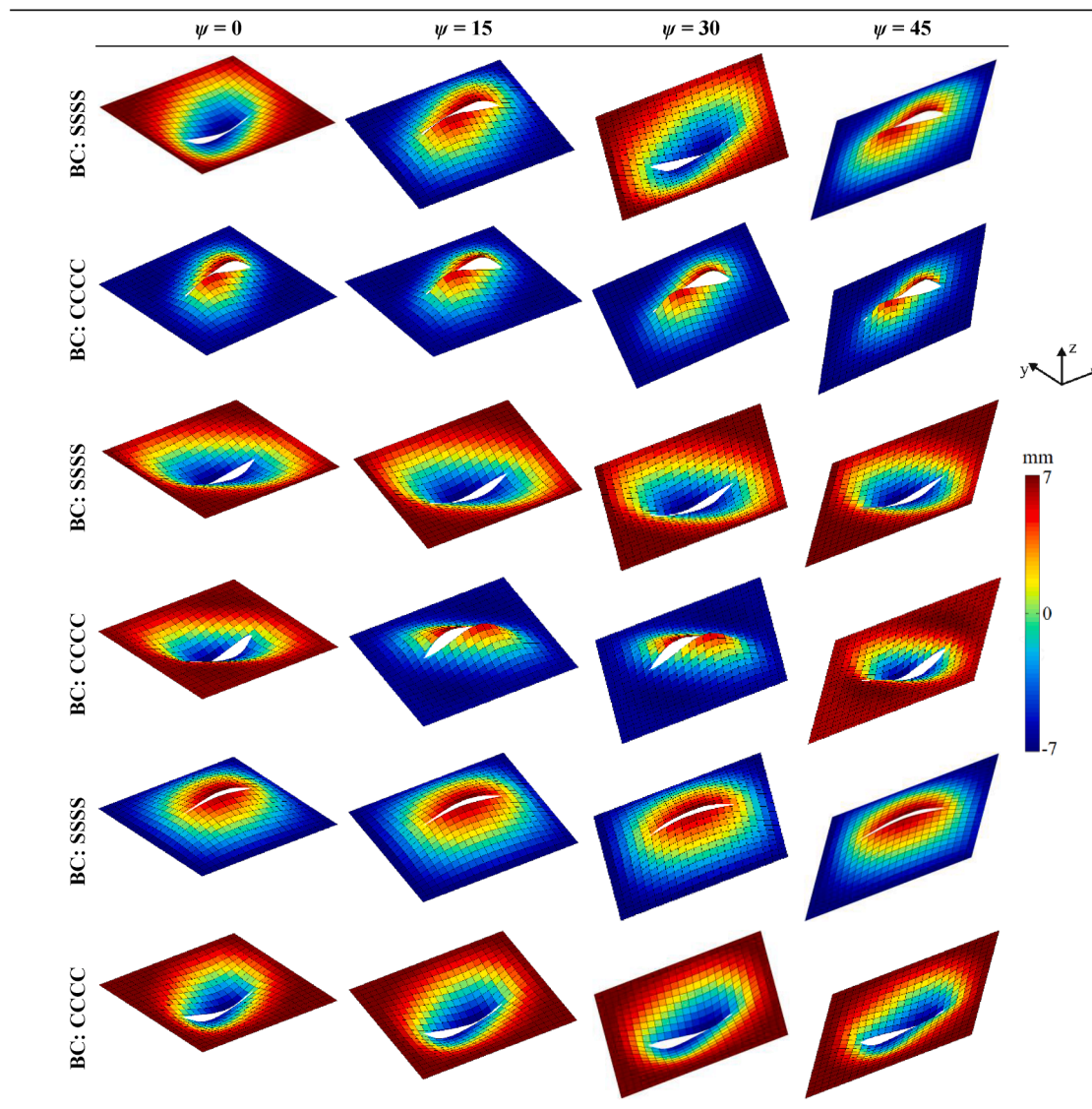
**Table 3**

Stress field ( $\sigma_{xx}$ ) for centrally cracked CNTRC skew plates under varying plate skew angles, loading conditions, and crack lengths, with parameters;  $\mu = 0.5$ ;  $\eta = 0.75$ ;  $V_{CNT} = 0.01$ .



**Table 4**

Critical buckling mode shapes for centrally cracked CNTRC skew plates considering varying plate skew angles, load and boundary conditions, with parameters:  $c^* = 0.5$ ;  $\mu = 0.5$ ;  $\eta = 0.75$ ;  $V_{CNT} = 0.01$ .

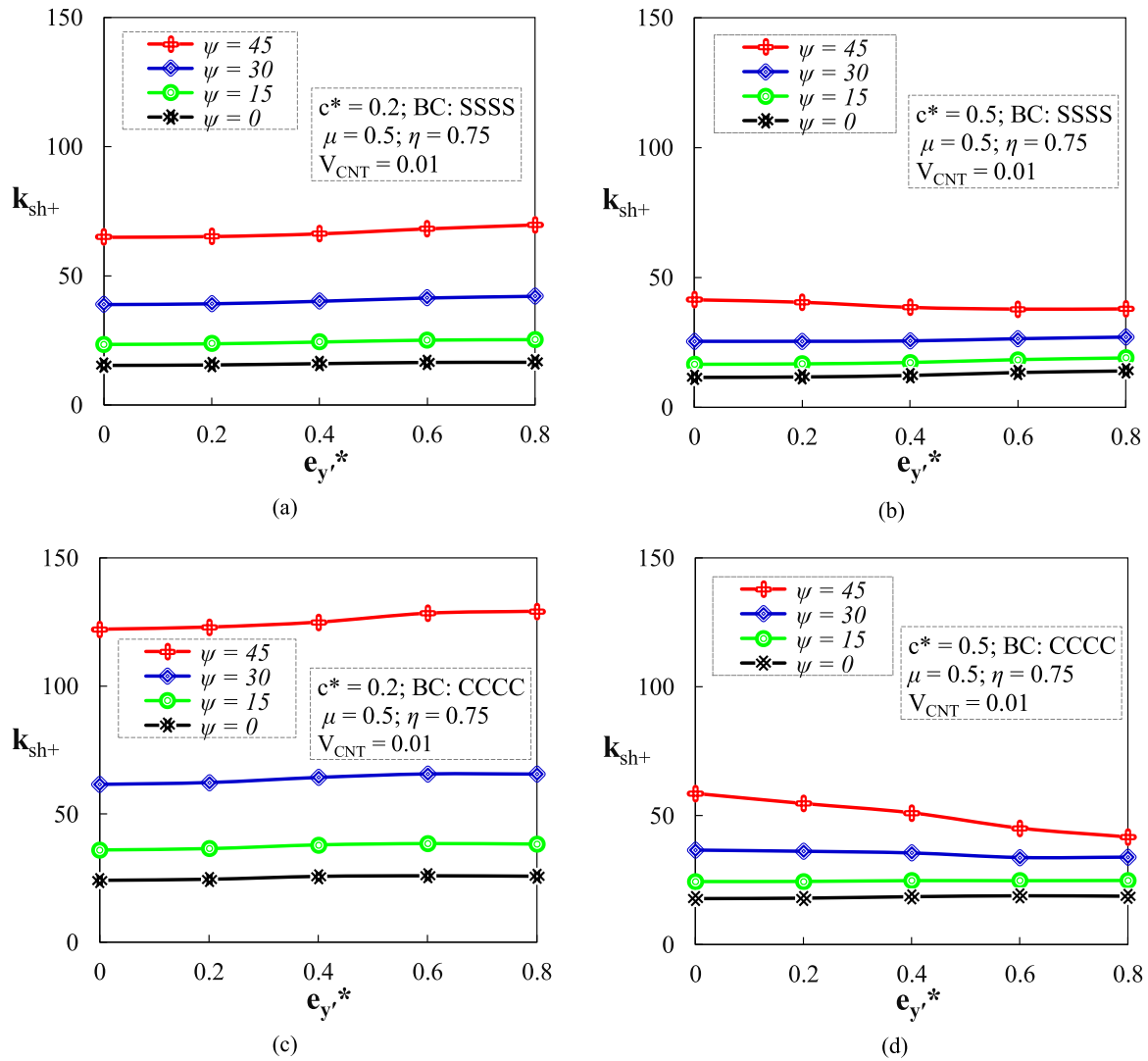


### 3.4. Numerical integration

The XFEM experiences singularities in the stiffness matrix's integrand, which involves a mix of  $\{1/r, 1/\sqrt{r}, 1, \sqrt{r}, r\}$  with harmonic functions and regular shape functions. By the XFEM, the singularity occurs in the integrand of related terms, in the stiffness matrix. The radial components are singular, complicating integration. To address this, crack tip elements are divided into triangular sub-domains, ensuring continuous integrands, with singular points at vertices. Béchet et al. [68] noted slow convergence with standard Gauss integration, necessitating improved quadrature schemes. Mapping the actual triangular interpolation to a reference cell transforms singular functions into regular ones, enhancing integration speed and convergence. They found that using nine integration points suffices for the XFEM stiffness matrix. Fig. 5 shows the Gauss points in different types of elements and the geometric transformation of the triangular sub-domain existing in this study. For further details, refer to [65,68].

### 3.5. Boundary conditions and loading scheme

To assess the buckling response of cracked CNTRC skew plates, four types of boundary conditions are considered: SSSS and CCCC, representing simply supported and clamped edges, respectively; CSCS, with two clamped and two simply supported edges; and CFCF, where two edges are free. Furthermore, three loading schemes are examined for the skew plates as shown in Fig. 6: positive shear load (Fig. 6a), negative shear load (Fig. 6b), and a combination of positive shear and compressive loads (Fig. 6c). It is important to note that for each loading conditions, the cracked CNTRC skew plate remains in a state of physical resistance, and under fixed edge load, the stress component values vary with the increase in the plane deflection angle. After calculating the geometric matrices and the stiffness of the plate and applying the boundary conditions, the critical buckling load is obtained from Eq. (53). The critical buckling load ( $\lambda_{cr}$ ) is the smallest value of the eigenvalues. The dimensionless critical buckling coefficient ( $k$ ) used in this study is obtained from Eq. (63),



**Fig. 13.** Critical buckling coefficient vs.  $y'$ -eccentricity ratio for cracked CNTRC skew plates under positive shear load: (a)  $c^* = 0.2$ , BC: SSSS; (b)  $c^* = 0.5$ , BC: SSSS; (c)  $c^* = 0.2$ , BC: CCCC; (d)  $c^* = 0.5$ , BC: CCCC.

$$k = 12(1 - \nu_m^2)\lambda_{cr}(2a)^2 / (\pi^2 E_m t^3) \quad (63)$$

### 3.6. Mesh convergence study

To study the convergence of the numerical results, several simulations were performed for uncracked and cracked CNTRC skew plate models with  $c^* = 0.5$  towards the buckling coefficient parameter. As can be seen in the graphs of Fig. 7, the values obtained for the buckling coefficient using a  $23 \times 23$  structural mesh converge well for all cases, and therefore this structural mesh was used for all cases.

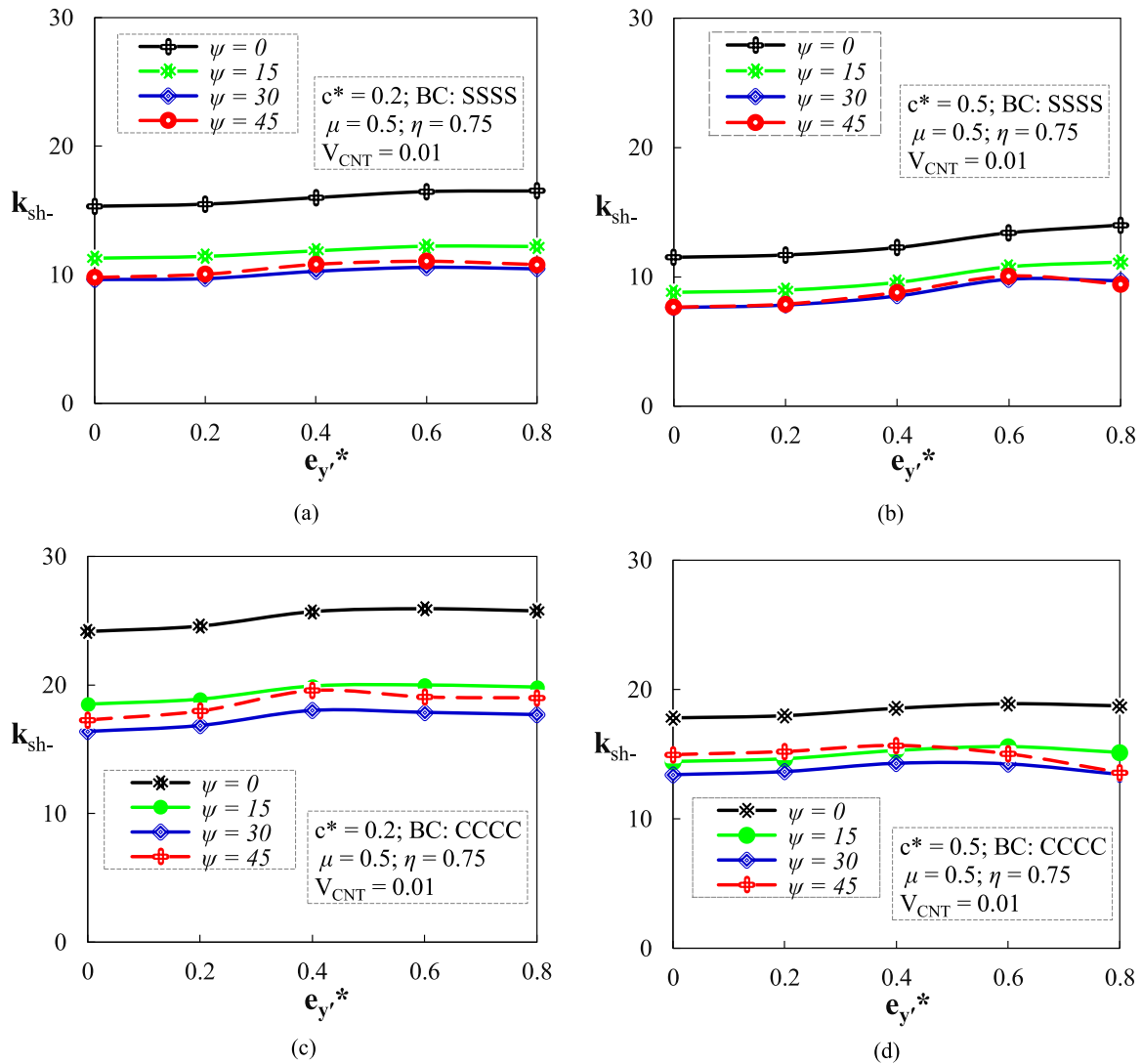
### 3.7. Model verification

To rigorously verify the capabilities and accuracy of the present approach, the study's results have been meticulously compared with reliable sources in the literature [12,13]. For instance, Table 2 meticulously compares the buckling coefficients of uncracked skew plates under various support conditions, clearly demonstrating the high accuracy of this study's numerical predictions. However, calculating buckling loads for a plate skew angle of  $45^\circ$  is challenging due to the bending moment singularity at corners. Furthermore, corner slope conditions significantly influence buckling at larger skew angles [12–14]. Nevertheless, plate bending finite elements, enhanced with

transformations, simplify the application of boundary conditions normal and tangential to the edges [17]. To further enhance the credibility of our findings, the buckling coefficients for cracked plates were also compared with the authors' previous experimental work [28]. This comparison revealed an acceptable level of accuracy, which unequivocally demonstrates the model's ability to predict the buckling behavior of skew plates in more complex scenarios, including those with cracks or subjected to combined loading conditions. This dual approach of comparing with both established theoretical benchmarks and prior experimental results provides comprehensive and robust validation for the proposed method.

## 4. Results and discussions

In this section, the linear buckling analysis of the eigenvalue of cracked skew plates developed in MATLAB program is performed. The effect of various parameters, such as volume fraction and agglomeration of CNT, crack length and location, boundary conditions, and plate skew angle under different loadings are investigated according to the critical buckling coefficient and the mode shape of CNTRC skew plates. The findings are organized into two sections: one for skew plates with center cracks and another for those with off-center cracks.



**Fig. 14.** Critical buckling coefficient vs.  $y'$ -eccentricity ratio for cracked CNTRC skew plates under negative shear load: (a)  $c^* = 0.2$ , BC: SSSS; (b)  $c^* = 0.5$ , BC: SSSS; (c)  $c^* = 0.2$ , BC: CCCC; (d)  $c^* = 0.5$ , BC: CCCC.

#### 4.1. CNTRC skew plates with center crack

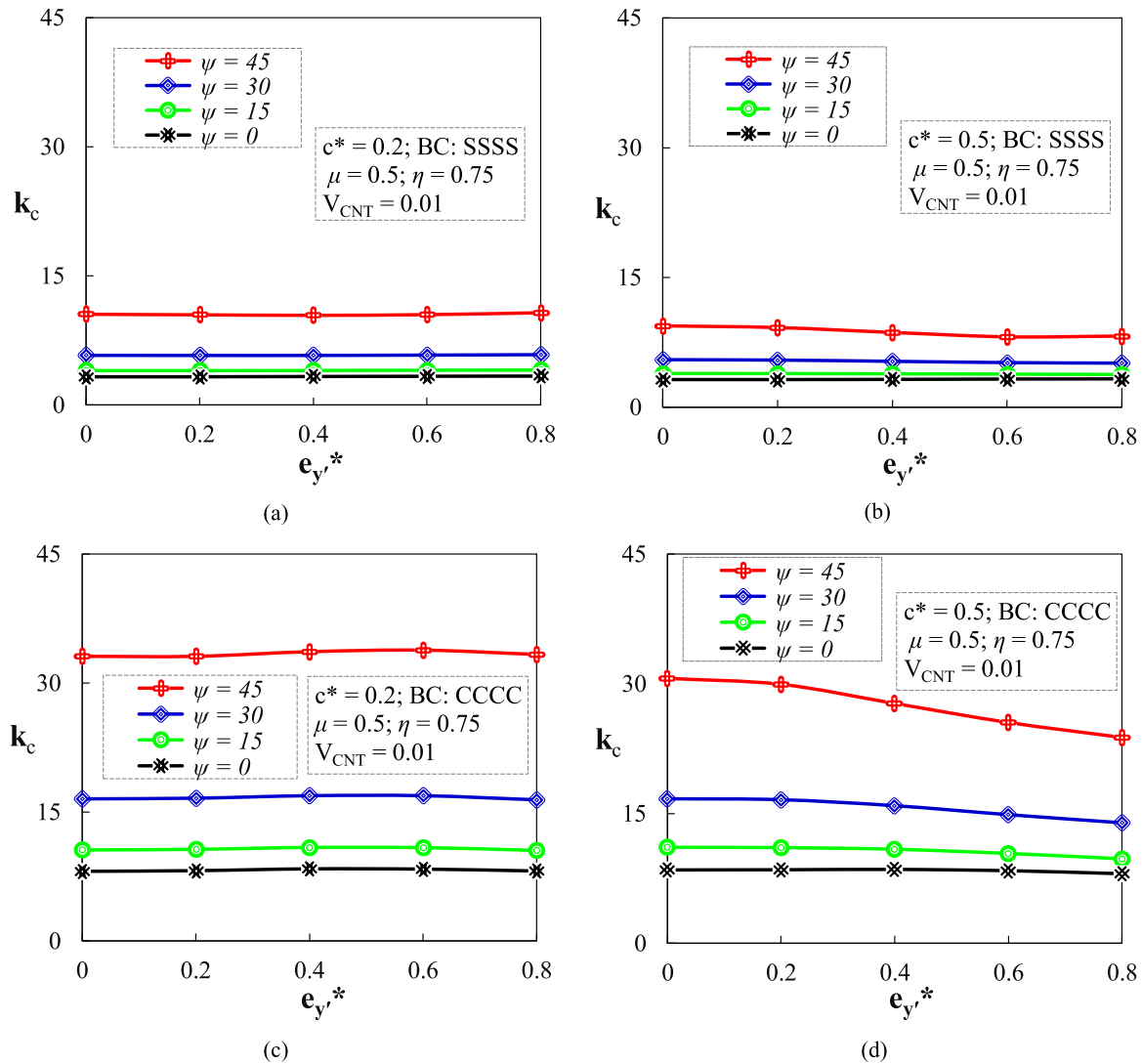
In this section, the effect of various parameters (agglomeration parameters and volume fraction of CNTs, crack length, and plate skew angle) on the buckling behavior and the mode shape of centrally cracked CNTRC skew plates, which are divided into several sub-sections, is examined. The findings are presented for skew plates under three loading conditions depicted in Fig. 6.

##### 4.1.1. The effect of agglomeration parameters and volume fraction of CNTs

The effect of CNTs agglomeration on the critical buckling coefficient of skew plates for four boundary conditions and both positive and negative shear loading is shown in Figs. 8 and 9. The results are presented for three different volume fractions of CNTs: null agglomeration ( $\mu = \eta = 1$ ), partial agglomeration ( $\mu = 0.5$ ,  $\eta = 0.75$ ), and complete agglomeration ( $\mu = 0.5$ ,  $\eta = 1$ ). As it is clear from Figs. 8 and 9, when CNTs are completely dispersed, the highest values of the critical buckling coefficient are observed in plates with various skew angles. This condition can be related to the elastic properties of the nanocomposite so that the uniform distribution of nanotubes in the matrix leads to the optimization of the effective properties of the nanocomposite. This distribution is crucial, as it ensures the effective interaction between the nanotubes and the surrounding matrix, ultimately contributing to

improved mechanical properties. Agglomerating CNTs in skew plates with equal volume fractions decreases the buckling coefficient. This results in a reduction of approximately 3 % with partial agglomeration and 13 % with complete agglomeration. On the other hand, as shown in Figs. 8 and 9, increasing the plate skew angle has a different effect on the critical buckling coefficients under positive and negative shear loading. This phenomenon can be attributed to the change in the stress distribution and effective stiffness of the plate with the change in the plate skew angle and loading direction. In skew plates, the asymmetric geometry causes coupling between different deformation modes, which behave differently under positive and negative shear loading. This phenomenon is caused by the complex interaction between the plate geometry, boundary conditions, loading direction, and stress field, each of which affects the buckling behavior of the plate separately and in interaction with each other.

Fig. 10 shows the Effect of agglomeration parameters  $\mu$  and  $\eta$  on the critical buckling coefficient of centrally cracked CNTRC skew plates under positive shear load for three different volume fractions. Increasing  $\mu$  in Fig. 10a results in a uniform distribution of inclusions, leading to more uniformly dispersed nanotubes and an increase in the critical buckling coefficient. Conversely, increasing  $\eta$  in Fig. 10b causes a non-uniform dispersion of the CNTs, resulting in a decrease in the critical buckling coefficient. Furthermore, Fig. 10 illustrates a significant



**Fig. 15.** Critical buckling coefficient vs.  $y'$ -eccentricity ratio for cracked CNTRC skew plates under combined load: (a)  $c^* = 0.2$ , BC: SSSS; (b)  $c^* = 0.5$ , BC: SSSS; (c)  $c^* = 0.2$ , BC: CCCC; (d)  $c^* = 0.5$ , BC: CCCC.

increase in the critical buckling coefficient with higher volume fractions. This trend is further highlighted in Fig. 11, which presents the buckling coefficients for the partial agglomeration mode and skew plates under positive shear load across four boundary conditions and skew angles. The results show that at  $V_{CNT} = 0.05$ , the buckling coefficient can be up to five times greater than in the case without CNTs.

#### 4.1.2. The effect of crack length

Fig. 12 illustrates the effect of crack length on the critical buckling coefficient of centrally cracked CNTRC skew plates with a volume fraction of 0.01 and partial agglomeration of CNTs under different loading conditions. As shown in this figure, for skew plates under shear loadings (positive and negative), the buckling coefficient decreases with the formation of a crack in the plate; such that the buckling coefficient consistently decreases with the crack length increase. The existence of a central crack with  $c^* = 0.5$  in the CNTRC skew plate can reduce the critical buckling coefficient by 57 % under positive shear loading conditions ( $\psi = 45^\circ$  and BC: CCCC) (Fig. 12b). For CNTRC skew plates under combined loading, the buckling coefficient may decrease (up to 15 %) or increase (up to 2.5 %) (Fig. 12e and f). In general, the effect of small cracks (relative crack lengths of 0.1 and 0.2) on the changes in the buckling coefficient of skew plates under all three loading conditions is less than 11 %.

#### 4.1.3. The effect of plate skew angle

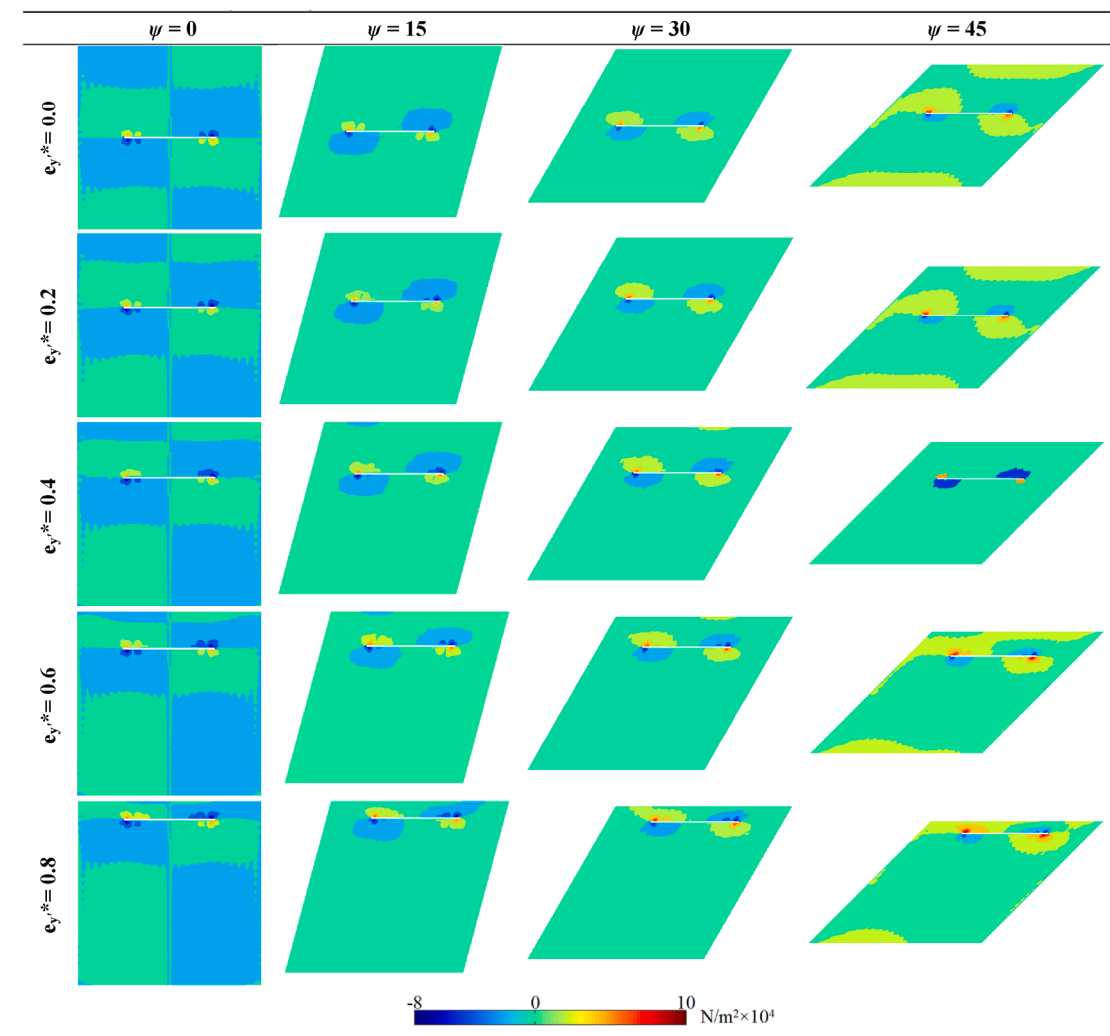
Subsequently, it can be seen from Fig. 12 that for CNTRC skew plates under positive and combined shear loads, the buckling coefficient increases with a certain trend as the plate skew angle increases. As the plate skew angle rises from  $0^\circ$  to  $45^\circ$  degrees, the buckling coefficient increases by approximately 3 to 5 times. On the other hand, for CNTRC skew plates under negative shear load, the buckling coefficient decreases by approximately 37 % as the skew angle increases from  $0^\circ$  to  $30^\circ$ , then increases by about 11 % as the angle rises from  $30^\circ$  to  $45^\circ$  (Fig. 12c and d).

#### 4.1.4. The stress field and mode shapes

For further investigation of centrally cracked CNTRC skew plates, different components of the stress field of these plates with varying angles of skew for two crack lengths and boundary conditions are shown in Table 3. This Table displays that in the cracked CNTRC skew plates under shear and combined loading, the stress distribution is axisymmetric over the surface of the skew plate. In addition, Table 4 presents the critical mode shapes of centrally cracked CNTRC skew plates for different support conditions, crack lengths, and plate skew angles. As can be seen, the support conditions of the CNTRC skew plate can play an important role in the critical buckling mode shape of skew plates with combined loading conditions. The buckling mode shapes of the cracked

**Table 5**

Stress field ( $\sigma_{xx}$ ) of cracked CNTRC skew plates under shear load, considering variations in plate skew angle and crack location, with  $c^* = 0.5$ ;  $\mu = 0.5$ ;  $\eta = 0.75$ ;  $V_{CNT} = 0.01$ .



skew plates are obtained through an eigenvalue analysis. The corresponding out-of-plane displacement fields illustrate the relative deformation patterns associated with each mode. Since these results stem from solving a linear eigenvalue problem, the displacement values are normalized and do not represent actual physical magnitudes. Therefore, while displacement units may be assigned for visualization purposes, they are arbitrary and only reflect the relative distribution of displacements across the skew plate.

#### 4.2. CNTRC plates with off-center crack

In this section, the influence of parameters, including crack length, plate skew angle and crack eccentricity, on the critical buckling coefficient and mode shapes of non-centrally CNTRC skew plates under simply and clamped boundary conditions, with a  $V_{CNT}$  of 0.01 and partial CNT agglomeration, is investigated. Buckling results for various parameters are presented under three loading conditions illustrated in Fig. 6.

##### 4.2.1. The effect of crack length

Figs. 13, 14, and 15 show the buckling coefficients of cracked CNTRC skew plates vs. the  $y'$ -eccentricity ratio parameter. It is observed that the buckling coefficient is always reduced by changing the relative crack

length from 0.2 to 0.5 and assuming that other effective parameters (BC,  $\psi$ ,  $e_y^*$ ) are constant for shear loadings. The maximum reduction of the coefficients for positive and negative shear loading is 68 % and 28 %, respectively, for the skew plate with the BC: CCCC and the  $\psi = 45^\circ$  (Figs. 13 and 14). For combined loading, depending on the parameter skew angle of the plate, the coefficients can increase (up to 5 %) or decrease (up to 33 %) (Fig. 15c and d).

##### 4.2.2. The effect of plate skew angle

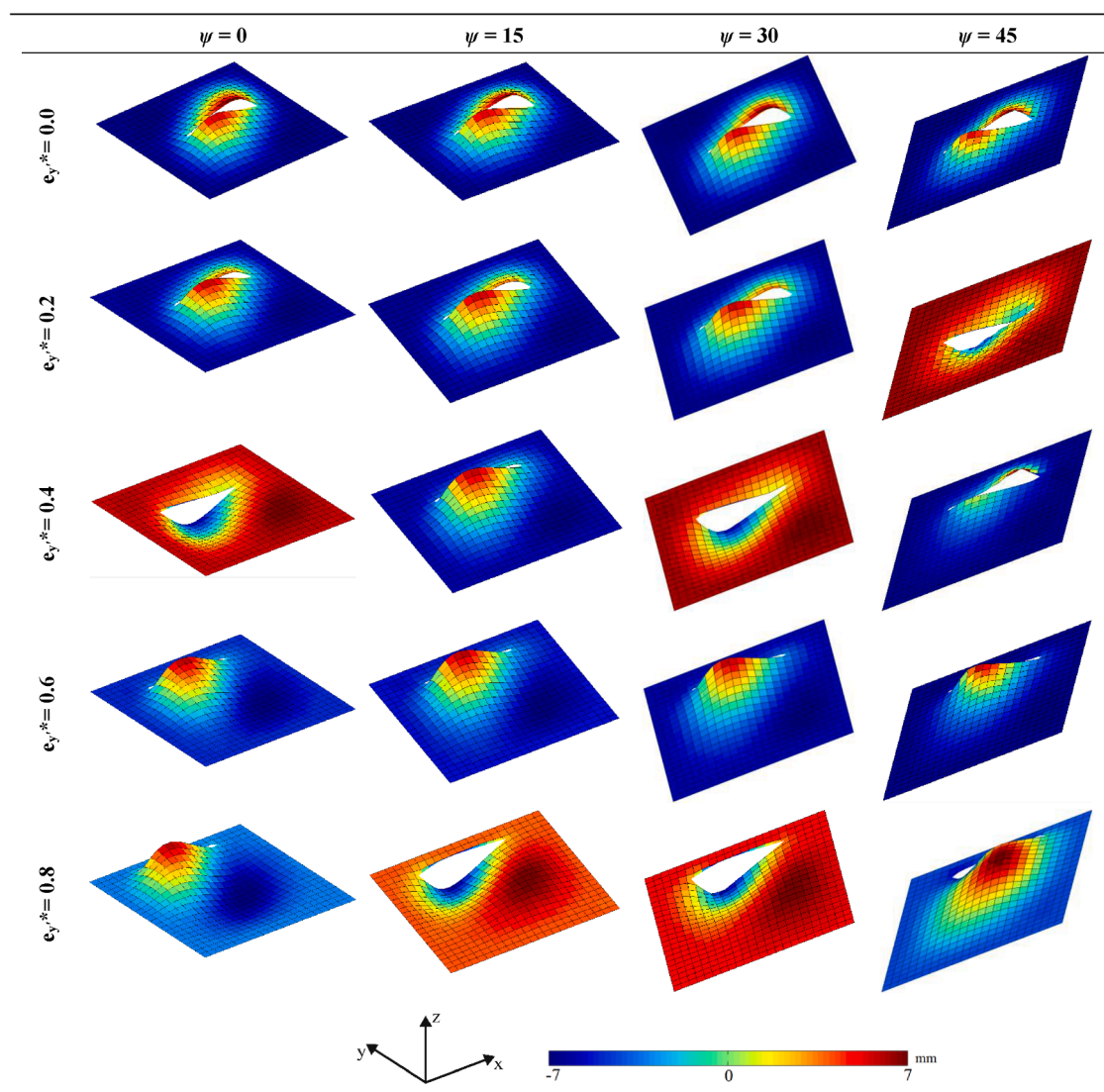
As it is clear from Figs. 13 and 15, with the increase of the skew angle of the plate from zero to  $45^\circ$  in a certain eccentricity, the buckling coefficient increases approximately 2 to 5 times for the conditions of positive shear and combined loadings. The highest coefficient belongs to the condition of positive shear loading for plates with  $c^* = 0.2$  and BC: CCCC. Furthermore, according to Fig. 14, CNTRC skew plates under negative shear load, the buckling coefficient decreases by approximately 37 % as the skew angle increases from  $0^\circ$  to  $30^\circ$ , and then increases by about 11 % as the angle rises from  $30^\circ$  to  $45^\circ$ .

##### 4.2.3. The effect of crack eccentricity in $y'$ -direction

Figs. 13 and 14 show that the centrally cracked CNTRC skew plates under shear loads (positive and negative) have lower buckling

**Table 6**

Critical buckling mode shapes for CNTRC skew plates with varying plate skew angle and crack location under positive shear load, and  $c^* = 0.5$ , BC: CCCC;  $\mu = 0.5$ ;  $\eta = 0.75$ ;  $V_{\text{CNT}} = 0.01$ .



coefficients than off-center cracked skew plates. While increasing crack eccentricity generally raises the buckling coefficient (up to 22 %), plates with a  $45^\circ$  skew angle and a larger crack exhibit the opposite trend: increasing eccentricity reduces the buckling coefficient (up to 29 %) (Figs. 13b,d and 14b,d). Besides, as illustrated in Fig. 15, cracked skew plates under combined load indicate varying buckling coefficients with increasing crack eccentricity, and the interplay of several other parameters influences these variations. Specifically, for clamped boundary conditions and  $c^* = 0.5$  at  $\psi = 45^\circ$ , the buckling coefficient of skew plates decreases by 22 % (Fig. 15d) and increases by 2 % at  $\psi = 0^\circ$  (Fig. 15a) as crack eccentricity increases. This demonstrates the sensitivity of the buckling behavior of a plate with an off-center crack to variations in the plate skew angle.

#### 4.2.4. The stress field and mode shapes

Table 5 illustrates the various components of the stress field for the cracked CNTRC skew plates with varying plate skew angles and eccentricity of the crack, the partial agglomeration of CNT, a volume fraction of 0.01, a crack length of 0.5, and clamped support conditions under shear loading. This figure shows that in the cracked CNTRC skew plates

with the off-center crack, the stress distribution is axisymmetric over the surface of the skew plate, similar to the skew plates with the center crack. Moreover, Table 6 presents the critical buckling mode shapes for CNTRC skew plates with different plate skew angles and crack locations under positive shear load. As can be seen, the  $y$ -eccentricity of the crack can affect the buckling mode shapes of the CNTRC skew plates. For instance, this change in the buckling mode shape of the skew plates is visible with increasing crack eccentricity from 0.2 to 0.4.

## 5. Conclusions

In this study, the effect of plate skew angle and cracking parameters on the buckling stability of cracked carbon nanotube-reinforced composite skew plates under three loading conditions was investigated for the first time using a complex code developed in MATLAB within the XFEM framework. The effective material properties of the nanocomposite skew plates were estimated using the Eshelby-Mori-Tanaka approach, considering null, partial, and complete agglomeration of single-walled carbon nanotubes in the epoxy resin matrix. The effect of these agglomeration states on the buckling behavior of skew

nanocomposites was examined. Equilibrium and stability equations for the cracked CNTRC skew plates under in-plane load were derived using Mindlin's first-order shear deformation theory. Key parameters, including CNT volume fraction and agglomeration, crack length and location, and boundary conditions, were analyzed. The results were validated against reliable literature sources to confirm the approach's capabilities and accuracy.

The results of this work indicated that achieving a uniform distribution of CNTs is crucial for obtaining high resistance against buckling in CNT-reinforced composites. Therefore, if the two agglomeration parameters are equal, that is, when the CNTs are fully dispersed (null agglomeration), the values of the critical buckling coefficient are the largest. Agglomeration of carbon nanotubes in CNTRC skew plates with equal volume fraction reduces the buckling coefficient by about 3 % for partial agglomeration and 13 % for complete agglomeration. In addition, for CNTRC skew plates, the volume fraction of CNTs can significantly influence buckling capacity, depending on the nanotubes' agglomeration case and other parameters.

According to the findings, cracking significantly affects the buckling capacity of CNTRC skew plates, depending on variations in crack length and location, plate skew angle, boundary conditions, and loading. A large center crack (relative crack length = 0.5) reduces the critical buckling coefficient of CNTRC skew plates by 57 % and 32 % under positive and negative shear loads, respectively. Under combined loading, the buckling coefficient of skew plates may decrease (up to 15 %) or increase (up to 2.5 %). Furthermore, for the CNTRC skew plates with the off-center crack under positive and negative shear loads, increasing the crack length from 0.2 to 0.5 reduces the buckling coefficient by 68 % and 28 %, respectively. With combined loading, the buckling coefficient may decrease (up to 28 %) or increase (up to 5 %).

As the skew angle of the CNTRC cracked plates increases from 0° to 45° under positive shear and combined loads, the buckling capacity increases by 2 to 5 times. On the other hand, for the skew plates under negative shear load, the buckling capacity decreases by approximately 37 % with the skew angle increasing from 0° to 30°, then increases by approximately 11 % with the angle increasing from 30° to 45°. The results of variations of the plate skew angle are valid for both center and off-center cracks, assuming constant parameters such as crack length, eccentricity, and boundary conditions.

Depending on the crack length, boundary conditions, and plate skew angle, cracked CNTRC plates exhibit different buckling capacities as the crack eccentricity increases. Under shear loading (positive and negative), the buckling coefficient of CNTRC cracked skew plates generally increases by up to 22 %, but can decrease by up to 29 % in some skew plates. Under combined loading, increasing crack eccentricity results in either a slight increase (around 2 %) or a significant decrease (up to 22 %) in the buckling coefficient, demonstrating parameter interaction.

Parametric studies revealed that complex interactions between various parameters significantly influence the buckling behavior in these systems. Specifically, the interplay between crack length and location is closely linked with factors such as plate skew angle, boundary conditions, and loading. These interactions clearly explain why, in some cases, buckling capacity increases, while in others, a change in the same parameter leads to a significant reduction. This interdependence confirms that no single parameter can be investigated in isolation, as a change in one factor fundamentally alters the entire stress distribution and the energy required to activate the instability mode. These findings emphasize the importance of simultaneously considering all geometric and loading variables when designing cracked CNT-reinforced composite skew plates.

## Author statement

The authors whose names are listed above certify that they have NO affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or

materials discussed in this manuscript.

This statement is signed by the corresponding author on behalf of all authors to indicate agreement that the above information is true and correct.

## CRediT authorship contribution statement

**Mohammad Hossein Taheri:** Writing – review & editing, Writing – original draft, Validation, Software, Resources, Investigation, Formal analysis, Data curation. **Parham Memarzadeh:** Visualization, Validation, Supervision, Software, Project administration, Methodology, Investigation, Formal analysis, Data curation, Conceptualization.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

## Data availability

Data will be made available on request.

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