• content
  • The Basics of Interest Rates: Simple and Compound Interest
  • The Basic Discount Factors: Present Value, Future Value, Annual Value
  • Economic Equivalence and Net Present Value
The Importance of Time

- A dollar in hand now is worth more than a dollar received in the future, because of its earning power, i.e., it can be invested to generate income.

- The purchasing power of money, i.e., the amount of goods that a certain amount can buy, changes with time also.

- Objective: To develop methods for establishing the equivalence of sums of money. It depends on the amounts, the time of occurrence of the sums of money, and the interest rate.

Simple Interest

- $P$: Principal amount
- $n$: Number of interest periods
- $i$: Interest rate
- $I$: Interest earned

Interest and principal become due at the end of $n$ periods:

$$I = P \times n \times i$$

The earned interest is proportional to the length of time and the principal amount was borrowed.
Compound Interest

• Interest is payable at the end of each interest period.

• If the interest is not paid, the borrower is charged interest on the total amount owed (principal plus interest).

• **Example**: $1,000 is borrowed for two years at 6% (compounded). A single payment will be made at the end of the second year.

  • Amount owed at the beginning of year 2: $1,060
  • Amount owed at the end of year 2: $1,060 × 1.06 = $1,000 × (1.06)^2 = $1,123.60
  • For simple interest, the amount owed at the end of year 2 would be: $1,000 + 1,000 × 2 × 0.06 = $1,120.00

The Basic Discount Factors

• **P** (*present value*) : Amount of money in present

• **F** (*future value*) : Value of money in future

• **A** (*annual value or "annuity") : A series of amounts that will be paid at the end of each year

• These three factors can be converted to each other
Cash Flow Diagram over Time

• Cash flow is a diagram that shows flow of cashes through periods (years)
• Each unit in cash flow diagram shows end of one period (usually one year)
• Up arrow = we receive $ ; Down arrow = we pay $

Example: Amount borrowed: $1,000
• Interest is paid at the end of each year at the rate of 10%
• The principal is due at the end of the fourth year.

Single-Payment Compound-Amount Factor

• A single payment is made after n periods.
• The interest earned at the end of each period is charged on the total amount owed (principal plus interest).
• $1 now is worth (F/P, i, n) at time n if invested at i%

\[ (F/P, i, n) = (1 + i)^n \]
Single-Payment Present-Worth Factor

\[
(P/F, i, n) = \frac{1}{(1+i)^n}
\]

- The reciprocal of the single-payment compound amount factor.
- Discount rate: \(i\)
- \(\$1\) \(n\) years in the future is worth \((P/F, i, n)\) now.

Equal-Payment-Series Compound-Amount Factor

- Equal payments, \(A\), occur at the end of each period.
- We will get back \((F/A, i, n)\) at the end of period \(n\) if funds are invested at an interest rate \(i\).
- \(F = A + A(1+i)^1 + A(1+i)^2 + \ldots + A(1+i)^{n-1}\)

\[
(F/A, i, n) = \frac{(1+i)^n - 1}{i}
\]
Equal-Payment-Series Sinking-Fund Factor

- Used to determine the payments $A$ required to accumulate a future amount $F$.

\[
(A/F, i, n) = \frac{i}{(1+i)^n - 1}
\]

- **Example:** We wish to deposit an amount $A$ every 6 months for 3 years so that we’ll have $10,000 at the end of this period. The interest rate is 5% per year. Find $A$.

- **Solution:**
  
  $n = 6$ deposits, $i = 2.5\%$ per 6-month period
  
  $F = 10,000$  
  
  \[
  (A/F, 0.025, 6) = 0.15655 \implies A = 10,000 \times 0.15655 = 1,565.50
  \]

Equal-Payment-Series Capital-Recovery Factor

- An amount $P$ is deposited now at an annual interest rate $i$.

- We will withdraw the principal plus the interest in a series of equal annual amounts $A$ over the next $n$ years.

- The principal will be worth $P(1+i)^n$ (slide \textsuperscript{31}) at the end of $n$ years. This amount is to be recovered by receiving $A$ every year $\implies$ the sinking-factor formula applies (slide \textsuperscript{61}) $\implies$

\[
A = P(1+i)^n \left[ \frac{i}{(1+i)^n - 1} \right] \implies (A/P, i, n) = \frac{i(1+i)^n}{(1+i)^n - 1}
\]
• **Example:** Your house mortgage is $300,000 for 30 years with an nominal annual rate of 7%. What is the monthly payment?

\[ n = 360 \text{ months }, \ i = 0.583\% \text{ per month} \]

\[ (A/P, 0.00583, 360) = 0.006650339 \]

\[ \Rightarrow A = 300,000 \times 0.006650339 = $1,995.10 \text{ per month} \]

---

**Equal-Payment-Series Present-Worth Factor**

\[ (P/A, i, n) = \frac{(1+i)^n - 1}{i(1+i)^n} \]

*\( n \) annual payment in \( n \) periods with interest rate \( i \) is worth \( P \) now.*
Summary of the Formulas

<table>
<thead>
<tr>
<th>Formula</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{F}{P}, i, n )</td>
<td>Single-Payment Compound-Amount Factor</td>
</tr>
<tr>
<td>( \frac{P}{F}, i, n )</td>
<td>Single-Payment Present-Worth Factor</td>
</tr>
<tr>
<td>( \frac{F}{A}, i, n )</td>
<td>Equal-Payment-Series Compound-Amount Factor</td>
</tr>
<tr>
<td>( \frac{A}{F}, i, n )</td>
<td>Equal-Payment-Series Sinking-Fund Factor</td>
</tr>
<tr>
<td>( \frac{A}{P}, i, n )</td>
<td>Equal-Payment-Series Capital-Recovery Factor</td>
</tr>
<tr>
<td>( \frac{P}{A}, i, n )</td>
<td>Equal-Payment-Series Present-Worth Factor</td>
</tr>
</tbody>
</table>

\[
\left( \frac{F}{P}, i, n \right) = (1 + i)^n \\
\left( \frac{P}{F}, i, n \right) = \frac{1}{(1 + i)^n} \\
\left( \frac{F}{A}, i, n \right) = \frac{(1 + i)^n - 1}{i} \\
\left( \frac{A}{F}, i, n \right) = \frac{i}{(1 + i)^n - 1} \\
\left( \frac{A}{P}, i, n \right) = \frac{(1 + i)^n - 1}{i(1 + i)^n} \\
\left( \frac{P}{A}, i, n \right) = \frac{(1 + i)^n - 1}{i(1 + i)^n}
\]

Economic Equivalence

**Example:** Consider the following cash flow:
You will receive $500 at the end of years 3 and 4 and $1,000 at the end of year 5. If the interest rate is 7%, what amount received at the present is equivalent to this cash flow?

**Solution:** In all problems like this, the first step is to draw cash flows diagram:

\[
P = ?
\]

\[
P = 500 \times \left( \frac{P}{F}, 0.07, 3 \right) + 500 \times \left( \frac{P}{F}, 0.07, 4 \right) + 1000 \times \left( \frac{P}{F}, 0.07, 5 \right)
\]

\[
P = \frac{500}{(1 + 0.07)^3} + \frac{500}{(1 + 0.07)^4} + \frac{1000}{(1 + 0.07)^5} = \$1,502.59
\]
Economic Equivalence – continued

- **Example**: Consider the following cash flow. If the interest rate is 6%, what amount received at the present is equivalent to this cash flow?

$\begin{align*}
P &= ? \\
\end{align*}$

- **Solution**: at first, we calculate present worth of 5 years Equal-Payment-Series:

\[ n = 5 \text{ years} \quad i = 6\% \]

\[ (P/A, 0.06, 5) = 4.212364 \]

\[ \Rightarrow P = A \times (P/A, 0.06, 5) = 500 \times 4.212364 = 2106.18 \]

- **Notice**: $P_1$ is in end of year 2. So the equivalent cash flow is below diagram:

\[ P = P \times (P/F, 0.06, 2) + 500 \times (P/F, 0.06, 5) \Rightarrow \]

\[ P = \frac{2106.18}{(1 + 0.06)^2} + \frac{500}{(1 + 0.06)^5} \approx 2248.12 \]
Example 4.4

An energy manager expects a boiler to last 7 years, and he thinks it will cost about $150,000 to replace at that time. How much money should the company deposit today in an account paying 10% per year in order to have $150,000 available in 7 years?

For this problem:

\[ F = 150,000 \]
\[ i = 10\% \]
\[ n = 7 \text{ years} \]
\[ P = ? \]

Using Equation 4.5,
\[ P = \frac{150,000}{(1.1)^7} = 76,974 \]

Using the present worth factor \( (P/F, 10\%, 7) \) and Table 4-4,
\[ P = \frac{150,000}{(P/F, 10\%, 7)} = 150,000(0.5132) = 76,980 \]

---

Example 4.5

An energy manager has $5,000 available today to purchase a high efficiency air conditioner with a life of six years. She would like to know what energy cost savings would be needed each year to justify this project if the company MARR is 10%.

\[ P = 5,000 \]
\[ i = 10\% \]
\[ n = 6 \]
\[ A = ? \]

This problem is the first type of conversion: find \( A \), given \( P \). Therefore:
\[ A = \frac{5,000}{(A/P, 10\%, 6)} \]
\[ = \frac{5,000}{(P/A, 10\%, 6)} \]
\[ = \frac{5,000}{0.5132} \]
\[ = 9,766 \]

This is the minimum acceptable rate of return (MARR). Therefore:

\[ A = \frac{5,000}{(A/P, 10\%, 6)} \]
\[ = \frac{5,000}{(P/A, 10\%, 6)} \]
\[ = \frac{5,000}{0.5132} \]
\[ = 9,766 \]

Thus, if the air conditioner produces annual energy cost savings of $1148 or greater, the company will earn its MARR at the least. If the savings is greater than $1148, the actual rate of return will be greater than 10%.
Example 4.6
A heat pump is expected to produce energy cost savings of $1,500 per year over a lifetime of 20 years. What is the equivalent present sum or present worth for this series of cash flows, if the company MARR is 10%?

\[ A = 1,500 \]
\[ i = 10\% \]
\[ n = 20 \]
\[ P = ? \]

This is the find \( P \), given \( A \) problem.

\[ P = 1,500\left(\frac{1}{P/A\,10,\,20}\right) = 1,500\left(0.3136\right) = 470.4 \]

This $12,770 is the worth in today's dollars of the series of annual savings over 20 years. It could also be considered as the incremental cost that could be paid for the heat pump compared to an electric resistance heater, and still produce a rate of return of 10% on the investment.

---

Example 4.7
A company needs to begin saving money for the new boiler in example 4.4. The company will make a deposit each year for 7 years to a savings account paying 10% annually. How large should the annual deposits be if they want to have $150,000 in the bank in 7 years?

\[ F = 150,000 \]
\[ i = 10\% \]
\[ n = 7 \]
\[ A = ? \]

This problem is characterized as a find \( A \), given \( F \) problem.

\[ A = 150,000\left(\frac{1}{A/F\,10,\,7}\right) \]
\[ = 150,000\left(0.1054\right) \]
\[ = 15,810 \]
Example 4.8
A high efficiency lighting project for a company is saving $10,000 a year in energy costs. If that $10,000 a year is deposited into an energy management savings account paying 10%, how much money will be available in 5 years to use to replace an old chiller with a new, high efficiency model?

\[ A = $10,000 \]
\[ i = 10\% \]
\[ n = 5 \]
\[ F = ? \]

This is the find F, given A problem.

\[ F = A \cdot F/A_{10,5} \]
\[ F = 10,000 \cdot 6.105 \]
\[ F = $61,050 \]

By placing the $10,000 a year energy cost savings into an account that earns 10% interest, the company will have $61,050 to spend on the next project in five years.

Example 4.11
An energy efficient air compressor is proposed by a vendor. The compressor will cost $30,000 installed, and will require $1,000 worth of maintenance each year for its life of 10 years. Energy costs will be $6,000 per year. A standard air compressor will cost $25,000 and will require $500 worth of maintenance each year. Its energy costs will be $10,000 per year. If your company uses a MARR of 10%, would you invest in the energy efficient air compressor?

Alternative 1: Energy efficient air compressor
Alternative 2: Standard air compressor
چند مثال مرتبط با مدیریت انرژی

<table>
<thead>
<tr>
<th>EOY</th>
<th>Alternative 1</th>
<th>Alternative 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$30,000</td>
<td>$25,000</td>
</tr>
<tr>
<td>1</td>
<td>$1,000 + $6,000 = $7,000</td>
<td>$600 + $10,000 = $10,500</td>
</tr>
<tr>
<td>2</td>
<td>$7,000</td>
<td>$10,500</td>
</tr>
<tr>
<td>3</td>
<td>$7,000</td>
<td>$10,500</td>
</tr>
<tr>
<td>4</td>
<td>$7,000</td>
<td>$10,500</td>
</tr>
<tr>
<td>5</td>
<td>$7,000</td>
<td>$10,500</td>
</tr>
<tr>
<td>6</td>
<td>$7,000</td>
<td>$10,500</td>
</tr>
<tr>
<td>7</td>
<td>$7,000</td>
<td>$10,500</td>
</tr>
<tr>
<td>8</td>
<td>$7,000</td>
<td>$10,500</td>
</tr>
<tr>
<td>9</td>
<td>$7,000</td>
<td>$10,500</td>
</tr>
<tr>
<td>10</td>
<td>$7,000</td>
<td>$10,500</td>
</tr>
</tbody>
</table>

آنالیز:

$LCC(\text{Alt 1}) = 30,000 + 7,000(P/A_{10,10})$

$LCC(\text{Alt 1}) = 30,000 + 7,000(6.1446)$

$LCC(\text{Alt 1}) = 73,012$

$LCC(\text{Alt 2}) = 25,000 + 10,500(P/A_{10,10})$

$LCC(\text{Alt 2}) = 25,000 + 10,500(6.1446)$

$LCC(\text{Alt 2}) = 89,518$

规定决策规则：在LCC分析中，选择具有最低LCC的方案。由于Alternative 1具有最低的LCC，它应该被选择。