A note on the use of Adomian decomposition method for high-order and system of nonlinear differential equations

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A B S T R A C T

This paper extends an earlier work [Hosseini MM, Nasabzadeh H. Modified Adomian decomposition method for specific second order ordinary differential equations. Appl Math Comput 2007;186:117–23] to high order and system of differential equations. Solution of these problems is considered by proposed modification of Adomian decomposition method. Furthermore, with providing some examples, the aforementioned cases are dealt with numerically.

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1. Introduction

It is well known that the Adomian decomposition method (ADM) and its modifications [3–5,7,10,12–15] are efficient methods to solve linear and nonlinear ODEs, DAEs, PDEs, SDEs, integral equations and integro-integral equations. The ADM has been applied to a wide class of problems in physics, biology and chemical reaction. The method provides the solution in a rapid convergent series with computable terms. It is the purpose of this paper to introduce a new reliable modification of ADM. For this reason, a new differential operator is defined which can be used for high-order and system of differential equations. In this manner, some examples are illustrated to show the advantages of using the proposed method to solve the initial value problems.

2. Modified ADM to solve high-order and system of differential equations

Consider the initial value problem in the n-order differential equation in the form:

\[
\begin{align*}
 y^{(n)} + P(x)y^{(n-1)} + Ny &= g(x), \\
 y(0) &= x_0, \\
 y^{(1)}(0) &= x_1, \ldots, y^{(n-1)}(0) &= x_{n-1},
\end{align*}
\]

where \( N \) is a nonlinear differential operator of order less than \( n - 1 \), \( P(x) \) and \( g(x) \) are given functions and \( x_0, x_1, \ldots, x_{n-1} \) are given constants.

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Here, we propose the new differential operator, as below:

\[ L = e^{-\int_{x_0}^{x} p(x) \, dx} \frac{d^n}{dx^n} \left( e^{\int_{x_0}^{x} p(x) \, dx} g^{(n-1)}(x) \right). \]  

(2)

So, the problem (1) can be written as

\[ Ly = g(x) - Ny. \]  

(3)

The inverse operator \( L^{-1} \) is therefore considered a \( n \)-fold integral operator, as below:

\[ L^{-1}(\cdot) = \int_{x_0}^{x} \cdots \int_{x_0}^{x} \int_{x_0}^{x} e^{-\int_{x_0}^{t} p(x) \, dx} \int_{x_0}^{x} e^{\int_{x_0}^{t} p(x) \, dx} g(\cdot) \, dx \, \cdots \, dx. \]  

(4)

By operating \( L^{-1} \) on (3), we have

\[ y(x) = \Phi(x) + L^{-1}g(x) - L^{-1}(Ny), \]  

(5)

such that

\[ L\Phi(x) = 0. \]

We write \( Ny = \sum_{n=0}^{\infty} A_n \) and \( y = \sum_{n=0}^{\infty} y_n \) where the components of \( A_n \) are the so-called Adomian polynomials, for each \( i \), \( A_i \) depends on \( y_0, y_1, \ldots, y_i \) only. Now by considering (5), we have

\[ y_n = \Phi(x) + L^{-1}g(x) - L^{-1}(Ny), \quad n = 0, 1, 2, \ldots, \]  

(6)

Through using Adomian decomposition method, the components \( y_n(x) \) can be determined as

\[ \begin{cases} y_0 = \Phi(x) + L^{-1}g(x), \\ y_{n+1} = -L^{-1}A_n, \quad n \geq 0. \end{cases} \]  

(7)

If the series converges in a suitable way, then it can be seen

\[ y = \lim_{M \to \infty} \Psi_M(x), \]  

where \( \Psi_M = \sum_{n=0}^{M} y_n \). Now an expression for the \( A_i \) is required. Specific algorithms were seen in [9,11] to formulate Adomian polynomials. The theoretical treatment of the convergence of ADM has been considered in [1,2,6,8].

The mentioned method can be used for solving system of differential equation in the following form (see Example 3):

\[ \begin{cases} y_1^{(n)} + p_1(x)y_1^{(n-1)} + p_2(x)y_1^{(n-2)} + \cdots + p_{n-1}(x)y_1^{(2)} + p_n(x)y_1^{(1)} + g_1(x), \\ y_2^{(n)} + p_1(x)y_2^{(n-1)} + p_2(x)y_2^{(n-2)} + \cdots + p_{n-1}(x)y_2^{(2)} + p_n(x)y_2^{(1)} + g_2(x), \\ \vdots \end{cases} \]  

and

\[ \begin{cases} y_n^{(n)} + p_1(x)y_n^{(n-1)} + p_2(x)y_n^{(n-2)} + \cdots + p_{n-1}(x)y_n^{(2)} + p_n(x)y_n^{(1)} + g_n(x). \end{cases} \]

To perform the ADM, in general, for an arbitrary natural number, \( v \), \( g(x) \), \( p(x) \), \( e^{\int_{x_0}^{x} p(x) \, dx} \) and \( e^{-\int_{x_0}^{x} p(x) \, dx} \) are expressed in Taylor series, at \( x = 0 \).

### 3. Test problems

In this section, nonsingular and singular 3-order ODEs and singular system of ODEs are considered and these problems are solved by standard and modified ADM which is presented in Section 2. The algorithms are performed by Maple 8.

**Example 1.** Consider the nonlinear initial value problem

\[ \begin{align*} y''' + e^x y'' + 4x^2 y' + x^2 y' & = g(x), \\ y(0) = y'(0) = y''(0) & = 0, \end{align*} \]  

(8)

where \( g(x) \) is compatible to exact solution

\[ y(x) = x^3 e^x. \]  

(9)

Here, we use Taylor series of \( g(x) \) with order 9,

\[ g(x) \approx 6 + 30x + 48x^2 + 45x^3 + \frac{171}{4} x^4 + \frac{164}{5} x^5 + \frac{529}{30} x^6 + \frac{243}{35} x^7 + \frac{2881}{1344} x^8. \]
Standard Adomian decomposition method: we put
\[ L(\cdot) = \frac{d^3}{dx^3}(\cdot), \]
so
\[ L^{-1}(\cdot) = \int_0^x \int_0^x \int_0^x (\cdot) dx dx dx. \]
In an operator form, Eq. (8) becomes
\[ Ly = -e^x y'' - 4x^2 y' - x^3 y + g(x). \] (10)
By applying \( L^{-1} \) to both sides of (10) we have
\[ y = y(0) + xy'(0) + \frac{x^2}{2} y''(0) - L^{-1}(e^x y') - 4L^{-1}(x^2 y') - L^{-1}(x^3 y) + L^{-1}(g(x)). \]
Proceeding as before we obtained the recursive relationship
\[ y_0 = L^{-1}(g(x)), \]
\[ y_{n+1} = -L^{-1}(f(x) \times y_n) - 4L^{-1}(x^2 y_n) - L^{-1}(x^3 A_n), \quad n \geq 0, \]
where \( f(x) \) is obtained by using Taylor series of \( e^x \) at \( x = 0 \) and \( A_n \)'s are Adomian polynomials of nonlinear term \( y^3 \), as below:
\[ f(x) \approx 1 + x + \frac{x^2}{2} + \cdots + \frac{x^8}{8!} \] (12)
and
\[ A_0 = y_0^3, \]
\[ A_1 = 3y_1 y_0^2, \]
\[ A_2 = 3y_2 y_0^2 + 3y_0 y_1^2, \]
\[ \vdots \]
So, by substituting (12) and (13) into (11), we have
\[ \begin{cases} y_0 = x^3 + \frac{1}{2} x^4 + \frac{1}{12} x^5 + \cdots, \\ y_0 + y_1 = x^3 + x^4 + \frac{9}{20} x^5 + \frac{11}{120} x^6 + \cdots, \\ y_0 + y_1 + \cdots + y_7 = x^3 + x^4 + \frac{1}{2} x^5 + \frac{1}{8} x^6 + \cdots + \left( \frac{1}{5040} x^{10} + \frac{41}{100320} x^{11} + \cdots \right) \end{cases} \] (14)
Note that, the Taylor series of exact solution (9) with order 9 is as below:
\[ y(x) = x^3 + x^4 + \frac{1}{2} x^5 + \frac{1}{6} x^6 + \cdots + \frac{1}{5040} x^{10} + \frac{1}{40320} x^{11} + \cdots \] (15)
It is easy to see that the standard ADM is slowly convergent to exact solution of this problem.

3.1. Modified Adomian decomposition method

Here, we use
\[ L^{-1}(\cdot) = \int_0^x \int_0^x \int_0^x (\cdot) dx dx dx, \] (16)
such that
\[ \int p(x) dx = \int e^x dx = e^x, \]
\[ e^{\int p(x) dx} = e^{e^x} \]
and
\[ e^{- \int p(x) dx} = e^{-e^x}. \]
Now by substituting the Taylor series of $e^r$ and $e^{-r}$ with order 9, into (16), we obtain
\[
p \approx e^r \approx 1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \frac{13}{30}x^5 + \frac{203}{720}x^6 + \frac{877}{5040}x^7 + \frac{23}{224}x^8,
\]
\[
q \approx e^{-r} \approx 1 - x + \frac{1}{6}x^3 + \frac{1}{24}x^4 - \frac{1}{60}x^5 - \frac{1}{80}x^6 - \frac{1}{560}x^7 + \frac{5}{4032}x^8.
\]
In addition
\[
L^{-1}(\cdot) = \int_0^x \int_0^x \int_0^x (q) \int_0^x (p) dx \, dx \, dx
\]
and according to (7) we have
\[
\begin{align*}
y_0 &= L^{-1} g(x), \\
y_{n+1} &= -L^{-1} A_n, \quad n \geq 0.
\end{align*}
\]
Thus,
\[
\begin{align*}
y_0 &= x^3 + x^4 + \frac{1}{6}x^5 + \frac{1}{6}x^6 + \frac{1}{2}x^7 + \cdots + \frac{1}{2040}x^{10} - \frac{3528}{221700}x^{11} + \cdots \\
y_0 + y_1 &= x^3 + x^4 + \frac{1}{2}x^5 + \frac{1}{6}x^6 + \cdots + \frac{1}{2040}x^{10} - \frac{3528}{221700}x^{11} + \cdots
\end{align*}
\]
The comparison between (14) and (17) shows that the rate of convergence of modified Adomian method is faster than standard Adomian method for this problem.

**Example 2.** Consider the nonlinear singular initial value problem
\[
y'' + \frac{2}{x} y' + 4xy' + \ln(y) = g(x),
\]
\[
y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2,
\]
where $g(x)$ is compatible to exact solution
\[
y(x) = \frac{1}{1 + x^2}.
\]
Here, we use Taylor series of $g(x)$ with order 9,
\[
g(x) \approx -\frac{4}{x} + 48x - 9x^2 - 180x^3 + \frac{33}{2}x^4 + 448x^5 - \frac{73}{3}x^6 - 900x^7 + \frac{129}{4}x^8.
\]

3.2. Standard Adomian decomposition method

According to example 1 it can be easily seen that the
\[
y_0 = L^{-1} g(x) = \int_0^x \int_0^x \int_0^x \left( -\frac{4}{x} + 48x - 9x^2 - 180x^3 + \frac{33}{2}x^4 + 448x^5 - \frac{73}{3}x^6 - 900x^7 + \frac{129}{4}x^8 \right) dx \, dx \, dx
\]
is undefined. So, the standard ADM can not solve the above problem.

3.3. Modified Adomian decomposition method

According to example 1 and by considering (6), we have
\[
\begin{align*}
A_0 &= y_0 \ln(y_0), \\
A_1 &= y_1 \ln(y_0), \\
A_2 &= y_2 \ln(y_0) + \frac{y_1^2}{70}, \\
&\vdots
\end{align*}
\]
Through applying modified Adomian decomposition method to problem (18), we obtain:
\[
\begin{align*}
y_0 &= 1 - x^2 + x^4 - \frac{9}{10}x^5 - x^6 + \cdots, \\
y_0 + y_1 &= 1 - x^2 + x^4 - x^6 + \frac{647}{600}x^8 + \cdots, \\
y_0 + y_1 + y_2 &= 1 - x^2 + x^4 - x^6 + x^8 - x^{10} - \frac{81}{70000}x^{11} + \cdots,
\end{align*}
\]
which $y_0 + y_1 + y_2$ is quite close to Taylor expansion of exact solution (19).
Example 3. Consider the nonlinear system of differential equation,

\[
\begin{align*}
    y'' + \tan(x)y' + z^2 &= g(x), \quad y(0) = 0, \quad y'(0) = 0, \\
    z'' + 100z' + y^2 &= h(x), \quad z(0) = 0, \quad z'(0) = 0, 
\end{align*}
\]

(20)

where \(g(x)\) and \(h(x)\) are compatible to exact solutions

\[
y(x) = x \sin(x) \quad \text{and} \quad z(x) = x \tan(x).
\]

Here, we use Taylor series of \(g(x), h(x)\) and \(\tan(x)\) with order 9.

Standard Adomian decomposition method. Here, we have

\[
y_0 = L^{-1}(g(x)) = x^2 + \frac{1}{24}x^6 + \frac{3}{224}x^8 + \cdots,
\]

(22)

\[
z_0 = L^{-1}(h(x)) = x^2 + \frac{100}{3}x^3 + \frac{1}{3}x^4 + \frac{20}{3}x^5 + \frac{6}{7}x^6 - \frac{40}{21}x^7 + \frac{121}{2520}x^8 + \cdots
\]

and

\[
y_{n+1} = -L^{-1}(f(x)y_n') - L^{-1}(A_n), \quad n \geq 0,
\]

(23)

\[
z_{n+1} = -L^{-1}(100z_n') - L^{-1}(B_n), \quad n \geq 0,
\]

where \(A_n\) and \(B_n\) are the Adomian polynomials of nonlinear terms \(y^2\) and \(z^2\). Also, \(f(x)\) denoted the taylor series of \(\tan(x)\) with order 9.

In this case, through considering (22) and (23), we have

\[
\begin{align*}
    y_0 &= x^2 + \frac{1}{24}x^6 + \frac{3}{224}x^8 + \cdots, \\
    y_0 + y_1 &= x^2 - \frac{1}{6}x^4 - \frac{1}{72}x^6 - \frac{100}{21}x^7 + \cdots, \\
    \vdots \\
    y_0 + y_1 + \cdots + y_6 &= x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 - \frac{1}{5040}x^8 + \frac{200000}{81}x^9 + \cdots
\end{align*}
\]

and

\[
\begin{align*}
    z_0 &= x^2 + \frac{100}{3}x^3 + \frac{1}{3}x^4 + \frac{20}{3}x^5 + \cdots, \\
    z_0 + z_1 &= x^2 - 833x^4 - \frac{4994}{72}x^5 + \frac{10}{21}x^7 + \cdots, \\
    \vdots \\
    z_0 + z_1 + \cdots + z_6 &= x^2 + \frac{1}{6}x^4 + \frac{2}{7}x^6 + \frac{17}{375}x^8 + \frac{31250000000}{367}x^9 + \cdots
\end{align*}
\]

So, the standard Adomian decomposition method converges to Taylor expansion of exact solution (21).

3.4. Modified Adomian decomposition method

By applying modified Adomian decomposition method to problem (20), we obtain:

\[
\begin{align*}
    y_0 &= x^2 + \frac{1}{6}x^4 + \frac{1}{3}x^6 + \frac{4}{30}x^8 + \cdots, \\
    y_0 + y_1 &= x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 - \frac{1}{5040}x^8 - \frac{1390}{181560}x^{10} + \cdots
\end{align*}
\]

and

\[
\begin{align*}
    z_0 &= x^2 + \frac{1}{3}x^4 + \frac{1}{6}x^6 + \cdots, \\
    z_0 + z_1 &= x^2 + \frac{1}{3}x^4 + \frac{2}{7}x^6 + \frac{17}{375}x^8 + \frac{599}{2735}x^{10} + \cdots
\end{align*}
\]

which is quite close to Taylor expansion of exact solution (21).

4. Conclusion

In this paper, modification of Adomian decomposition method was proposed. The merit of this method is that it is more efficient than the standard Adomian decomposition method when high order and system of differential equations are given. The advantage of using the proposed algorithm of this paper is clearly demonstrated for Examples 1–3. The obtained results show that the rate of convergence of modified Adomian decomposition method is higher than standard Adomian decomposition method for these problems.
References