Principles of Measurement Systems
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To Pauline, Sarah and Victoria
Contents

Preface to the fourth edition xi
Acknowledgements xiii

Part A General Principles 1

1 The General Measurement System 3
1.1 Purpose and performance of measurement systems 3
1.2 Structure of measurement systems 4
1.3 Examples of measurement systems 5
1.4 Block diagram symbols 7

2 Static Characteristics of Measurement System Elements 9
2.1 Systematic characteristics 9
2.2 Generalised model of a system element 15
2.3 Statistical characteristics 17
2.4 Identification of static characteristics – calibration 21

3 The Accuracy of Measurement Systems in the Steady State 35
3.1 Measurement error of a system of ideal elements 35
3.2 The error probability density function of a system of non-ideal elements 36
3.3 Error reduction techniques 41

4 Dynamic Characteristics of Measurement Systems 51
4.1 Transfer function G(s) for typical system elements 51
4.2 Identification of the dynamics of an element 58
4.3 Dynamic errors in measurement systems 65
4.4 Techniques for dynamic compensation 70

5 Loading Effects and Two-port Networks 77
5.1 Electrical loading 77
5.2 Two-port networks 84

6 Signals and Noise in Measurement Systems 97
6.1 Introduction 97
6.2 Statistical representation of random signals 98
6.3 Effects of noise and interference on measurement circuits 107
6.4 Noise sources and coupling mechanisms 110
6.5 Methods of reducing effects of noise and interference 113
### Part B  Typical Measurement System Elements

#### 8 Sensing Elements
8.1 Resistive sensing elements 149
8.2 Capacitive sensing elements 160
8.3 Inductive sensing elements 165
8.4 Electromagnetic sensing elements 170
8.5 Thermoelectric sensing elements 172
8.6 Elastic sensing elements 177
8.7 Piezoelectric sensing elements 182
8.8 Piezoresistive sensing elements 188
8.9 Electrochemical sensing elements 190
8.10 Hall effect sensors 196

#### 9 Signal Conditioning Elements
9.1 Deflection bridges 205
9.2 Amplifiers 214
9.3 A.C. carrier systems 224
9.4 Current transmitters 228
9.5 Oscillators and resonators 235

#### 10 Signal Processing Elements and Software
10.1 Analogue-to-digital (A/D) conversion 247
10.2 Computer and microcontroller systems 260
10.3 Microcontroller and computer software 264
10.4 Signal processing calculations 270

#### 11 Data Presentation Elements
11.1 Review and choice of data presentation elements 285
11.2 Pointer–scale indicators 287
11.3 Digital display principles 289
11.4 Light-emitting diode (LED) displays 292
11.5 Cathode ray tube (CRT) displays 295
11.6 Liquid crystal displays (LCDs) 299
11.7 Electroluminescence (EL) displays 302
11.8 Chart recorders 304
11.9 Paperless recorders 306
11.10 Laser printers 307
Part C Specialised Measurement Systems  311

12 Flow Measurement Systems  313
12.1 Essential principles of fluid mechanics  313
12.2 Measurement of velocity at a point in a fluid  319
12.3 Measurement of volume flow rate  321
12.4 Measurement of mass flow rate  339
12.5 Measurement of flow rate in difficult situations  342

13 Intrinsically Safe Measurement Systems  351
13.1 Pneumatic measurement systems  353
13.2 Intrinsically safe electronic systems  362

14 Heat Transfer Effects in Measurement Systems  367
14.1 Introduction  367
14.2 Dynamic characteristics of thermal sensors  369
14.3 Constant-temperature anemometer system for fluid velocity measurements  374
14.4 Katharometer systems for gas thermal conductivity and composition measurement  378

15 Optical Measurement Systems  385
15.1 Introduction: types of system  385
15.2 Sources  387
15.3 Transmission medium  393
15.4 Geometry of coupling of detector to source  398
15.5 Detectors and signal conditioning elements  403
15.6 Measurement systems  409

16 Ultrasonic Measurement Systems  427
16.1 Basic ultrasonic transmission link  427
16.2 Piezoelectric ultrasonic transmitters and receivers  428
16.3 Principles of ultrasonic transmission  436
16.4 Examples of ultrasonic measurement systems  447

17 Gas Chromatography  461
17.1 Principles and basic theory  461
17.2 Typical gas chromatograph  465
17.3 Signal processing and operations sequencing  468

18 Data Acquisition and Communication Systems  475
18.1 Time division multiplexing  476
18.2 Typical data acquisition system  477
18.3 Parallel digital signals  478
18.4 Serial digital signals  479
18.5 Error detection and correction  487
18.6 Frequency shift keying  490
18.7 Communication systems for measurement  493
19 The Intelligent Multivariable Measurement System
19.1 The structure of an intelligent multivariable system 503
19.2 Modelling methods for multivariable systems 507

Answers to Numerical Problems 515
Index 521
Measurement is an essential activity in every branch of technology and science. We need to know the speed of a car, the temperature of our working environment, the flow rate of liquid in a pipe, the amount of oxygen dissolved in river water. It is important, therefore, that the study of measurement forms part of engineering and science courses in further and higher education. The aim of this book is to provide the fundamental principles of measurement which underlie these studies.

The book treats measurement as a coherent and integrated subject by presenting it as the study of measurement systems. A measurement system is an information system which presents an observer with a numerical value corresponding to the variable being measured. A given system may contain four types of element: sensing, signal conditioning, signal processing and data presentation elements.

The book is divided into three parts. Part A (Chapters 1 to 7) examines general systems principles. This part begins by discussing the static and dynamic characteristics that individual elements may possess and how they are used to calculate the overall system measurement error, under both steady and unsteady conditions. In later chapters, the principles of loading and two-port networks, the effects of interference and noise on system performance, reliability, maintainability and choice using economic criteria are explained. Part B (Chapters 8 to 11) examines the principles, characteristics and applications of typical sensing, signal conditioning, signal processing and data presentation elements in wide current use. Part C (Chapters 12 to 19) examines a number of specialised measurement systems which have important industrial applications. These are flow measurement systems, intrinsically safe systems, heat transfer, optical, ultrasonic, gas chromatography, data acquisition, communication and intelligent multivariable systems.

The fourth edition has been substantially extended and updated to reflect new developments in, and applications of, technology since the third edition was published in 1995. Chapter 1 has been extended to include a wider range of examples of basic measurement systems. New material on solid state sensors has been included in Chapter 8; this includes resistive gas, electrochemical and Hall effect sensors. In Chapter 9 there is now a full analysis of operational amplifier circuits which are used in measurement systems. The section on frequency to digital conversion in Chapter 10 has been expanded; there is also new material on microcontroller structure, software and applications. Chapter 11 has been extensively updated with new material on digital displays, chart and paperless recorders and laser printers. The section on vortex flowmeters in Chapter 12 has been extended and updated. Chapter 19 is a new chapter on intelligent multivariable measurement systems which concentrates on structure and modelling methods. There are around 35 additional problems in this new edition; many of these are at a basic, introductory level.
Each chapter in the book is clearly divided into sections. The topics to be covered are introduced at the beginning and reviewed in a conclusion at the end. Basic and important equations are highlighted, and a number of references are given at the end of each chapter; these should provide useful supplementary reading. The book contains about 300 line diagrams and tables and about 140 problems. At the end of the book there are answers to all the numerical problems and a comprehensive index.

This book is primarily aimed at students taking modules in measurement and instrumentation as part of degree courses in instrumentation/control, mechanical, manufacturing, electrical, electronic, chemical engineering and applied physics. Much of the material will also be helpful to lecturers and students involved in HNC/HND and foundation degree courses in technology. The book should also be useful to professional engineers and technicians engaged in solving practical measurement problems.

I would like to thank academic colleagues, industrial contacts and countless students for their helpful comments and criticism over many years. Thanks are again especially due to my wife Pauline for her constant support and help with the preparation of the manuscript.

John P. Bentley
Guisborough, December 2003
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and Sons Ltd. (Bailey, P.L., 1976); Figure 8.25(c) from pH facts – the glass electrode in Kent Technical Review, Kent Industrial Measurements Ltd., E.I.L Analytical Instruments (Thompson, W.); Figure 8.26(a) from Electrical Engineering: Principles and Applications, 2nd edition, reprinted by permission of Pearson Education Inc., Upper Saddle River, NJ, USA (Hambley, A.R.); Table 10.5 from Appendix A, MCS BASIC-52 User’s Manual, Intel Corporation; Figure 11.10(c) from Instrumentation T 292 Block 6, part 2 Displays, 1986, The Open University Press; Figures 11.12(a) and (b) from Trident Displays technical literature on EL displays, Trident Microsystems Ltd. and M.S. Caddy and D. Weber; Figure 12.11(a) from Kent Process Control Ltd., Flow Products; Figure 15.10 from Optical Fibre Communications (Keiser, G., 1983), and Figure 15.12(b) from Measurement Systems: Application and Design (Doebelin, E.O., 1976), McGraw-Hill Book Co. (USA); Table 16.1 from Piezoelectric transducers in Methods of Experimental Physics, Vol. 19, Academic Press (O’Donnell, M., Busse, L.J. and Miller, J.G., 1981); Table 16.2 from Ultrasonics: Methods and Applications, Butterworth and Co. (Blitz, J., 1971); Figure 16.15 from ultrasonic image of Benjamin Stefan Morton, Nottingham City Hospital NHS Trust and Sarah Morton; Figure 17.5 from Process gas chromatography in Talanta 1967, Vol. 14, Pergamon Press Ltd. (Pine, C.S.F., 1967); and Figure 18.5.2 from Technical Information on Kent P4000 Telemetry Systems, 1985, Kent Automation Systems Ltd.

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1 The General Measurement System

1.1 Purpose and performance of measurement systems

We begin by defining a **process** as a system which generates **information**. Examples are a chemical reactor, a jet fighter, a gas platform, a submarine, a car, a human heart, and a weather system.

Table 1.1 lists **information variables** which are commonly generated by processes: thus a car generates displacement, velocity and acceleration variables, and a chemical reactor generates temperature, pressure and composition variables.

<table>
<thead>
<tr>
<th>Acceleration</th>
<th>Density</th>
</tr>
</thead>
<tbody>
<tr>
<td>Velocity</td>
<td>Viscosity</td>
</tr>
<tr>
<td>Displacement</td>
<td>Composition</td>
</tr>
<tr>
<td>Force–Weight</td>
<td>pH</td>
</tr>
<tr>
<td>Pressure</td>
<td>Humidity</td>
</tr>
<tr>
<td>Torque</td>
<td>Temperature</td>
</tr>
<tr>
<td>Volume</td>
<td>Heat/Light flux</td>
</tr>
<tr>
<td>Mass</td>
<td>Current</td>
</tr>
<tr>
<td>Flow rate</td>
<td>Voltage</td>
</tr>
<tr>
<td>Level</td>
<td>Power</td>
</tr>
</tbody>
</table>

We then define the **observer** as a person who needs this information from the process. This could be the car driver, the plant operator or the nurse.

The purpose of the **measurement system** is to link the observer to the process, as shown in Figure 1.1. Here the observer is presented with a number which is the current value of the information variable.

We can now refer to the information variable as a **measured variable**. The input to the measurement system is the **true value** of the variable; the system output is the **measured value** of the variable. In an ideal measurement system, the measured
value would be equal to the true value. The **accuracy** of the system can be defined as the closeness of the measured value to the true value. A perfectly accurate system is a theoretical ideal and the accuracy of a real system is quantified using **measurement system error** $E$, where

$$ E = \text{measured value} - \text{true value} $$

$$ E = \text{system output} - \text{system input} $$

Thus if the measured value of the flow rate of gas in a pipe is 11.0 m$^3$/h and the true value is 11.2 m$^3$/h, then the error $E = -0.2$ m$^3$/h. If the measured value of the rotational speed of an engine is 3140 rpm and the true value is 3133 rpm, then $E = +7$ rpm. Error is the main performance indicator for a measurement system. The procedures and equipment used to establish the true value of the measured variable will be explained in Chapter 2.

### 1.2 Structure of measurement systems

The measurement system consists of several elements or blocks. It is possible to identify four types of element, although in a given system one type of element may be missing or may occur more than once. The four types are shown in Figure 1.2 and can be defined as follows.

#### Sensing element

This is in contact with the process and gives an output which depends in some way on the variable to be measured. Examples are:

- Thermocouple where millivolt e.m.f. depends on temperature
- Strain gauge where resistance depends on mechanical strain
- Orifice plate where pressure drop depends on flow rate.

If there is more than one sensing element in a system, the element in contact with the process is termed the primary sensing element, the others secondary sensing elements.

#### Signal conditioning element

This takes the output of the sensing element and converts it into a form more suitable for further processing, usually a d.c. voltage, d.c. current or frequency signal. Examples are:

- Deflection bridge which converts an impedance change into a voltage change
- Amplifier which amplifies millivolts to volts
- Oscillator which converts an impedance change into a variable frequency voltage.
Signal processing element
This takes the output of the conditioning element and converts it into a form more suitable for presentation. Examples are:

- Analogue-to-digital converter (ADC) which converts a voltage into a digital form for input to a computer
- Computer which calculates the measured value of the variable from the incoming digital data.

Typical calculations are:

- Computation of total mass of product gas from flow rate and density data
- Integration of chromatograph peaks to give the composition of a gas stream
- Correction for sensing element non-linearity.

Data presentation element
This presents the measured value in a form which can be easily recognised by the observer. Examples are:

- Simple pointer–scale indicator
- Chart recorder
- Alphanumeric display
- Visual display unit (VDU).

Examples of measurement systems
Figure 1.3 shows some typical examples of measurement systems.

Figure 1.3(a) shows a temperature system with a thermocouple sensing element; this gives a millivolt output. Signal conditioning consists of a circuit to compensate for changes in reference junction temperature, and an amplifier. The voltage signal is converted into digital form using an analogue-to-digital converter, the computer corrects for sensor non-linearity, and the measured value is displayed on a VDU.

In Figure 1.3(b) the speed of rotation of an engine is sensed by an electromagnetic tachogenerator which gives an a.c. output signal with frequency proportional to speed. The Schmitt trigger converts the sine wave into sharp-edged pulses which are then counted over a fixed time interval. The digital count is transferred to a computer which calculates frequency and speed, and the speed is presented on a digital display.

The flow system of Figure 1.3(c) has an orifice plate sensing element; this gives a differential pressure output. The differential pressure transmitter converts this into a current signal and therefore combines both sensing and signal conditioning stages. The ADC converts the current into digital form and the computer calculates the flow rate, which is obtained as a permanent record on a chart recorder.

The weight system of Figure 1.3(d) has two sensing elements: the primary element is a cantilever which converts weight into strain; the strain gauge converts this into a change in electrical resistance and acts as a secondary sensor. There are two signal conditioning elements: the deflection bridge converts the resistance change into
Figure 1.3 Examples of measurement systems.
millivolts and the amplifier converts millivolts into volts. The computer corrects for non-linearity in the cantilever and the weight is presented on a digital display.

The word ‘transducer’ is commonly used in connection with measurement and instrumentation. This is a manufactured package which gives an output voltage (usually) corresponding to an input variable such as pressure or acceleration. We see therefore that such a transducer may incorporate both sensing and signal conditioning elements; for example a weight transducer would incorporate the first four elements shown in Figure 1.3(d).

It is also important to note that each element in the measurement system may itself be a system made up of simpler components. Chapters 8 to 11 discuss typical examples of each type of element in common use.

1.4 Block diagram symbols

A block diagram approach is very useful in discussing the properties of elements and systems. Figure 1.4 shows the main block diagram symbols used in this book.

![Figure 1.4 Block diagram symbols.](image)

**Conclusion**

This chapter has defined the purpose of a measurement system and explained the importance of **system error**. It has shown that, in general, a system consists of four types of element: **sensing**, **signal conditioning**, **signal processing** and **data presentation** elements. Typical examples have been given.
2 Static Characteristics of Measurement System Elements

In the previous chapter we saw that a measurement system consists of different types of element. The following chapters discuss the characteristics that typical elements may possess and their effect on the overall performance of the system. This chapter is concerned with static or steady-state characteristics; these are the relationships which may occur between the output \( O \) and input \( I \) of an element when \( I \) is either at a constant value or changing slowly (Figure 2.1).

### 2.1 Systematic characteristics

Systematic characteristics are those that can be exactly quantified by mathematical or graphical means. These are distinct from statistical characteristics which cannot be exactly quantified and are discussed in Section 2.3.

**Range**

The input range of an element is specified by the minimum and maximum values of \( I \), i.e. \( I_{\text{MIN}} \) to \( I_{\text{MAX}} \). The output range is specified by the minimum and maximum values of \( O \), i.e. \( O_{\text{MIN}} \) to \( O_{\text{MAX}} \). Thus a pressure transducer may have an input range of 0 to \( 1 \times 10^4 \) Pa and an output range of 4 to 20 mA; a thermocouple may have an input range of 100 to 250 °C and an output range of 4 to 10 mV.

**Span**

Span is the maximum variation in input or output, i.e. input span is \( I_{\text{MAX}} - I_{\text{MIN}} \) and output span is \( O_{\text{MAX}} - O_{\text{MIN}} \). Thus in the above examples the pressure transducer has an input span of \( 1 \times 10^4 \) Pa and an output span of 16 mA; the thermocouple has an input span of 150 °C and an output span of 6 mV.

**Ideal straight line**

An element is said to be linear if corresponding values of \( I \) and \( O \) lie on a straight line. The *ideal straight line* connects the minimum point \( A(I_{\text{MIN}}, O_{\text{MIN}}) \) to maximum point \( B(I_{\text{MAX}}, O_{\text{MAX}}) \) (Figure 2.2) and therefore has the equation:
10 STATIC CHARACTERISTICS OF MEASUREMENT SYSTEM ELEMENTS

\[ O - O_{\text{MIN}} = \left( \frac{O_{\text{MAX}} - O_{\text{MIN}}}{I_{\text{MAX}} - I_{\text{MIN}}} \right) (I - I_{\text{MIN}}) \]  

[2.1]

*Ideal straight line equation*

\[ O_{\text{IDEAL}} = KI + a \]  

[2.2]

where:

\[ K = \text{ideal straight-line slope} = \frac{O_{\text{MAX}} - O_{\text{MIN}}}{I_{\text{MAX}} - I_{\text{MIN}}} \]

and

\[ a = \text{ideal straight-line intercept} = O_{\text{MIN}} - KI_{\text{MIN}} \]

Thus the ideal straight line for the above pressure transducer is:

\[ O = 1.6 \times 10^{-3} I + 4.0 \]

The ideal straight line defines the ideal characteristics of an element. Non-ideal characteristics can then be quantified in terms of deviations from the ideal straight line.

**Non-linearity**

In many cases the straight-line relationship defined by eqn [2.2] is not obeyed and the element is said to be **non-linear**. Non-linearity can be defined (Figure 2.2) in terms of a function \( N(I) \) which is the difference between actual and ideal straight-line behaviour, i.e.

\[ N(I) = O(I) - (KI + a) \]  

[2.3]

or

\[ O(I) = KI + a + N(I) \]  

[2.4]

Non-linearity is often quantified in terms of the maximum non-linearity \( \bar{N} \) expressed as a percentage of full-scale deflection (f.s.d.), i.e. as a percentage of span. Thus:

\[ \text{Max. non-linearity as a percentage of f.s.d.} = \frac{\bar{N}}{O_{\text{MAX}} - O_{\text{MIN}}} \times 100\% \]  

[2.5]

---

**Figure 2.2** Definition of non-linearity.
As an example, consider a pressure sensor where the maximum difference between actual and ideal straight-line output values is 2 mV. If the output span is 100 mV, then the maximum percentage non-linearity is 2% of f.s.d.

In many cases $O(I)$ and therefore $N(I)$ can be expressed as a polynomial in $I$:

$$O(I) = a_0 + a_1 I + a_2 I^2 + \ldots + a_m I^m = \sum_{q=0}^{m} a_q I^q$$  \[2.6\]

An example is the temperature variation of the thermoelectric e.m.f. at the junction of two dissimilar metals. For a copper–constantan (Type T) thermocouple junction, the first four terms in the polynomial relating e.m.f. $E(T)$, expressed in $\mu$V, and junction temperature $T$ °C are:

$$E(T) = 38.74 T + 3.319 \times 10^{-2} T^2 + 2.071 \times 10^{-4} T^3$$
$$- 2.195 \times 10^{-6} T^4 + \text{higher-order terms up to } T^8$$  \[2.7a\]

for the range 0 to 400 °C.[1] Since $E = 0 \mu$V at $T = 0$ °C and $E = 20,869 \mu$V at $T = 400$ °C, the equation to the ideal straight line is:

$$E_{\text{IDEAL}} = 52.17T$$  \[2.7b\]

and the non-linear correction function is:

$$N(T) = E(T) - E_{\text{IDEAL}}$$
$$= -13.43 T + 3.319 \times 10^{-2} T^2 + 2.071 \times 10^{-4} T^3$$
$$- 2.195 \times 10^{-6} T^4 + \text{higher-order terms}$$  \[2.7c\]

In some cases expressions other than polynomials are more appropriate: for example the resistance $R(T)$ ohms of a thermistor at $T$ °C is given by:

$$R(T) = 0.04 \exp \left( \frac{3300}{T + 273} \right)$$  \[2.8\]

**Sensitivity**

This is the change $\Delta O$ in output $O$ for unit change $\Delta I$ in input $I$, i.e. it is the ratio $\Delta O/\Delta I$. In the limit that $\Delta I$ tends to zero, the ratio $\Delta O/\Delta I$ tends to the derivative $dO/dI$, which is the rate of change of $O$ with respect to $I$. For a linear element $dO/dI$ is equal to the slope or gradient $K$ of the straight line; for the above pressure transducer the sensitivity is $1.6 \times 10^{-3}$ mA/Pa. For a non-linear element $dO/dI = K + dN/dI$, i.e. sensitivity is the **slope or gradient** of the output versus input characteristics $O(I)$. Figure 2.3 shows the e.m.f. versus temperature characteristics $E(T)$ for a Type T thermocouple (eqn [2.7a]). We see that the gradient and therefore the sensitivity vary with temperature: at 100 °C it is approximately 35 $\mu$V/°C and at 200 °C approximately 42 $\mu$V/°C.

**Environmental effects**

In general, the output $O$ depends not only on the signal input $I$ but on environmental inputs such as ambient temperature, atmospheric pressure, relative humidity, supply voltage, etc. Thus if eqn [2.4] adequately represents the behaviour of the element under ‘standard’ environmental conditions, e.g. 20 °C ambient temperature,
1000 millibars atmospheric pressure, 50% RH and 10 V supply voltage, then the equation must be modified to take account of deviations in environmental conditions from ‘standard’. There are two main types of environmental input.

A **modifying** input $I_M$ causes the linear sensitivity of an element to change. $K$ is the sensitivity at standard conditions when $I_M = 0$. If the input is changed from the standard value, then $I_M$ is the deviation from standard conditions, i.e. (new value – standard value). The sensitivity changes from $K$ to $K + K_M I_M$, where $K_M$ is the change in sensitivity for unit change in $I_M$. Figure 2.4(a) shows the modifying effect of ambient temperature on a linear element.

An **interfering** input $I_I$ causes the straight line intercept or zero bias to change. $a$ is the zero bias at standard conditions when $I_I = 0$. If the input is changed from the standard value, then $I_I$ is the deviation from standard conditions, i.e. (new value – standard value). The zero bias changes from $a$ to $a + K_I I_I$, where $K_I$ is the change in zero bias for unit change in $I_I$. Figure 2.4(b) shows the interfering effect of ambient temperature on a linear element.

$K_M$ and $K_I$ are referred to as environmental coupling constants or sensitivities. Thus we must now correct eqn [2.4], replacing $K_I$ with $(K + K_M I_M) I_I$ and replacing $a$ with $a + K_I I_I$, to give:

$$O = KI + a + N(I) + K_M I_M I_I + K_I I_I$$  \[2.9\]
An example of a modifying input is the variation $\Delta V_S$ in the supply voltage $V_S$ of the potentiometric displacement sensor shown in Figure 2.5. An example of an interfering input is provided by variations in the reference junction temperature $T_2$ of the thermocouple (see following section and Section 8.5).

**Hysteresis**

For a given value of $I$, the output $O$ may be different depending on whether $I$ is increasing or decreasing. Hysteresis is the difference between these two values of $O$ (Figure 2.6), i.e.

$$H(I) = O(I)_\downarrow - O(I)_\uparrow$$  \[2.10\]

Again hysteresis is usually quantified in terms of the maximum hysteresis $\hat{H}$ expressed as a percentage of f.s.d., i.e. span. Thus:

$$\text{Maximum hysteresis as a percentage of f.s.d.} = \frac{\hat{H}}{O_{\text{MAX}} - O_{\text{MIN}}} \times 100\%$$  \[2.11\]

A simple gear system (Figure 2.7) for converting linear movement into angular rotation provides a good example of hysteresis. Due to the ‘backlash’ or ‘play’ in the gears the angular rotation $\theta$, for a given value of $x$, is different depending on the direction of the linear movement.

**Resolution**

Some elements are characterised by the output increasing in a series of discrete steps or jumps in response to a continuous increase in input (Figure 2.8). Resolution is defined as the largest change in $I$ that can occur without any corresponding change in $O$. 

\[ O_{\text{OUT}} = (V_S + \Delta V_S)x = V_Sx + \Delta V_Sx \]
Thus in Figure 2.8 resolution is defined in terms of the width $\Delta I_R$ of the widest step; resolution expressed as a percentage of f.s.d. is thus:

$$\frac{\Delta I_R}{I_{\text{MAX}} - I_{\text{MIN}}} \times 100\%$$

A common example is a wire-wound potentiometer (Figure 2.8); in response to a continuous increase in $x$ the resistance $R$ increases in a series of steps, the size of each step being equal to the resistance of a single turn. Thus the resolution of a 100 turn potentiometer is 1%. Another example is an analogue-to-digital converter (Chapter 10); here the output digital signal responds in discrete steps to a continuous increase in input voltage; the resolution is the change in voltage required to cause the output code to change by the least significant bit.

**Wear and ageing**

These effects can cause the characteristics of an element, e.g. $K$ and $a$, to change slowly but systematically throughout its life. One example is the stiffness of a spring $k(t)$ decreasing slowly with time due to wear, i.e.

$$k(t) = k_0 - bt$$  \[2.12\]

where $k_0$ is the initial stiffness and $b$ is a constant. Another example is the constants $a_1$, $a_2$, etc. of a thermocouple, measuring the temperature of gas leaving a cracking furnace, changing systematically with time due to chemical changes in the thermocouple metals.

**Error bands**

Non-linearity, hysteresis and resolution effects in many modern sensors and transducers are so small that it is difficult and not worthwhile to exactly quantify each individual effect. In these cases the manufacturer defines the performance of the element in terms of error bands (Figure 2.9). Here the manufacturer states that for any value of $I$, the output $O$ will be within $\pm h$ of the ideal straight-line value $O_{\text{IDEAL}}$. Here an exact or systematic statement of performance is replaced by a statistical statement in terms of a probability density function $p(O)$. In general a probability density function $p(x)$ is defined so that the integral $\int_{x_1}^{x_2} p(x) \, dx$ (equal to the area under the curve in Figure 2.10 between $x_1$ and $x_2$) is the probability $P_{x_1,x_2}$ of $x$ lying between $x_1$ and $x_2$ (Section 6.2). In this case the probability density function is rectangular (Figure 2.9), i.e.
2.2 Generalised model of a system element

We note that the area of the rectangle is equal to unity: this is the probability of $O$ lying between $O_{\text{ideal}} - h$ and $O_{\text{ideal}} + h$.

\[
p(O) = \begin{cases} 
\frac{1}{2h} & O_{\text{ideal}} - h \leq O \leq O_{\text{ideal}} + h \\
0 & O > O_{\text{ideal}} + h \\
0 & O_{\text{ideal}} - h > O 
\end{cases}
\]  

[2.13]

We note that the area of the rectangle is equal to unity; this is the probability of $O$ lying between $O_{\text{ideal}} - h$ and $O_{\text{ideal}} + h$.

### 2.2 Generalised model of a system element

If hysteresis and resolution effects are not present in an element but environmental and non-linear effects are, then the steady-state output $O$ of the element is in general given by eqn [2.9], i.e.:

\[
O = K_I + a + N(I) + K_{ah}I_0I_a + K_{ii}I_i
\]  

[2.9]

Figure 2.11 shows this equation in block diagram form to represent the static characteristics of an element. For completeness the diagram also shows the transfer function $G(s)$, which represents the dynamic characteristics of the element. The meaning of transfer function will be explained in Chapter 4 where the form of $G(s)$ for different elements will be derived.

Examples of this general model are shown in Figure 2.12(a), (b) and (c), which summarise the static and dynamic characteristics of a strain gauge, thermocouple and accelerometer respectively.
Figure 2.11 General model of element.

![Figure 2.11 General model of element.](image)

Figure 2.12 Examples of element characteristics:
(a) Strain gauge
(b) Copper–constantan thermocouple
(c) Accelerometer.

![Figure 2.12 Examples of element characteristics.](image)
2.3 Statistical characteristics

2.3.1 Statistical variations in the output of a single element with time – repeatability

Suppose that the input $I$ of a single element, e.g. a pressure transducer, is held constant, say at 0.5 bar, for several days. If a large number of readings of the output $O$ are taken, then the expected value of 1.0 volt is not obtained on every occasion; a range of values such as 0.99, 1.01, 1.00, 1.02, 0.98, etc., scattered about the expected value, is obtained. This effect is termed a lack of repeatability in the element. Repeatability is the ability of an element to give the same output for the same input, when repeatedly applied to it. Lack of repeatability is due to random effects in the element and its environment. An example is the vortex flowmeter (Section 12.2.4): for a fixed flow rate $Q = 1.4 \times 10^{-2} \text{ m}^3 \text{ s}^{-1}$, we would expect a constant frequency output $f = 209 \text{ Hz}$. Because the output signal is not a perfect sine wave, but is subject to random fluctuations, the measured frequency varies between 207 and 211 Hz.

The most common cause of lack of repeatability in the output $O$ is random fluctuations with time in the environmental inputs $I_{\mu}, I_{\iota}$; if the coupling constants $K_{\mu}, K_{\iota}$ are non-zero, then there will be corresponding time variations in $O$. Thus random fluctuations in ambient temperature cause corresponding time variations in the resistance of a strain gauge or the output voltage of an amplifier; random fluctuations in the supply voltage of a deflection bridge affect the bridge output voltage.

By making reasonable assumptions for the probability density functions of the inputs $I, I_{\mu}$ and $I_{\iota}$ (in a measurement system random variations in the input $I$ to
a given element can be caused by random effects in the previous element), the probability density function of the element output $O$ can be found. The most likely probability density function for $I$, $I_M$ and $I_i$ is the normal or Gaussian distribution function (Figure 2.13):

$$p(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(x - \bar{x})^2}{2\sigma^2} \right]$$  \hspace{1cm} [2.14]

where: $\bar{x} = \text{mean or expected value (specifies centre of distribution)}$

$\sigma = \text{standard deviation (specifies spread of distribution)}$

Equation [2.9] expresses the independent variable $O$ in terms of the independent variables $I$, $I_M$ and $I_i$. Thus if $\Delta O$ is a small deviation in $O$ from the mean value $\bar{O}$, caused by deviations $\Delta I$, $\Delta I_M$ and $\Delta I_i$ from respective mean values $I$, $I_M$ and $I_i$, then:

$$\Delta O = \left( \frac{\partial O}{\partial I} \right) \Delta I + \left( \frac{\partial O}{\partial I_M} \right) \Delta I_M + \left( \frac{\partial O}{\partial I_i} \right) \Delta I_i$$  \hspace{1cm} [2.15]

Thus $\Delta O$ is a linear combination of the variables $\Delta I$, $\Delta I_M$ and $\Delta I_i$; the partial derivatives can be evaluated using eqn [2.9]. It can be shown that if a dependent variable $y$ is a linear combination of independent variables $x_1$, $x_2$ and $x_3$, i.e.

$$y = a_1 x_1 + a_2 x_2 + a_3 x_3$$  \hspace{1cm} [2.16]

and if $x_1$, $x_2$ and $x_3$ have normal distributions with standard deviations $\sigma_1$, $\sigma_2$ and $\sigma_3$ respectively, then the probability distribution of $y$ is also normal with standard deviation $\sigma$ given by:

$$\sigma = \sqrt{a_1^2 \sigma_1^2 + a_2^2 \sigma_2^2 + a_3^2 \sigma_3^2}$$  \hspace{1cm} [2.17]

From eqns [2.15] and [2.17] we see that the standard deviation of $\Delta O$, i.e. of $O$ about mean $\bar{O}$, is given by:
2.3 Statistical Characteristics

Standard deviation of output for a single element

\[
\sigma_0 = \left( \frac{\partial O}{\partial I} \right)^2 \sigma_I^2 + \left( \frac{\partial O}{\partial I_M} \right)^2 \sigma_{I_M}^2 + \left( \frac{\partial O}{\partial I_I} \right)^2 \sigma_{I_I}^2
\]  
[2.18]

where \( \sigma_I, \sigma_{I_M} \) and \( \sigma_{I_I} \) are the standard deviations of the inputs. Thus \( \sigma_0 \) can be calculated using eqn [2.18] if \( \sigma_I, \sigma_{I_M} \) and \( \sigma_{I_I} \) are known; alternatively if a calibration test (see following section) is being performed on the element then \( \sigma_0 \) can be estimated directly from the experimental results. The corresponding mean or expected value \( \bar{O} \) of the element output is given by:

Mean value of output for a single element

\[
\bar{O} = K \bar{I} + a + N(\bar{I}) + K_M \bar{I}_M + K_I \bar{I}_I
\]  
[2.19]

and the corresponding probability density function is:

\[
p(O) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp\left[ -\frac{(O - \bar{O})^2}{2\sigma_0^2} \right]
\]  
[2.20]

2.3.2 Statistical variations amongst a batch of similar elements – tolerance

Suppose that a user buys a batch of similar elements, e.g. a batch of 100 resistance temperature sensors, from a manufacturer. If he then measures the resistance \( R_0 \) of each sensor at 0 °C he finds that the resistance values are not all equal to the manufacturer’s quoted value of 100.0 Ω. A range of values such as 99.8, 100.1, 99.9, 100.0 and 100.2 Ω, distributed statistically about the quoted value, is obtained. This effect is due to small random variations in manufacture and is often well represented by the normal probability density function given earlier. In this case we have:

\[
p(R_0) = \frac{1}{\sigma_{R_0} \sqrt{2\pi}} \exp\left[ -\frac{(R_0 - \bar{R}_0)^2}{2\sigma_{R_0}^2} \right]
\]  
[2.21]

where \( \bar{R}_0 \) = mean value of distribution = 100 Ω and \( \sigma_{R_0} \) = standard deviation, typically 0.1 Ω. However, a manufacturer may state in his specification that \( R_0 \) lies within ±0.15 Ω of 100 Ω for all sensors, i.e. he is quoting tolerance limits of ±0.15 Ω. Thus in order to satisfy these limits he must reject for sale all sensors with \( R_0 < 99.85 \) Ω and \( R_0 > 100.15 \) Ω, so that the probability density function of the sensors bought by the user now has the form shown in Figure 2.14.

The user has two choices:

(a) He can design his measurement system using the manufacturer’s value of \( R_0 = 100.0 \) Ω and accept that any individual system, with \( R_0 = 100.1 \) Ω say, will have a small measurement error. This is the usual practice.

(b) He can perform a calibration test to measure \( R_0 \) as accurately as possible for each element in the batch. This theoretically removes the error due to uncertainty in \( R_0 \) but is time-consuming and expensive. There is also a small remaining uncertainty in the value of \( R_0 \) due to the limited accuracy of the calibration equipment.
This effect is found in any batch of ‘identical’ elements; significant variations are found in batches of thermocouples and thermistors, for example. In the general case we can say that the values of parameters, such as linear sensitivity $K$ and zero bias $a$, for a batch of elements are distributed statistically about mean values $\bar{K}$ and $\bar{a}$.

### 2.3.3 Summary

In the general case of a batch of several ‘identical’ elements, where each element is subject to random variations in environmental conditions with time, both inputs $I, I_M$ and $I_I$ and parameters $K, a$, etc., are subject to statistical variations. If we assume that each statistical variation can be represented by a normal probability density function, then the probability density function of the element output $O$ is also normal, i.e.:

$$p(O) = \frac{1}{\sigma_0 \sqrt{2\pi}} \exp \left( -\frac{(O - \bar{O})^2}{2\sigma_0^2} \right)$$  \hspace{1cm} [2.20]$$

where the mean value $\bar{O}$ is given by:

$$\bar{O} = \bar{K}I + \bar{N}(I) + \bar{a} + \bar{R}_M I_M I + \bar{R}_I I_I$$  \hspace{1cm} [2.22]$$

and the standard deviation $\sigma_0$ is given by:

$$\sigma_0 = \sqrt{\left( \frac{\partial O}{\partial I} \right)^2 \sigma_I^2 + \left( \frac{\partial O}{\partial I_M} \right)^2 \sigma_{I_M}^2 + \left( \frac{\partial O}{\partial I_I} \right)^2 \sigma_{I_I}^2 + \left( \frac{\partial O}{\partial K} \right)^2 \sigma_K^2 + \left( \frac{\partial O}{\partial a} \right)^2 \sigma_a^2 + \ldots}$$  \hspace{1cm} [2.23]$$

Tables 2.1 and 2.2 summarise the static characteristics of a chromel–alumel thermocouple and a millivolt to current temperature transmitter. The thermocouple is
2.4 Identification of static characteristics – calibration

2.4.1 Standards

The static characteristics of an element can be found experimentally by measuring corresponding values of the input $I$, the output $O$, and the environmental inputs $I_M$ and $I_n$ when $I$ is either at a constant value or changing slowly. This type of experiment is referred to as calibration, and the measurement of the variables $I$, $O$, $I_M$ and $I_n$ must be accurate if meaningful results are to be obtained. The instruments and techniques used to quantify these variables are referred to as standards (Figure 2.15).
The accuracy of a measurement of a variable is the closeness of the measurement to the true value of the variable. It is quantified in terms of measurement error, i.e. the difference between the measured value and the true value (Chapter 3). Thus the accuracy of a laboratory standard pressure gauge is the closeness of the reading to the true value of pressure. This brings us back to the problem, mentioned in the previous chapter, of how to establish the true value of a variable. We define the true value of a variable as the measured value obtained with a standard of ultimate accuracy. Thus the accuracy of the above pressure gauge is quantified by the difference between the gauge reading, for a given pressure, and the reading given by the ultimate pressure standard. However, the manufacturer of the pressure gauge may not have access to the ultimate standard to measure the accuracy of his products.

In the United Kingdom the manufacturer is supported by the National Measurement System. Ultimate or primary measurement standards for key physical variables such as time, length, mass, current and temperature are maintained at the National Physical Laboratory (NPL). Primary measurement standards for other important industrial variables such as the density and flow rate of gases and liquids are maintained at the National Engineering Laboratory (NEL). In addition there is a network of laboratories and centres throughout the country which maintain transfer or intermediate standards. These centres are accredited by UKAS (United Kingdom Accreditation Service). Transfer standards held at accredited centres are calibrated against national primary and secondary standards, and a manufacturer can calibrate his products against the transfer standard at a local centre. Thus the manufacturer of pressure gauges can calibrate his products against a transfer standard, for example a deadweight tester. The transfer standard is in turn calibrated against a primary or secondary standard, for example a pressure balance at NPL. This introduces the concept of a traceability ladder, which is shown in simplified form in Figure 2.16.

The element is calibrated using the laboratory standard, which should itself be calibrated using the transfer standard, and this in turn should be calibrated using the primary standard. Each element in the ladder should be significantly more accurate than the one below it.

NPL are currently developing an Internet calibration service. This will allow an element at a remote location (for example at a user’s factory) to be calibrated directly against a national primary or secondary standard without having to be transported to NPL. The traceability ladder is thereby collapsed to a single link between element and national standard. The same input must be applied to element and standard. The measured value given by the standard instrument is then the true value of the input to the element, and this is communicated to the user via the Internet. If the user measures the output of the element for a number of true values of input, then the characteristics of the element can be determined to a known, high accuracy.
Having introduced the concepts of standards and traceability we can now discuss different types of standards in more detail. The International System of Units (SI) comprises seven base units, which are listed and defined in Table 2.3. The units of all physical quantities can be derived from these base units. Table 2.4 lists common physical quantities and shows the derivation of their units from the base units. In the United Kingdom the National Physical Laboratory (NPL) is responsible for the

<table>
<thead>
<tr>
<th>Physical Quantity</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time:</td>
<td>second (s)</td>
<td>The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium-133 atom.</td>
</tr>
<tr>
<td>Length:</td>
<td>metre (m)</td>
<td>The metre is the length of the path travelled by light in vacuum during a time interval of 1/299 792 458 of a second.</td>
</tr>
<tr>
<td>Mass:</td>
<td>kilogram (kg)</td>
<td>The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.</td>
</tr>
<tr>
<td>Electric current:</td>
<td>ampere (A)</td>
<td>The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to 2 × 10⁻⁷ newton per metre of length.</td>
</tr>
<tr>
<td>Thermodynamic temperature:</td>
<td>kelvin (K)</td>
<td>The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.</td>
</tr>
<tr>
<td>Amount of substance:</td>
<td>mole (mol)</td>
<td>The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon-12.</td>
</tr>
<tr>
<td>Luminous intensity:</td>
<td>candela (cd)</td>
<td>The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540 × 10¹² hertz and that has a radiant intensity in that direction of (1/683) watt per steradian.</td>
</tr>
</tbody>
</table>
### Examples of SI derived units expressed in terms of base units

<table>
<thead>
<tr>
<th>Quantity</th>
<th>SI unit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Name</strong></td>
<td><strong>Symbol</strong></td>
</tr>
<tr>
<td>area</td>
<td>square metre</td>
</tr>
<tr>
<td>volume</td>
<td>cubic metre</td>
</tr>
<tr>
<td>speed, velocity</td>
<td>metre per second</td>
</tr>
<tr>
<td>acceleration</td>
<td>metre per second squared</td>
</tr>
<tr>
<td>wave number</td>
<td>1 per metre</td>
</tr>
<tr>
<td>density, mass density</td>
<td>kilogram per cubic metre</td>
</tr>
<tr>
<td>specific volume</td>
<td>cubic metre per kilogram</td>
</tr>
<tr>
<td>current density</td>
<td>ampere per square metre</td>
</tr>
<tr>
<td>magnetic field strength</td>
<td>ampere per metre</td>
</tr>
<tr>
<td>concentration (of amount of substance)</td>
<td>mole per cubic metre</td>
</tr>
<tr>
<td>luminance</td>
<td>candela per square metre</td>
</tr>
</tbody>
</table>

### SI derived units with special names

<table>
<thead>
<tr>
<th>Quantity</th>
<th>SI unit</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Name</strong></td>
<td><strong>Symbol</strong></td>
</tr>
<tr>
<td>plane angleb</td>
<td>radian</td>
</tr>
<tr>
<td>solid angleb</td>
<td>steradian</td>
</tr>
<tr>
<td>frequency</td>
<td>hertz</td>
</tr>
<tr>
<td>force</td>
<td>newton</td>
</tr>
<tr>
<td>pressure, stress</td>
<td>pascal</td>
</tr>
<tr>
<td>energy, work quantity of heat</td>
<td>joule</td>
</tr>
<tr>
<td>power, radiant flux</td>
<td>watt</td>
</tr>
<tr>
<td>electric charge, quantity of electricity</td>
<td>coulomb</td>
</tr>
<tr>
<td>electric potential, potential difference, electromotive force</td>
<td>volt</td>
</tr>
<tr>
<td>capacitance</td>
<td>farad</td>
</tr>
<tr>
<td>electric resistance</td>
<td>ohm</td>
</tr>
<tr>
<td>electric conductance</td>
<td>siemens</td>
</tr>
<tr>
<td>magnetic flux</td>
<td>weber</td>
</tr>
<tr>
<td>magnetic flux density</td>
<td>tesla</td>
</tr>
<tr>
<td>inductance</td>
<td>henry</td>
</tr>
<tr>
<td>Celsius temperature</td>
<td>degree Celsius</td>
</tr>
<tr>
<td>luminous flux</td>
<td>lumen</td>
</tr>
<tr>
<td>illuminance</td>
<td>lux</td>
</tr>
<tr>
<td>activity (of a radionuclide)</td>
<td>becquerel</td>
</tr>
<tr>
<td>absorbed dose, specific energy imparted, kerma</td>
<td>gray</td>
</tr>
<tr>
<td>dose equivalent</td>
<td>sievert</td>
</tr>
</tbody>
</table>
2.4 IDENTIFICATION OF STATIC CHARACTERISTICS – CALIBRATION

The NPL is therefore the custodian of ultimate or primary standards in the UK. There are secondary standards held at United Kingdom Accreditation Service (UKAS) centres. These have been calibrated against NPL standards and are available to calibrate transfer standards.

Examples of SI derived units expressed by means of special names

<table>
<thead>
<tr>
<th>Quantity</th>
<th>SI unit</th>
<th>Name</th>
<th>Symbol</th>
<th>Expression in terms of SI base units</th>
</tr>
</thead>
<tbody>
<tr>
<td>dynamic viscosity</td>
<td>pascal second</td>
<td>Pa s</td>
<td>m−1 kg s−1</td>
<td></td>
</tr>
<tr>
<td>moment of force</td>
<td>newton metre</td>
<td>N m</td>
<td>m2 kg s−2</td>
<td></td>
</tr>
<tr>
<td>surface tension</td>
<td>newton per metre</td>
<td>N/m</td>
<td>kg s−2</td>
<td></td>
</tr>
<tr>
<td>heat flux density, irradiance</td>
<td>watt per square metre</td>
<td>W/m²</td>
<td>kg s−3</td>
<td></td>
</tr>
<tr>
<td>heat capacity, entropy specific</td>
<td>joule per kelvin</td>
<td>J/K</td>
<td>m2 kg s−2 K−1</td>
<td></td>
</tr>
<tr>
<td>energy density</td>
<td>joule per kilogram</td>
<td>J/kg</td>
<td>m2 s−2</td>
<td></td>
</tr>
<tr>
<td>thermal conductivity</td>
<td>watt per metre kelvin</td>
<td>W/(m K)</td>
<td>m kg s−3 K−1</td>
<td></td>
</tr>
<tr>
<td>electric field strength</td>
<td>joule per cubic metre</td>
<td>J/m³</td>
<td>m−1 kg s−2</td>
<td></td>
</tr>
<tr>
<td>electric charge density</td>
<td>coulomb per cubic metre</td>
<td>C/m³</td>
<td>m−3 s A</td>
<td></td>
</tr>
<tr>
<td>electric flux density</td>
<td>coulomb per square metre</td>
<td>C/m²</td>
<td>m−2 s A</td>
<td></td>
</tr>
<tr>
<td>permittivity</td>
<td>farad per metre</td>
<td>F/m</td>
<td>m−3 kg−1 s4 A2</td>
<td></td>
</tr>
<tr>
<td>permeability</td>
<td>henry per metre</td>
<td>H/m</td>
<td>m kg s−2 A−2</td>
<td></td>
</tr>
<tr>
<td>molar energy</td>
<td>joule per mole</td>
<td>J/mol</td>
<td>m2 kg s−2 mol−1</td>
<td></td>
</tr>
<tr>
<td>molar entropy, molar heat capacity</td>
<td>joule per mole kelvin</td>
<td>J/(mol K)</td>
<td>m2 kg s−2 K−1 mol−1</td>
<td></td>
</tr>
<tr>
<td>exposure (X and γ rays)</td>
<td>coulomb per kilogram</td>
<td>C/kg</td>
<td>kg−1 s A</td>
<td></td>
</tr>
<tr>
<td>absorbed dose rate</td>
<td>gray per second</td>
<td>Gy/s</td>
<td>m² s−3</td>
<td></td>
</tr>
</tbody>
</table>

Examples of SI derived units formed by using the radian and steradian

<table>
<thead>
<tr>
<th>Quantity</th>
<th>SI unit</th>
<th>Name</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>angular velocity</td>
<td>radian per second</td>
<td>rad/s</td>
<td></td>
</tr>
<tr>
<td>angular acceleration</td>
<td>radian per second squared</td>
<td>rad/s²</td>
<td></td>
</tr>
<tr>
<td>radiant intensity</td>
<td>watt per steradian</td>
<td>W/sr</td>
<td></td>
</tr>
<tr>
<td>radiance</td>
<td>watt per square metre steradian</td>
<td>W m² sr⁻¹</td>
<td></td>
</tr>
</tbody>
</table>

Notes:

a Acceptable forms are, for example, m · kg · s⁻², m kg s⁻²; m/s, m/s or m · s⁻¹.
b The CIPM (1995) decided that the radian and steradian should henceforth be designated as dimensionless derived units.

At NPL, the metre is realised using the wavelength of the 633 nm radiation from an iodine-stabilised helium–neon laser. The reproducibility of this primary standard
**Figure 2.17** Traceability ladders: (a) length (adapted from Scarr\(^{[5]}\))
(b) mass (reprinted by permission of the Council of the Institution of Mechanical Engineers from Hayward\(^{[6]}\)).

<table>
<thead>
<tr>
<th>Responsibility</th>
<th>Accuracy</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>BIPM and NPL</td>
<td>(3 \times 10^{11})</td>
<td>Wavelength of 633 nm radiation He-Ne laser</td>
</tr>
<tr>
<td>NPL</td>
<td>(1 \times 10^{7})</td>
<td>Wavelength of secondary laser sources</td>
</tr>
<tr>
<td>NPL or UKAS or industry</td>
<td>(1 \times 10^{6})</td>
<td>Laser interferometric calibration of reference quality length standards</td>
</tr>
<tr>
<td>UKAS or industry</td>
<td>(1 \times 10^{5})</td>
<td>Comparative calibration of working quality length standards</td>
</tr>
<tr>
<td>UKAS or industry</td>
<td>(1 \times 10^{4})</td>
<td>Calibration of gauges and measuring equipment</td>
</tr>
<tr>
<td>BIPM – International Bureau of Weights and Measures</td>
<td></td>
<td></td>
</tr>
<tr>
<td>NPL – National Physical Laboratory</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UKAS – United Kingdom Accreditation Service</td>
<td></td>
<td></td>
</tr>
<tr>
<td>The international kilogram (Paris)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>National primary standard kg (London)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>National secondary standard weight (London)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>National transfer standard weight (portable)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Local standard weight (held by weights and measures dept)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Your own laboratory standard weight</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Your weighing instrument</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

is about 3 parts in \(10^{11}\), and the wavelength of the radiation has been accurately related to the definition of the metre in terms of the velocity of light. The primary standard is used to calibrate secondary laser interferometers which are in turn used to calibrate precision length bars, gauges and tapes. A simplified traceability ladder for length\(^{[5]}\) is shown in Figure 2.17(a).

The international prototype of the **kilogram** is made of platinum–iridium and is kept at the International Bureau of Weights and Measures (BIPM) in Paris. The British national copy is kept at NPL and is used in conjunction with a precision balance to calibrate secondary and transfer kilogram standards. Figure 2.17(b) shows a simplified traceability ladder for mass and weight.\(^{[6]}\) The weight of a mass \(m\) is the force \(mg\) it experiences under the acceleration of gravity \(g\); thus if the local value of \(g\) is known accurately, then a force standard can be derived from mass standards.

At NPL deadweight machines covering a range of forces from 450 N to 30 MN are used to calibrate strain-gauge load cells and other weight transducers.

The **second** is realised by caesium beam standards to about 1 part in \(10^{13}\); this is equivalent to one second in 300 000 years! A uniform timescale, synchronised to 0.1 microsecond, is available worldwide by radio transmissions; this includes satellite broadcasts.

The **ampere** has traditionally been the electrical base unit and has been realised at NPL using the Ayrton-Jones current balance; here the force between two current-carrying coils is balanced by a known weight. The accuracy of this method is limited by the large deadweight of the coils and formers and the many length measurements necessary. For this reason the two electrical base units are now chosen to be the **farad** and the **volt** (or watt); the other units such as the ampere, ohm, henry and joule are derived from these two base units with time or frequency units, using...
Ohm’s law where necessary. The farad is realised using a calculable capacitor based on the Thompson–Lampard theorem. Using a.c. bridges, capacitance and frequency standards can then be used to calibrate standard resistors. The primary standard for the volt is based on the Josephson effect in superconductivity; this is used to calibrate secondary voltage standards, usually saturated Weston cadmium cells. The ampere can then be realised using a modified current balance. As before, the force due to a current $I$ is balanced by a known weight $mg$, but also a separate measurement is made of the voltage $e$ induced in the coil when moving with velocity $u$. Equating electrical and mechanical powers gives the simple equation:

$$el = mgu$$

Accurate measurements of $m$, $u$ and $e$ can be made using secondary standards traceable back to the primary standards of the kilogram, metre, second and volt.

Ideally temperature should be defined using the thermodynamic scale, i.e. the relationship $PV = R\theta$ between the pressure $P$ and temperature $\theta$ of a fixed volume $V$ of an ideal gas. Because of the limited reproducibility of real gas thermometers the International Practical Temperature Scale (IPTS) was devised. This consists of:

(a) several highly reproducible fixed points corresponding to the freezing, boiling or triple points of pure substances under specified conditions;
(b) standard instruments with a known output versus temperature relationship obtained by calibration at fixed points.

The instruments interpolate between the fixed points.

The numbers assigned to the fixed points are such that there is exactly 100 K between the freezing point (273.15 K) and boiling point (373.15 K) of water. This means that a change of 1 K is equal to a change of 1 °C on the older Celsius scale. The exact relationship between the two scales is:

$$\theta \text{ K} = T \text{ °C} + 273.15$$

Table 2.5 shows the primary fixed points, and the temperatures assigned to them, for the 1990 version of the International Temperature Scale – ITS90. In addition there are primary standard instruments which interpolate between these fixed points. Platinum resistance detectors (PRTD – Section 8.1) are used up to the freezing point of silver (961.78 °C) and radiation pyrometers (Section 15.6) are used at higher temperatures. The resistance $R_T \Omega$ of a PRTD at temperature $T \text{ °C}$ (above 0 °C) can be specified by the quadratic equation

$$R_T = R_0(1 + \alpha T + \beta T^2)$$

where $R_0$ is the resistance at 0 °C and $\alpha$, $\beta$ are constants. By measuring the resistance at three adjacent fixed points (e.g. water, gallium and indium), $R_0$, $\alpha$ and $\beta$ can be calculated. An interpolating equation can then be found for the standard, which relates $R_T$ to $T$ and is valid between the water and indium fixed points. This primary standard PRTD, together with its equation, can then be used to calibrate secondary and transfer standards, usually PRTDs or thermocouples depending on temperature range.

The standards available for the base quantities, i.e. length, mass, time, current and temperature, enable standards for derived quantities to be realised. This is illustrated in the methods for calibrating liquid flowmeters. The actual flow rate through the meter is found by weighing the amount of water collected in a given time, so that the accuracy of the flow rate standard depends on the accuracies of weight and time standards.
National primary and secondary standards for pressure above 110 kPa use pressure balances. Here the force $pA$ due to pressure $p$ acting over an area $A$ is balanced by the gravitational force $mg$ acting on a mass $m$, i.e.

$$pA = mg$$

or

$$p = mg/A$$

Thus standards for pressure can be derived from those for mass and length, though the local value of gravitational acceleration $g$ must also be accurately known.

### 2.4.3 Experimental measurements and evaluation of results

The calibration experiment is divided into three main parts.

**$O$ versus $I$ with $I_M = I_N = 0$**

Ideally this test should be held under ‘standard’ environmental conditions so that $I_M = I_N = 0$; if this is not possible all environmental inputs should be measured. $I$ should be increased slowly from $I_{\text{MIN}}$ to $I_{\text{MAX}}$ and corresponding values of $I$ and $O$ recorded at intervals of 10% span (i.e. 11 readings), allowing sufficient time for the output to settle out before taking each reading. A further 11 pairs of readings should be taken with $I$ decreasing slowly from $I_{\text{MAX}}$ to $I_{\text{MIN}}$. The whole process should be repeated for two further ‘ups’ and ‘downs’ to yield two sets of data: an ‘up’ set $(I_i, O_i)$, $i = 1, 2, \ldots, n$ ($n = 33$).

Computer software regression packages are readily available which fit a polynomial, i.e.

$$O(I) = \sum_{i=0}^{m} a_i I^i$$

to a set of $n$ data points. These packages use a ‘least squares’ criterion. If $d_i$ is the deviation of the polynomial value $O(I_i)$ from the data value $O_i$, then $d_i = O(I_i) - O_i$. The program finds a set of coefficients $a_0, a_1, a_2, \ldots$. 

<table>
<thead>
<tr>
<th>Equilibrium state</th>
<th>$\theta$ K</th>
<th>$T$ °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triple point of hydrogen</td>
<td>13.8033</td>
<td>−259.3467</td>
</tr>
<tr>
<td>Boiling point of hydrogen at a pressure of 33 321.3 Pa</td>
<td>17.035</td>
<td>−256.115</td>
</tr>
<tr>
<td>Boiling point of hydrogen at a pressure of 101 292 Pa</td>
<td>20.27</td>
<td>−252.88</td>
</tr>
<tr>
<td>Triple point of neon</td>
<td>24.5561</td>
<td>−248.5939</td>
</tr>
<tr>
<td>Triple point of oxygen</td>
<td>54.3584</td>
<td>−218.7916</td>
</tr>
<tr>
<td>Triple point of argon</td>
<td>83.8058</td>
<td>−189.3442</td>
</tr>
<tr>
<td>Triple point of mercury</td>
<td>234.3156</td>
<td>−38.8344</td>
</tr>
<tr>
<td>Triple point of water</td>
<td>273.16</td>
<td>0.01</td>
</tr>
<tr>
<td>Melting point of gallium</td>
<td>302.9146</td>
<td>29.7646</td>
</tr>
<tr>
<td>Freezing point of indium</td>
<td>429.7485</td>
<td>156.5985</td>
</tr>
<tr>
<td>Freezing point of tin</td>
<td>505.078</td>
<td>231.928</td>
</tr>
<tr>
<td>Freezing point of zinc</td>
<td>692.677</td>
<td>419.527</td>
</tr>
<tr>
<td>Freezing point of aluminium</td>
<td>933.473</td>
<td>660.323</td>
</tr>
<tr>
<td>Freezing point of silver</td>
<td>1234.93</td>
<td>961.78</td>
</tr>
<tr>
<td>Freezing point of gold</td>
<td>1337.33</td>
<td>1064.18</td>
</tr>
<tr>
<td>Freezing point of copper</td>
<td>1357.77</td>
<td>1084.62</td>
</tr>
</tbody>
</table>

Table 2.5 The primary fixed points defining the International Temperature Scale 1990 – ITS90.
such that the sum of the squares of the deviations, i.e. $\sum_{i=1}^{n} d_i^2$, is a minimum. This involves solving a set of linear equations.\[^9\]

In order to detect any hysteresis, separate regressions should be performed on the two sets of data $(I_i, O_i)$, $(I_j, O_j)$, to yield two polynomials:

$$O(I)_{\uparrow} = \sum_{q=0}^{m} a_q^\uparrow I^q$$
$$O(I)_{\downarrow} = \sum_{q=0}^{m} a_q^\downarrow I^q$$

If the hysteresis is significant, then the separation of the two curves will be greater than the scatter of data points about each individual curve (Figure 2.18(a)). Hysteresis $H(I)$ is then given by eqn [2.10], i.e. $H(I) = O(I)_{\downarrow} - O(I)_{\uparrow}$. If, however, the scatter of points about each curve is greater than the separation of the curves (Figure 2.18(b)), then $H$ is not significant and the two sets of data can then be combined to give a single polynomial $O(I)$. The slope $K$ and zero bias $a$ of the ideal straight line joining the minimum and maximum points $(I_{\text{MIN}}, O_{\text{MIN}})$ and $(I_{\text{MAX}}, O_{\text{MAX}})$ can be found from eqn [2.3]. The non-linear function $N(I)$ can then be found using eqn [2.4]:

$$N(I) = O(I) - (KI + a)$$ \[2.24\]

Temperature sensors are often calibrated using appropriate fixed points rather than a standard instrument. For example, a thermocouple may be calibrated between 0 and 500 °C by measuring e.m.f.’s at ice, steam and zinc points. If the e.m.f.–temperature relationship is represented by the cubic $E = a_1 T + a_2 T^2 + a_3 T^3$, then the coefficients $a_1, a_2, a_3$ can be found by solving three simultaneous equations (see Problem 2.1).

**O versus $I_M$, $I_f$ at constant $I$**

We first need to find which environmental inputs are interfering, i.e. which affect the zero bias $a$. The input $I$ is held constant at $I = I_{\text{MIN}}$, and one environmental input is changed by a known amount, the rest being kept at standard values. If there is a resulting change $\Delta O$ in $O$, then the input $I$ is interfering and the value of the corresponding coefficient $K_j$ is given by $K_j = \Delta O/\Delta I$. If there is no change in $O$, then the input is not interfering; the process is repeated until all interfering inputs are identified and the corresponding $K_j$ values found.

We now need to identify modifying inputs, i.e. those which affect the sensitivity of the element. The input $I$ is held constant at the mid-range value $\frac{1}{2}(I_{\text{MIN}} + I_{\text{MAX}})$ and each environmental input is varied in turn by a known amount. If a change in input...
produces a change $\Delta O$ in $O$ and is not an interfering input, then it must be a modifying input $I_M$ and the value of the corresponding coefficient $K_M$ is given by

$$K_M = \frac{1}{I} \frac{\Delta O}{\Delta I_M} = \frac{2}{I_{MIN} + I_{MAX}} \frac{\Delta O}{\Delta I_M}$$

[2.25]

Suppose a change in input produces a change $\Delta O$ in $O$ and it has already been identified as an interfering input with a known value of $K_I$. Then we must calculate a non-zero value of $K_M$ before we can be sure that the input is also modifying. Since

$$\Delta O = K_I \Delta I_{I,M} + K_M \Delta I_{I,M} \frac{I_{MIN} + I_{MAX}}{2}$$

then

$$K_M = \frac{2}{I_{MIN} + I_{MAX}} \left[ \frac{\Delta O}{\Delta I_{I,M}} - K_I \right]$$

[2.26]

**Repeatability test**

This test should be carried out in the normal working environment of the element, e.g. out on the plant, or in a control room, where the environmental inputs $I_M$ and $I_I$ are subject to the random variations usually experienced. The signal input $I$ should be held constant at mid-range value and the output $O$ measured over an extended period, ideally many days, yielding a set of values $O_k, k = 1, 2, \ldots, N$. The mean value of the set can be found using

$$\bar{O} = \frac{1}{N} \sum_{k=1}^{N} O_k$$

[2.27]

and the standard deviation (root mean square of deviations from the mean) is found using

$$\sigma_0 = \sqrt{\frac{1}{N} \sum_{k=1}^{N} (O_k - \bar{O})^2}$$

[2.28]

A histogram of the values $O_k$ should then be plotted in order to estimate the probability density function $p(O)$ and to compare it with the normal form (eqn [2.20]). A repeatability test on a pressure transducer can be used as an example of the construction of a histogram. Suppose that $N = 50$ and readings between 0.975 and 1.030 V are obtained corresponding to an expected value of 1.000 V. The readings are grouped into equal intervals of width 0.005 V and the number in each interval is found, e.g. 12, 10, 8, etc. This number is divided by the total number of readings, i.e. 50, to give the probabilities 0.24, 0.20, 0.16, etc. of a reading occurring in a given interval. The probabilities are in turn divided by the interval width 0.005 V to give the probability densities 48, 40 and 32 V$^{-1}$ plotted in the histogram (Figure 2.19). We note that the area of each rectangle represents the probability that a reading lies within the interval, and that the total area of the histogram is equal to unity. The mean and standard deviation are found from eqns [2.27] and [2.28] to be $\bar{O} = 0.999$ V and $\sigma_0 = 0.010$ V respectively. Figure 2.19 also shows a normal probability density function with these values.
Conclusion

The chapter began by discussing the static or steady-state characteristics of measurement system elements. Systematic characteristics such as non-linearity and environmental effects were first explained. This led to the generalised model of an element. Statistical characteristics, i.e. repeatability and tolerance, were then discussed. The last section explained how these characteristics can be measured experimentally, i.e. calibration and the use and types of standards.

References


Problems

2.1 The e.m.f. at a thermocouple junction is 645 $\mu$V at the steam point, 3375 $\mu$V at the zinc point and 9149 $\mu$V at the silver point. Given that the e.m.f.–temperature relationship is of the form $E(T) = a_1 T + a_2 T^2 + a_3 T^3$ ($T$ in °C), find $a_1$, $a_2$ and $a_3$.

2.2 The resistance $R(\theta)$ of a thermistor at temperature $\theta$ K is given by $R(\theta) = \alpha \exp(\beta/\theta)$. Given that the resistance at the ice point ($\theta = 273.15$ K) is 9.00 kΩ and the resistance at the steam point is 0.50 kΩ, find the resistance at 25 °C.

2.3 A displacement sensor has an input range of 0.0 to 3.0 cm and a standard supply voltage $V_s = 0.5$ volts. Using the calibration results given in the table, estimate:

(a) The maximum non-linearity as a percentage of f.s.d.
(b) The constants $K_i$, $K_{str}$ associated with supply voltage variations.
(c) The slope $K$ of the ideal straight line.

<table>
<thead>
<tr>
<th>Displacement $x$ cm</th>
<th>0.0</th>
<th>0.5</th>
<th>1.0</th>
<th>1.5</th>
<th>2.0</th>
<th>2.5</th>
<th>3.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output voltage millivolts ($V_s = 0.5$)</td>
<td>0.0</td>
<td>16.5</td>
<td>32.0</td>
<td>44.0</td>
<td>51.5</td>
<td>55.5</td>
<td>58.0</td>
</tr>
<tr>
<td>Output voltage millivolts ($V_s = 0.6$)</td>
<td>0.0</td>
<td>21.0</td>
<td>41.5</td>
<td>56.0</td>
<td>65.0</td>
<td>70.5</td>
<td>74.0</td>
</tr>
</tbody>
</table>

2.4 A liquid level sensor has an input range of 0 to 15 cm. Use the calibration results given in the table to estimate the maximum hysteresis as a percentage of f.s.d.

<table>
<thead>
<tr>
<th>Level $h$ cm</th>
<th>0.0</th>
<th>1.5</th>
<th>3.0</th>
<th>4.5</th>
<th>6.0</th>
<th>7.5</th>
<th>9.0</th>
<th>10.5</th>
<th>12.0</th>
<th>13.5</th>
<th>15.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output volts $h$ increasing</td>
<td>0.00</td>
<td>0.35</td>
<td>1.42</td>
<td>2.40</td>
<td>3.43</td>
<td>4.35</td>
<td>5.61</td>
<td>6.50</td>
<td>7.77</td>
<td>8.85</td>
<td>10.2</td>
</tr>
<tr>
<td>Output volts $h$ decreasing</td>
<td>0.14</td>
<td>1.25</td>
<td>2.32</td>
<td>3.55</td>
<td>4.43</td>
<td>5.70</td>
<td>6.78</td>
<td>7.80</td>
<td>8.87</td>
<td>9.65</td>
<td>10.2</td>
</tr>
</tbody>
</table>

2.5 A repeatability test on a vortex flowmeter yielded the following 35 values of frequency corresponding to a constant flow rate of $1.4 \times 10^{-2}$ m$^3$s$^{-1}$: 208.6; 208.3; 208.7; 208.5; 208.8; 207.6; 208.9; 209.1; 208.2; 208.4; 208.1; 209.2; 209.6; 208.6; 208.5; 207.4; 210.2; 209.2; 208.7; 208.4; 207.7; 208.9; 208.7; 208.0; 209.0; 208.1; 209.3; 208.2; 206.8; 209.4; 207.6; 208.1; 208.8; 209.2; 209.7 Hz.

(a) Using equal intervals of width 0.5 Hz, plot a histogram of probability density values.
(b) Calculate the mean and standard deviation of the data.
(c) Sketch a normal probability density function with the mean and standard deviation calculated in (b) on the histogram drawn in (a).

2.6 A platinum resistance sensor is used to interpolate between the triple point of water (0 °C), the boiling point of water (100 °C) and the freezing point of zinc (419.6 °C). The corresponding resistance values are 100.0 Ω, 138.5 Ω and 253.7 Ω. The algebraic form of the interpolation equation is:

$$R_T = R_0(1 + \alpha T + \beta T^2)$$

where $R_T$ Ω = resistance at $T$ °C
$R_0$ Ω = resistance at 0 °C
$\alpha$, $\beta$ = constants.

Find the numerical form of the interpolation equation.
The following results were obtained when a pressure transducer was tested in a laboratory under the following conditions:

I Ambient temperature 20 °C, supply voltage 10 V (standard)
II Ambient temperature 20 °C, supply voltage 12 V
III Ambient temperature 25 °C, supply voltage 10 V

<table>
<thead>
<tr>
<th>Input (barg)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
<td>II</td>
<td>III</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Output (mA)</td>
<td>4</td>
<td>4</td>
<td>6</td>
<td>7.2</td>
<td>8.4</td>
<td>9.2</td>
</tr>
<tr>
<td></td>
<td>10.4</td>
<td>12.8</td>
<td>12.4</td>
<td>13.6</td>
<td>17.2</td>
<td>15.6</td>
</tr>
<tr>
<td></td>
<td>16.8</td>
<td>21.6</td>
<td>18.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>28</td>
<td>22</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Determine the values of $K_M$, $K_i$, $a$ and $K$ associated with the generalised model equation $O = (K + K_M I_o)I + a + K I_o$.
(b) Predict an output value when the input is 5 barg, $V_S = 12$ V and ambient temperature is 25 °C.

**Basic problems**

2.8 A force sensor has an output range of 1 to 5 V corresponding to an input range of 0 to $2 \times 10^5$ N. Find the equation of the ideal straight line.

2.9 A differential pressure transmitter has an input range of 0 to $2 \times 10^4$ Pa and an output range of 4 to 20 mA. Find the equation to the ideal straight line.

2.10 A non-linear pressure sensor has an input range of 0 to 10 bar and an output range of 0 to 5 V. The output voltage at 4 bar is 2.20 V. Calculate the non-linearity in volts and as a percentage of span.

2.11 A non-linear temperature sensor has an input range of 0 to 400 °C and an output range of 0 to 20 mV. The output signal at 100 °C is 4.5 mV. Find the non-linearity at 100 °C in millivolts and as a percentage of span.

2.12 A thermocouple used between 0 and 500 °C has the following input–output characteristics:

<table>
<thead>
<tr>
<th>Input $T$ °C</th>
<th>0</th>
<th>100</th>
<th>200</th>
<th>300</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output $E$ µV</td>
<td>0</td>
<td>5268</td>
<td>10 777</td>
<td>16 325</td>
<td>27 388</td>
</tr>
</tbody>
</table>

(a) Find the equation of the ideal straight line.
(b) Find the non-linearity at 100 °C and 300 °C in µV and as a percentage of f.s.d.

2.13 A force sensor has an input range of 0 to 10 kN and an output range of 0 to 5 V at a standard temperature of 20 °C. At 30 °C the output range is 0 to 5.5 V. Quantify this environmental effect.

2.14 A pressure transducer has an output range of 1.0 to 5.0 V at a standard temperature of 20 °C, and an output range of 1.2 to 5.2 V at 30 °C. Quantify this environmental effect.

2.15 A pressure transducer has an input range of 0 to $10^6$ Pa and an output range of 4 to 20 mA at a standard ambient temperature of 20 °C. If the ambient temperature is increased to 30 °C, the range changes to 4.2 to 20.8 mA. Find the values of the environmental sensitivities $K_i$ and $K_M$. 


An analogue-to-digital converter has an input range of 0 to 5 V. Calculate the resolution error both as a voltage and as a percentage of f.s.d. if the output digital signal is:

(a) 8-bit binary
(b) 16-bit binary.

A level transducer has an output range of 0 to 10 V. For a 3 metre level, the output voltage for a falling level is 3.05 V and for a rising level 2.95 V. Find the hysteresis as a percentage of span.
3 The Accuracy of Measurement Systems in the Steady State

In Chapter 1 we saw that the input to a measurement system is the true value of the variable being measured. Also, if the measurement system is complete, the system output is the measured value of the variable. In Chapter 2 we defined accuracy in terms of measurement error, i.e. the difference between the measured and true values of a variable. It follows therefore that accuracy is a property of a complete measurement system rather than a single element. Accuracy is quantified using measurement error $E$ where:

$$E = \text{measured value} - \text{true value} = \text{system output} - \text{system input} \quad [3.1]$$

In this chapter we use the static model of a single element, developed previously, to calculate the output and thus the measurement error for a complete system of several elements. The chapter concludes by examining methods of reducing system error.

3.1 Measurement error of a system of ideal elements

Consider the system shown in Figure 3.1 consisting of $n$ elements in series. Suppose each element is ideal, i.e. perfectly linear and not subject to environmental inputs. If we also assume the intercept or bias is zero, i.e. $a = 0$, then:

$$O_i = K_i I_i \quad [3.2]$$

for $i = 1, \ldots, n$, where $K_i$ is the linear sensitivity or slope (eqn [2.3]). It follows that $O_2 = K_2 I_2 = K_2 K_1 I_1$, $O_3 = K_3 I_3 = K_3 K_2 K_1 I_1$, and for the whole system:

$$O = O_n = K_1 K_2 K_3 \ldots K_i \ldots K_n I \quad [3.3]$$

Figure 3.1.
THE ACCURACY OF MEASUREMENT SYSTEMS IN THE STEADY STATE

If the measurement system is complete, then \( E = O - I \), giving:

\[
E = (K_1K_2K_3 \ldots K_n - 1)I
\]

This if

\[
K_1K_2K_3 \ldots K_n = 1
\]

we have \( E = 0 \) and the system is perfectly accurate. The temperature measurement system shown in Figure 3.2 appears to satisfy the above condition. The indicator is simply a moving coil voltmeter (see Chapter 11) with a scale marked in degrees Celsius so that an input change of 1 V causes a change in deflection of 25 °C. This system has \( K_1K_2K_3 = 40 \times 10^{-6} \times 10^3 \times 25 = 1 \) and thus appears to be perfectly accurate. The system is not accurate, however, because none of the three elements present is ideal. The thermocouple is non-linear (Chapter 2), so that as the input temperature changes the sensitivity is no longer \( 40 \mu \text{V} \, ^\circ\text{C}^{-1} \). Also changes in reference junction temperature (Figure 2.12(b)) cause the thermocouple e.m.f. to change. The output voltage of the amplifier is also affected by changes in ambient temperature (Chapter 9). The sensitivity \( K_3 \) of the indicator depends on the stiffness of the restoring spring in the moving coil assembly. This is affected by changes in environmental temperature and wear, causing \( K_3 \) to deviate from \( 25 \degree \text{C} \, \text{V}^{-1} \). Thus the condition \( K_1K_2K_3 = 1 \) cannot be always satisfied and the system is in error.

In general the error of any measurement system depends on the non-ideal characteristics – e.g. non-linearity, environmental and statistical effects – of every element in the system. Thus in order to quantify this error as precisely as possible we need to use the general model for a single element developed in Sections 2.2 and 2.3.

3.2 The error probability density function of a system of non-ideal elements

In Chapter 2 we saw that the probability density function of the output \( p(O) \) of a single element can be represented by a normal distribution (eqn [2.20]). The mean value \( \bar{O} \) of the distribution is given by eqn [2.22], which allows for non-linear and environmental effects. The standard deviation \( \sigma_0 \) is given by [2.23], which allows for statistical variations in inputs \( I, I_m \) and \( I_i \) with time, and statistical variations in parameters \( K, a, \) etc., amongst a batch of similar elements. These equations apply to each element in a measurement system of \( n \) elements and can be used to calculate the system error probability density function \( p(E) \) as shown in Table 3.1.

Equations [3.6] (based on [2.22]) show how to calculate the mean value of the output of each element in turn, starting with \( \bar{O}_1 \) for the first and finishing with \( \bar{O}_n = \bar{O} \) for the \( n \)th. The mean value \( \bar{E} \) of the system error is simply the difference between the mean value of system output and mean value of system input (eqn [3.7]). Since the probability densities of the outputs of the individual elements are normal, then, using the result outlined in Section 2.3, the probability density function of the
3.2 THE ERROR PROBABILITY DENSITY FUNCTION OF A SYSTEM OF NON-IDEAL ELEMENTS

Table 3.1 General calculation of system $p(E)$. 

<table>
<thead>
<tr>
<th>$I = I_1$</th>
<th>$O_1 = I_2$</th>
<th>$O_2 = I_3$</th>
<th>...</th>
<th>$O_{n-1} = I_n$</th>
<th>$O_n = O$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>...</td>
<td>$n$</td>
<td></td>
</tr>
</tbody>
</table>

Mean values of element outputs 

$I_i = I$

$I_i = \bar{O}_i = \bar{K}_i I_i + \bar{N}_i(I_i) + \bar{a}_i + \bar{K}_{mi} I_m I_i + \bar{K}_i I_i$

$E = \bar{O}_e = \bar{K}_e I_e + \bar{N}_e(I_e) + \bar{a}_e + \bar{K}_{me} I_m I_e + \bar{K}_e I_e$

[3.6]

Mean value of system error

$E = O - I$

[3.7]

Standard deviations of element outputs

$\sigma_i^2 = \sigma^{02}_i$

$\sigma_i^2 = \sigma^{02}_i + \left( \frac{\partial O_i}{\partial I_i} \right)^2 \sigma_i^{02}_i + \left( \frac{\partial O_i}{\partial I_m} \right)^2 \sigma_i^{02}_i + \left( \frac{\partial O_i}{\partial K_i} \right)^2 \sigma_i^{02}_i + \ldots$

$\sigma_i^2 = \sigma^{02}_i + \left( \frac{\partial O_i}{\partial I_i} \right)^2 \sigma_i^{02}_i + \left( \frac{\partial O_i}{\partial I_m} \right)^2 \sigma_i^{02}_i + \left( \frac{\partial O_i}{\partial K_i} \right)^2 \sigma_i^{02}_i + \ldots$

$\sigma_i^2 = \sigma^{02}_i + \left( \frac{\partial O_i}{\partial I_i} \right)^2 \sigma_i^{02}_i + \left( \frac{\partial O_i}{\partial I_m} \right)^2 \sigma_i^{02}_i + \left( \frac{\partial O_i}{\partial K_i} \right)^2 \sigma_i^{02}_i + \ldots$

$\sigma_i^2 = \sigma^{02}_i + \left( \frac{\partial O_i}{\partial I_i} \right)^2 \sigma_i^{02}_i + \left( \frac{\partial O_i}{\partial I_m} \right)^2 \sigma_i^{02}_i + \left( \frac{\partial O_i}{\partial K_i} \right)^2 \sigma_i^{02}_i + \ldots$

[3.8]

Standard deviation of system error

$\sigma_e = \sigma_i$

[3.9]

Error probability density function

$p(E) = \frac{1}{\sigma_e \sqrt{2\pi}} \exp \left[ -\frac{1}{2\sigma_e^2} (E - \bar{E})^2 \right]$

[3.10]

system output $O$ and system error $E$ is also normal (eqn [3.10]). Equations [3.8] (based on [2.23]) show how to calculate the standard deviation of the output of each element in turn, starting with $\sigma_i$, for the first, and finishing with $\sigma_o$, for the $n$th. We note that the standard deviation of the system input is zero and that the standard deviation of the error is equal to that of the system output (eqn [3.9]).

We now use the temperature measurement system shown in Figure 3.3 as an example of the calculation of $\bar{E}$ and $\sigma_e$. This consists of a platinum resistance temperature detector, current transmitter and recorder.

Table 3.2 gives the models for each of the elements in the temperature measurement system. In (a), the platinum resistance temperature detector is characterised by
The accuracy of measurement systems in the steady state

Figure 3.3 Temperature measurement system.

(a) Platinum resistance temperature detector
Model equation

\[ R_T = R_0(1 + \alpha T + \beta T^2) \]

Individual mean values

\[ R_0 = 100.0 \, \Omega, \ \alpha = 3.909 \times 10^{-3}, \ \beta = -5.897 \times 10^{-7} \]

(between 100 and 130 °C)

Individual standard deviations

\[ \sigma_{R_0} = 4.33 \times 10^{-2}, \ \sigma_{\alpha} = 0.0, \ \sigma_{\beta} = 0.0 \]

Partial derivatives

\[ \frac{\partial R_T}{\partial R_0} = 1.449 \text{ at } T = 117 \, ^\circ C \]

Overall mean value

\[ \bar{R}_T = \bar{R}_0(1 + \bar{\alpha} T + \bar{\beta} T^2) \]

Overall standard deviation

\[ \sigma_{\bar{R}_T} = \left( \frac{\partial R_T}{\partial R_0} \right)^2 \sigma_{R_0}^2 + \left( \frac{\partial R_T}{\partial \alpha} \right)^2 \sigma_{\alpha}^2 + \left( \frac{\partial R_T}{\partial \beta} \right)^2 \sigma_{\beta}^2 \]

(b) Current transmitter
Model equation

\[ i = \bar{K}R_T + \bar{K}_m R_T \Delta T_a + \bar{K}_i \Delta T_a + \bar{a} \]

Individual mean values

\[ \bar{K} = 1.4134, \ \bar{K}_m = 1.4134 \times 10^{-4}, \ \bar{K}_i = -1.637 \times 10^{-2} \]

\[ \bar{a} = -191.76, \ \Delta T_a = -10 \]

Individual standard deviations

\[ \sigma_{\bar{K}} = 0.0, \ \sigma_{\bar{K}_m} = 0.0, \ \sigma_{\bar{K}_i} = 0.0 \]

\[ \sigma_{\bar{a}} = 0.24, \ \sigma_{\Delta T_a} = 6.7 \]

Partial derivatives

\[ \frac{\partial i}{\partial R_T} = 1.413, \ \frac{\partial i}{\partial \Delta T_a} = 4.11 \times 10^{-3}, \ \frac{\partial i}{\partial a} = 1.00 \]

Overall mean value

\[ \bar{i} = \bar{K}\bar{R}_T + \bar{K}_m \bar{R}_T \Delta \bar{T}_a + \bar{K}_i \Delta \bar{T}_a + \bar{a} \]

Overall standard deviation

\[ \sigma_i^2 = \left( \frac{\partial i}{\partial R_T} \right)^2 \sigma_{\bar{R}_T}^2 + \left( \frac{\partial i}{\partial \Delta T_a} \right)^2 \sigma_{\Delta T_a}^2 + \left( \frac{\partial i}{\partial a} \right)^2 \sigma_{\bar{a}}^2 \]

(c) Recorder
Model equation

\[ T_M = \bar{K}i + a \]

Individual mean values

\[ \bar{K} = 1.875, \ \bar{a} = 92.50 \]

(100 to 130 °C record for 4 to 20 mA input)

Individual standard deviations

\[ \sigma_{\bar{K}} = 0.0, \ \sigma_{\bar{a}} = 0.10 \]

Partial derivatives

\[ \frac{\partial T_M}{\partial i} = 1.875, \ \frac{\partial T_M}{\partial a} = 1.00 \]

Overall mean value

\[ \bar{T}_m = \bar{K}\bar{i} + \bar{a} \]

Overall standard deviation

\[ \sigma_{\bar{T}_m}^2 = \left( \frac{\partial \bar{T}_M}{\partial \bar{i}} \right)^2 \sigma_{\bar{i}}^2 + \left( \frac{\partial \bar{T}_M}{\partial \bar{a}} \right)^2 \sigma_{\bar{a}}^2 \]
3.2 THE ERROR PROBABILITY DENSITY FUNCTION OF A SYSTEM OF NON-IDEAL ELEMENTS

The error probability density function of a system of non-ideal elements includes a small amount of non-linearity and a spread of values of $R_0$ (resistance at 0 °C). The current transmitter (b) is linear but temperature acts as both a modifying and an interfering input. The zero bias and sensitivity are adjustable: we cannot be certain that the transmitter will be set up exactly as stated in the table, and this is reflected in the non-zero value of $\sigma_a$. In (c) the recorder is linear but again calibration uncertainties are modelled by a non-zero value of $\sigma_a$.

Table 3.3 summarises the calculation of $\bar{E}$ and $\sigma_E$ for the temperature system.

<table>
<thead>
<tr>
<th>Mean $\bar{E}$</th>
<th>$\bar{T} = 117 °C$</th>
<th>$R_f = 144.93 \Omega$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{i} = 13.04 mA$</td>
<td>$\bar{T}_m = 116.95 °C$</td>
<td></td>
</tr>
<tr>
<td>$\bar{E} = \bar{T}_m - \bar{T} = -0.005 °C$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Standard deviation $\sigma_E$

\[
\sigma_E^2 = \sum \left( \frac{\partial \bar{E}}{\partial a_i} \right)^2 \sigma_{a_i}^2 + \sum \left( \frac{\partial \bar{E}}{\partial \Delta T_a} \right)^2 \sigma_{\Delta T_a}^2 + \sum \left( \frac{\partial \bar{E}}{\partial \Delta i} \right)^2 \sigma_i^2
\]

Table 3.3

<table>
<thead>
<tr>
<th>Summary of calculation of $\bar{E}$ and $\sigma_E$ for temperature system.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean $\bar{E}$</td>
</tr>
<tr>
<td>$\bar{i} = 13.04 mA$</td>
</tr>
<tr>
<td>$\bar{E} = \bar{T}_m - \bar{T} = -0.005 °C$</td>
</tr>
</tbody>
</table>

Modelling using error bands

In Section 2.1 we saw that in situations where element non-linearity, hysteresis and environmental effects are small, their overall effect is quantified using error bands. Here a systematic statement of the exact element input/output relationship (e.g. eqn [2.9]) is replaced by a statistical statement. The element output is described by a rectangular probability density function, of width $2\Delta$, centred on the ideal straight line value $O_{\text{ideal}} = K_i + a_i$. If every element in the system is described in this way, then the mean output value $\bar{O}$ will have the ideal value $\bar{O}_i = K_i \bar{i}_i + a_i$ for each element in the system. In the special case $a_i = 0$ for all $i$, $\bar{O}_i = K_i \bar{i}_i$, the mean value of system output is given by

$$\bar{O} = K_1K_2 \ldots K_i \ldots K_n \bar{i}$$

and the mean value of error $\bar{E} = \bar{O} - \bar{i}$. In the special case $K_1K_2 \ldots K_i \ldots K_n = 1$, $\bar{E} = 0$ (Table 3.4). The error probability density function $p(E)$ is the result of
combining $n$ rectangular distributions, each of width $2h_i$, $i = 1, 2, \ldots, n$. If $n > 3$, then the resultant distribution $p(E)$ approximates to a normal distribution; the larger the value of $n$ the closer the distribution is to normal.

In order to calculate the standard deviation $\sigma_o$ of $p(E)$ we return to eqns [2.16] and [2.17]. If a dependent variable $y$ is a linear combination of independent random variables $x_1, x_2, x_3$, i.e.

\[
y = a_1x_1 + a_2x_2 + a_3x_3
\]

then the standard deviation $\sigma$ of the probability distribution of $y$ is given by

\[
\sigma^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + a_3^2\sigma_3^2
\]  

[2.16]

Equation [2.17] applies whatever the probability distributions of the individual $x_1, x_2, x_3$ are, provided that the individual standard deviations $\sigma_1, \sigma_2, \sigma_3$ are small.\(^{[1]}\)

Consider the $i$th element in Table 3.4 with sensitivity $K_i$ and error band width $2h_i$. The standard deviation $\sigma_{O_i}$ of the output $O_i$ is equal to

\[
\sigma_{O_i}^2 = \sigma_o^2 \text{ due to input } + \sigma_i^2 \text{ due to element}
\]

If $\sigma_i$ is the standard deviation of the input $I_i$, then this contributes $K_i\sigma_i$ to the output standard deviation. The standard deviation of the rectangular distribution of width $2h_i$, due to the element, is $h_i/\sqrt{3}$. We therefore have:

\[
\sigma_{O_i}^2 = K_i^2\sigma_i^2 + h_i^2/3
\]

Table 3.4 gives the calculation procedure for a complete system of $n$ elements described by error bands.
3.3 ERROR REDUCTION TECHNIQUES

Error reduction techniques

In the two previous sections we saw that the error of a measurement system depends on the non-ideal characteristics of every element in the system. Using the calibration techniques of Section 2.4, we can identify which elements in the system have the most dominant non-ideal behaviour. We can then devise compensation strategies for these elements which should produce significant reductions in the overall system error. This section outlines compensation methods for non-linear and environmental effects.

One of the most common methods of correcting a non-linear element is to introduce a compensating non-linear element into the system. This method is illustrated in Figure 3.4. Given a non-linear element, described by \( U(I) \), we need a compensating element \( C(U) \), such that the overall characteristics \( C[U(I)] \) of the elements together are as close to the ideal straight line as possible. The method is illustrated in Figure 3.4 by the use of a deflection bridge to compensate for the non-linear characteristics of a thermistor. A detailed procedure for the design of the bridge is given in Section 9.1.

The most obvious method of reducing the effects of environmental inputs is that of isolation, i.e. to isolate the transducer from environmental changes so that effectively \( I_0 = I = 0 \). Examples of this are the placing of the reference junction of a thermocouple in a temperature-controlled enclosure, and the use of spring mountings to isolate a transducer from the vibrations of the structure to which it is attached. Another obvious method is that of zero environmental sensitivity, where the element is completely insensitive to environmental inputs, i.e. \( K_0 = K_I = 0 \). An example of this is the use of a metal alloy with zero temperature coefficients of expansion and resistance as a strain gauge element. Such an ideal material is difficult to find, and in practice the resistance of a metal strain gauge is affected slightly by changes in ambient temperature.

A more successful method of coping with environmental inputs is that of opposing environmental inputs. Suppose that an element is affected by an environmental

---

**Figure 3.4** Compensating non-linear element.

![Figure 3.4](image-url)
input; then a second element, subject to the same environmental input, is deliberately introduced into the system so that the two effects tend to cancel out. This method is illustrated for interfering inputs in Figure 3.5 and can be easily extended to modifying inputs.

An example is compensation for variations in the temperature $T_2$ of the reference junction of a thermocouple. For a copper–constantan thermocouple (Figure 2.12(b)), we have $K_{II}$ equal to $-38.74 T_2 \mu V$ so that a compensating element giving an output equal to $+38.74 T_2 \mu V$ is required. The design of the reference junction compensation element is discussed in Sections 8.5 and 9.1 and Problem 9.2.

An example of a differential system (Figure 3.5(b)) is the use of two matched strain gauges in adjacent arms of a bridge to provide compensation for ambient temperature changes. One gauge is measuring a tensile strain $+e$ and the other an equal compressive strain $-e$. The bridge effectively subtracts the two resistances so that the strain effect is doubled and the environmental effects cancel out.

The use of high-gain negative feedback is an important method of compensating for modifying inputs and non-linearity. Figure 3.6 illustrates the technique for a force transducer. The voltage output of a force-sensing element, subject to a modifying input, is amplified by a high-gain amplifier. The amplifier output is fed back to an element (e.g. a coil and permanent magnet) which provides a balancing force to oppose the input force.
Ignoring the effects of the modifying input for the moment we have:

\[
\Delta F = F_{IN} - F_B \quad [3.11]
\]

\[
V_{OUT} = KK_A \Delta F \quad [3.12]
\]

\[
F_B = K_F V_{OUT} \quad [3.13]
\]

i.e.

\[
\frac{V_{OUT}}{KK_A} = F_{IN} - K_F V_{OUT}
\]

giving

\[
V_{OUT} = \frac{KK_A}{1 + KK_A K_F} F_{IN} \quad [3.14]
\]

If the amplifier gain \(K_A\) is made large such that the condition

\[
KK_A K_F \gg 1 \quad [3.15]
\]

is satisfied, then \(V_{OUT} = \frac{1}{K_F} F_{IN}\). This means that the system output depends only on the gain \(K_F\) of the feedback element and is independent of the gains \(K\) and \(K_A\) in the forward path. This means that, provided the above condition is obeyed, changes in \(K\) and \(K_A\) due to modifying inputs and/or non-linear effects have negligible effect on \(V_{OUT}\). This can be confirmed by repeating the above analysis with \(K\) replaced by \(K + K_M I_M\), giving

\[
V_{OUT} = \frac{(K + K_M I_M)K_A}{1 + (K + K_M I_M)K_F} F_{IN} \quad [3.16]
\]

which again reduces to \(V_{OUT} = \frac{F_{IN}}{K_F}\) if \((K + K_M I_M)K_A K_F \gg 1\). We now, of course, have to ensure that the gain \(K_F\) of the feedback element does not change due to non-linear or environmental effects. Since the amplifier provides most of the required power, the feedback element can be designed for low power-handling capacity,
THE ACCURACY OF MEASUREMENT SYSTEMS IN THE STEADY STATE

The accuracy of measurement systems in the steady state can be improved by using devices with greater linearity and less susceptibility to environmental inputs. A commonly used device which employs this principle is discussed in Section 9.4. The rapid fall in the cost of digital integrated circuits in recent years has meant that microcontrollers are now widely used as signal-processing elements in measurement systems (Chapter 10). This means that the powerful techniques of computer estimation of measured value can now be used. For this method, a good model of the elements in the system is required. In Sections 2.1 and 2.2 we saw that the steady-state output $O$ of an element is in general given by an equation of the form:

**Direct equation**

$$O = KI + a + N(I) + K_M I_M I + K_I I_I$$  \[2.9\]

This is the direct equation (Figure 3.7(a)); here $O$ is the dependent variable, which is expressed in terms of the independent variables $I$, $I_M$, and $I_I$. In Section 2.4.2 we saw how the direct equation could be derived from sets of data obtained in a calibration experiment. In Section 3.2 eqn [2.9] was used to derive the error probability density function for a complete measurement system.

The steady-state characteristics of an element can also be represented by an alternative equation. This is the inverse equation (Figure 3.7(b)); here the signal input $I$ is the dependent variable and the output $O$ and environmental inputs $I$ and $I_M$ are the independent variables. The general form of this equation is:

**Inverse equation**

$$I = K'O + N'(O) + a' + K_M I_M O + K'I$$  \[3.17\]
where the values of $K'$, $N'$, $a'$, etc., are quite different from those for the direct equation. For example, the direct and inverse equations for a copper–constantan (type T) thermocouple with reference junction at 0 °C are:

**DIRECT**

$$E = 3.845 \times 10^{-2}T + 4.682 \times 10^{-5}T^2 - 3.789 \times 10^{-8}T^3 + 1.652 \times 10^{-11}T^4 \text{ mV}$$

**INVERSE**

$$T = 25.55E - 0.5973E^2 + 2.064 \times 10^{-2}E^3 - 3.205 \times 10^{-4}E^4 \text{ °C} \quad [3.18]$$

where $E$ is the thermocouple e.m.f. and $T$ the measured junction temperature between 0 and 400 °C. Both equations were derived using a least squares polynomial fit to IEC 584.1 data; for the direct equation $E$ is the dependent variable and $T$ the independent variable; for the inverse equation $T$ is the dependent variable and $E$ the independent variable. While the direct equation is more useful for error estimation, the inverse equation is more useful for error reduction.

The use of the inverse equation in computer estimation of measured value is best implemented in a number of stages; with reference to Figure 3.8(a), these are:

1. Treat the uncompensated system as a single element. Using the calibration procedure of Section 2.4.2 (or any other method of generating data) the parameters $K'$, $a'$, etc. in the inverse model equation:

$$I = K'U + N'(U) + a' + K'M'I_M + K'I_I$$  \[3.19\]

representing the overall behaviour of the uncompensated system can be found. This procedure will enable major environmental inputs $I_M$ and $I_I$ to be identified (there may be more than one of each type).

2. The uncompensated system should be connected to the estimator. This consists firstly of a computer which stores the model parameters $K'$, $a'$, $N'( )$, etc. If errors due to environmental inputs are considered significant, then environmental sensors to provide the computer with estimates $I'_M$, $I'_I$, of these inputs are also necessary. The output $U$ of the uncompensated system is also fed to the computer.

3. The computer then calculates an initial estimate $I'$ of $I$ using the inverse equation:

$$I' = K'U + N'(U) + a' + K'M'I'_M + K'I'_I$$  \[3.20\]

4. The data presentation element then displays the measured value $O$, which should be close to $I'$. In applications not requiring the highest accuracy the procedure can be terminated at this stage.

5. If high accuracy is required, then it may be possible to further improve the estimator by calibrating the complete system. Values of system output $O$ are measured for a range of known standard inputs $I$ and the corresponding values of system error $E = O - I$ are calculated. These error values will be mainly due to random effects but may also contain a small systematic component which can be corrected for.

6. An attempt should now be made to fit the data set $(O_i, E_i), i = 1, 2, \ldots n$, by a least squares straight line of the form:

$$E = kO + b$$  \[3.21\]

where $b$ is any residual zero error and $k$ specifies any residual scale error. There is little point in attempting a polynomial fit at this stage.
7. The correlation coefficient:

\[
    r = \frac{\sum_{i=1}^{n} O_i E_i}{\sqrt{\sum_{i=1}^{n} O_i^2 \times \sum_{i=1}^{n} E_i^2}}
\]  \hspace{1cm} [3.22]

between \( E \) and \( O \) data should now be evaluated. If the magnitude of \( r \) is greater than 0.5, then there is reasonable correlation between the \( E \) and \( O \) data; this means the systematic error of eqn [3.7] is present and we can proceed to stage 8 to correct for it. If the magnitude of \( r \) is less than 0.5, then there is no correlation between the \( E \) and \( O \) data; this means that the errors \( E \) are purely random and no correction can be made.

8. If appropriate, eqn [3.21] can be used to calculate an improved measured value:

\[
    O' = O - E = O - (kO + b)
\]  \hspace{1cm} [3.23]

The displacement measurement system of Figure 3.8(b) shows this method. The uncompensated system consists of an inductive displacement sensor, an oscillator (Section 9.5) and a Schmitt trigger (Section 10.1.4). The sensor has a non-linear

---

**Figure 3.8** Computer estimation of measured value using inverse model equation:

(a) Principles
(b) Example of displacement measurement system.
relation between inductance \( L \) and displacement \( x \), the oscillator a non-linear relation between frequency \( f \) and inductance \( L \). This means that the inverse model equation, relating displacement \( x \) and frequency \( f \) of the Schmitt trigger output signal, has the non-linear form shown. The estimator consists of a 16-bit pulse counter and a computer. The computer reads the state of the counter at the beginning and end of a fixed time interval and thus measures the frequency \( f \) of the pulse signal. The computer then calculates \( x \) from the inverse model equation using model coefficients stored in memory.

Conclusion

This chapter has shown how to find the error of a complete measurement system under steady-state conditions. Measurement error was first defined and then the error probability density function was derived, firstly for a general system of non-ideal elements and then for the typical example of a temperature measurement system. The last section discussed a range of methods for error reduction.

Reference


Problems

3.1 A measurement system consists of a chromel–alumel thermocouple (with cold junction compensation), a millivolt-to-current converter and a recorder. Table Prob. 1 gives the model equations, and parameters for each element. Assuming that all probability distributions are normal, calculate the mean and standard deviation of the error probability distribution, when the input temperature is 117 °C.

Table Prob. 1.

<table>
<thead>
<tr>
<th></th>
<th>Chromel–alumel thermocouple</th>
<th>e.m.f-to-current converter</th>
<th>Recorder</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model equation</td>
<td>( E = C_0 + C_1T + C_2T^2 )</td>
<td>( i = K_1E + K_ME\Delta T_a + K_2\Delta T_a + a_1 )</td>
<td>( T_M = K_2i + a_2 )</td>
</tr>
<tr>
<td>Mean values</td>
<td>( C_0 = 0.00 )</td>
<td>( K_1 = 3.893 )</td>
<td>( K_2 = 6.25 )</td>
</tr>
<tr>
<td></td>
<td>( C_1 = 4.017 \times 10^{-2} )</td>
<td>( \Delta T_a = -10 )</td>
<td>( \bar{a}_1 = -3.864 )</td>
</tr>
<tr>
<td></td>
<td>( C_2 = 4.66 \times 10^{-6} )</td>
<td>( \Delta T_a )</td>
<td>( \bar{K}_M = 1.95 \times 10^{-4} )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( K_1 = 2.00 \times 10^{-3} )</td>
<td></td>
</tr>
<tr>
<td>Standard deviations</td>
<td>( \sigma_{C_0} = 6.93 \times 10^{-2} )</td>
<td>( \sigma_{C_1} = 0.14 ), ( \sigma_{C_2} = 10 )</td>
<td>( \sigma_{M} = 0.30 )</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{C_1} = 0 )</td>
<td>( \sigma_{K_1} = \sigma_{K_2} = 0 )</td>
<td>( \sigma_{K_1} = 0 )</td>
</tr>
<tr>
<td></td>
<td>( \sigma_{C_2} = 0 )</td>
<td></td>
<td>( \sigma_{K_2} = 0 )</td>
</tr>
</tbody>
</table>
Table Prob. 2.

<table>
<thead>
<tr>
<th>Element</th>
<th>Linear sensitivity $K$</th>
<th>Error bandwidth $\pm h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pressure sensor</td>
<td>$10^{-4} \text{ Pa}^{-1}$</td>
<td>$\pm0.005 \Omega$</td>
</tr>
<tr>
<td>Deflection bridge</td>
<td>$4 \times 10^{-7} \text{ mV} \Omega^{-1}$</td>
<td>$\pm5 \times 10^{-4} \text{ mV}$</td>
</tr>
<tr>
<td>Amplifier</td>
<td>$10^3 \text{ mV} \text{ V}^{-1}$</td>
<td>$\pm0.5 \text{ mV}$</td>
</tr>
<tr>
<td>Recorder</td>
<td>$250 \text{ Pa} \text{ mV}^{-1}$</td>
<td>$\pm100 \text{ Pa}$</td>
</tr>
</tbody>
</table>

3.2

A pressure measurement system consists of a pressure sensor, deflection bridge, amplifier and recorder. Table Prob. 2 gives the linear sensitivities and error bandwidths for each element in the system.

(a) Calculate the standard deviation $\sigma_E$ of the error distribution function.

(b) Given that the recorder is incorrectly adjusted so that its sensitivity is $225 \text{ Pa} \text{ mV}^{-1}$, calculate the mean error $\bar{E}$ for an input pressure of $5 \times 10^3 \text{ Pa}$.

3.3

Figure Prob. 3 shows a block diagram of a force transducer using negative feedback. The elastic sensor gives a displacement output for a force input; the displacement sensor gives a voltage output for a displacement input. $V_s$ is the supply voltage for the displacement sensor.

(a) Calculate the output voltage $V_0$ when
   (i) $V_s = 1.0 \text{ V}$, $F = 50 \text{ N}$
   (ii) $V_s = 1.5 \text{ V}$, $F = 50 \text{ N}$.

(b) Comment on the practical significance of the variation of the supply voltage $V_s$.

3.4

Figure Prob. 4 is a block diagram of a voltmeter. The motor produces a torque $T$ proportional to voltage $V$, and the output angular displacement $\theta$ of the spring is proportional to $T$. The stiffness $K_s$ of the spring can vary by $\pm10\%$ about the nominal value of $5 \times 10^{-2} \text{ rad} \text{ N}^{-1} \text{ m}^{-1}$.

Given that the following are available:

(i) a d.c. voltage amplifier of gain 1000
(ii) a voltage subtraction unit
(iii) a stable angular displacement transducer of sensitivity $100 \text{ V} \text{ rad}^{-1}$,

(a) draw a block diagram of a modified system using these components, which reduces the effect of changes in $K_s$;
(b) calculate the effect of a $10\%$ increase in $K_s$ on the sensitivity of the modified system.
A temperature measurement system consists of a thermocouple, an amplifier, an 8-bit analogue-to-digital converter and a microcontroller with display facilities. Table Prob. 5 gives the model equations and parameters for each element in the system. The temperature of the thermocouple measurement junction is $T_1$ °C and the temperature of the reference junction is $T_2$ °C. The microcontroller corrects for $T_2$ having a non-zero mean value.

(a) Estimate the mean and standard deviation of the error probability density function when the input temperature $T_1$ is 100 °C. Treat rectangular distribution as normal with $\sigma = h/\sqrt{3}$.

(b) Explain briefly what modifications should be made to the system to reduce the quantities calculated in (a).

Table Prob. 5.

<table>
<thead>
<tr>
<th>Thermocouple</th>
<th>Amplifier</th>
<th>Analogue-to-digital converter</th>
<th>Microcontroller with display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model equations $E_{T_1, T_2} = a_0 + a_1(T_1 - T_2) + a_2(T_1^2 - T_2^2)$</td>
<td>$V = K_1 E_{T_1, T_2} + b_1$</td>
<td>$n = K_2 V + b_2$</td>
<td>$T_n = K_3 n + b_3$</td>
</tr>
<tr>
<td>Mean values</td>
<td>$\bar{a}_0 = 0$</td>
<td>$\bar{K}_1 = 255$</td>
<td>$\bar{K}_2 = 0.1$</td>
</tr>
<tr>
<td>$\bar{a}_1 = 4.3796 \times 10^{-2}$</td>
<td>$\bar{b}_1 = 0.0$</td>
<td>$\bar{b}_2 = 0.0$</td>
<td>$\bar{b}_3 = 20$</td>
</tr>
<tr>
<td>$\bar{a}_2 = -1.7963 \times 10^{-5}$</td>
<td>$\bar{T}_1 = 20$</td>
<td>$n$ rounded to nearest integer</td>
<td></td>
</tr>
<tr>
<td>Statistical distributions</td>
<td>Normal with $\sigma_a = 0.05$</td>
<td>Normal with $\sigma_n = 5.0$</td>
<td>$b_3$ has a rectangular distribution of $\sigma_{b_3} = 0.0$</td>
</tr>
<tr>
<td>$\sigma_{a_1} = 2.0$</td>
<td>$\sigma_{n_1} = 0.0$</td>
<td>$\sigma_{b_3} = 0.5$</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{a_2} = 0.0$</td>
<td>$\sigma_{n_2} = 0.0$</td>
<td>$\sigma_{b_3} = 0.0$</td>
<td></td>
</tr>
</tbody>
</table>

A fluid velocity measurement system consists of a pitot tube, a differential pressure transmitter, an 8-bit analogue-to-digital converter and a microcontroller with display facilities. Table Prob. 6 gives the model equations and parameters for each element in the system. The microcontroller calculates the measured value of velocity assuming a constant density.

(a) Estimate the mean and standard deviation of the error probability density function assuming the true value of velocity $v_T$ is 14.0 m s$^{-1}$. Use the procedure of Table 3.1, treating the rectangular distributions as normal with $\sigma = h/\sqrt{3}$.

(b) Explain briefly what modifications could be made to the system to reduce the quantities calculated in (a).

Table Prob. 6.

<table>
<thead>
<tr>
<th>Pitot tube</th>
<th>Differential pressure transmitter</th>
<th>Analogue-to-digital converter</th>
<th>Microcontroller with display</th>
</tr>
</thead>
<tbody>
<tr>
<td>Model equations $\Delta P = \frac{1}{2} \rho v_T^2$</td>
<td>$i = K_1 \Delta P + a_1$</td>
<td>$n = K_2 d + a_2$</td>
<td>$v_M = K_3 \sqrt{n - 51}$</td>
</tr>
<tr>
<td>Mean values</td>
<td>$\rho = 1.2$</td>
<td>$\bar{K}_1 = 0.064$</td>
<td>$\bar{K}_2 = 12.80$</td>
</tr>
<tr>
<td>$\bar{a}_1 = 4.0$</td>
<td>$\bar{b}_1 = 0.0$</td>
<td>$\bar{b}_2 = 0.0$</td>
<td>$n$ rounded off to nearest integer</td>
</tr>
<tr>
<td>$h_p = 0.1$</td>
<td>$h_m = 0.04$</td>
<td>$h_{a_2} = 0.5$</td>
<td>$h_{b_3} = 0.0$</td>
</tr>
<tr>
<td>Half-widths of rectangular distribution</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A temperature measurement system consists of a thermistor, bridge and recorder. Table Prob. 7 gives the model equations, mean values and standard deviations for each element in the system. Assuming all probability distributions are normal, calculate the mean and standard deviation of the error probability density function for a true input temperature of 320 K.

**Table Prob. 7.**

<table>
<thead>
<tr>
<th></th>
<th>Thermistor</th>
<th>Bridge</th>
<th>Recorder</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Model equations</strong></td>
<td>$R_0 = K_1 \exp \left( \frac{\beta}{\theta} \right)$</td>
<td>$V_0 = V_S \left{ 1 + \frac{3.3}{\theta} - a_1 \right}$</td>
<td>$\theta_M = K_2 V_0 + a_2$</td>
</tr>
<tr>
<td><strong>Mean values</strong></td>
<td>$\bar{K}_1 = 5 \times 10^{-4} \text{k}\Omega$</td>
<td>$\bar{V}_S = -3.00 \text{ V}$</td>
<td>$\bar{K}_2 = 50.0 \text{ K/V}$</td>
</tr>
<tr>
<td></td>
<td>$\bar{\beta} = 3 \times 10^{1} \text{ K}$</td>
<td>$\bar{a}_1 = 0.77$</td>
<td>$\bar{a}_2 = 300 \text{ K}$</td>
</tr>
<tr>
<td><strong>Standard deviations</strong></td>
<td>$\sigma_{K_1} = 0.5 \times 10^{-4}$</td>
<td>$\sigma_{V_S} = 0.03$</td>
<td>$\sigma_{K_2} = 0.0$</td>
</tr>
<tr>
<td></td>
<td>$\sigma_{\beta} = 0$</td>
<td>$\sigma_{a_1} = 0.01$</td>
<td>$\sigma_{a_2} = 3.0$</td>
</tr>
</tbody>
</table>

**Basic problems**

3.8 A force measurement system consists of four elements with sensitivities $10^{-2}$, $5 \times 10^{-2}$, $10^{3}$ and 1.9. Find the system error for a true value input of 10 kN.

3.9 A level measurement system consists of three linear elements with sensitivities of 0.050, 21.5 and 0.99. Find the system error for a true value input of 5.0 metres.
4 Dynamic Characteristics of Measurement Systems

If the input signal $I$ to an element is changed suddenly, from one value to another, then the output signal $O$ will not instantaneously change to its new value. For example, if the temperature input to a thermocouple is suddenly changed from 25 °C to 100 °C, some time will elapse before the e.m.f. output completes the change from 1 mV to 4 mV. The ways in which an element responds to sudden input changes are termed its dynamic characteristics, and these are most conveniently summarised using a transfer function $G(s)$. The first section of this chapter examines the dynamics of typical elements and derives the corresponding transfer function. The next section examines how standard test signals can be used to identify $G(s)$ for an element. If the input signal to a multi-element measurement system is changing rapidly, then the waveform of the system output signal is in general different from that of the input signal. Section 4.3 explains how this dynamic error can be found, and the final section outlines dynamic compensation methods that can be used to minimise errors.

4.1 Transfer function $G(s)$ for typical system elements

4.1.1 First-order elements

A good example of a first-order element is provided by a temperature sensor with an electrical output signal, e.g. a thermocouple or thermistor. The bare element (not enclosed in a sheath) is placed inside a fluid (Figure 4.1). Initially at time $t = 0^-$ (just before $t = 0$), the sensor temperature is equal to the fluid temperature, i.e. $T(0^-) = T_F(0^-)$. If the fluid temperature is suddenly raised at $t = 0$, the sensor is no longer in a steady state, and its dynamic behaviour is described by the heat balance equation:

$$\text{rate of heat inflow} - \text{rate of heat outflow} = \text{rate of change of sensor heat content}$$

*Figure 4.1 Temperature sensor in fluid.*

Assuming that $T_F > T$, then the rate of heat outflow will be zero, and the rate of heat inflow $W$ will be proportional to the temperature difference $(T_F - T)$. From Chapter 14 we have
where $U \, W \, m^{-2} \, ^\circ C^{-1}$ is the overall heat transfer coefficient between fluid and sensor and $A \, m^2$ is the effective heat transfer area. The increase of heat content of the sensor is $MC(T - T(0-))$ joules, where $M \, kg$ is the sensor mass and $C \, J \, kg^{-1} \, ^\circ C^{-1}$ is the specific heat of the sensor material. Thus, assuming $M$ and $C$ are constants:

rate of increase of sensor heat content = $MC \frac{d}{dt} [T - T(0-)]$ \hspace{1cm} [4.3]

Defining $\Delta T = T - T(0-)$ and $\Delta T_f = T_f - T_f(0-)$ to be the deviations in temperatures from initial steady-state conditions, the differential equation describing the sensor temperature changes is

$UA(\Delta T_f - \Delta T) = MC \frac{d\Delta T}{dt}$

i.e.

$\frac{MC}{UA} \frac{d\Delta T}{dt} + \Delta T = \Delta T_f$ \hspace{1cm} [4.4]

This is a linear differential equation in which $d\Delta T/dt$ and $\Delta T$ are multiplied by constant coefficients; the equation is first order because $d\Delta T/dt$ is the highest derivative present. The quantity $MC/UA$ has the dimensions of time:

$\frac{kg \times J \times kg^{-1} \times ^\circ C^{-1}}{W \times m^2 \times ^\circ C^{-1} \times m^2} = \frac{J}{W} = \text{seconds}$

and is referred to as the time constant $\tau$ for the system. The differential equation is now:

$$\tau \frac{d\Delta T}{dt} + \Delta T = \Delta T_f$$ \hspace{1cm} [4.5]

While the above differential equation is a perfectly adequate description of the dynamics of the sensor, it is not the most useful representation. The transfer function based on the Laplace transform of the differential equation provides a convenient framework for studying the dynamics of multi-element systems. The Laplace transform $f(s)$ of a time-varying function is defined by:

$$f(s) = \int_{0}^{\infty} e^{-st} f(t) \, dt$$ \hspace{1cm} [4.6]

where $s$ is a complex variable of the form $\sigma + j\omega$ where $j = \sqrt{-1}$.

Table 4.1 gives Laplace transforms for some common standard functions $f(t)$. In order to find the transfer function for the sensor we must find the Laplace transform of eqn [4.5]. Using Table 4.1 we have:

$$\tau [s\Delta T(s) - \Delta T(0-)] + \Delta T_f(s) = \Delta T_f(s)$$ \hspace{1cm} [4.7]
**4.1 TRANSFER FUNCTION G(S) FOR TYPICAL SYSTEM ELEMENTS**

where $\Delta T(0-)$ is the temperature deviation at initial conditions prior to $t = 0$. By definition, $\Delta T(0-) = 0$, giving:

$$\tau s \Delta \bar{T}(s) + \Delta \bar{T}(s) = \Delta \bar{T}_p(s)$$

i.e.

$$(\tau s + 1) \Delta \bar{T}(s) = \Delta \bar{T}_p(s) \quad [4.8]$$

The transfer function $G(s)$ of an element is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input, provided the initial conditions are zero. Thus:

---

**Table 4.1 Laplace transforms of common time functions $f(t)$.$^a$**

<table>
<thead>
<tr>
<th>Function</th>
<th>Symbol</th>
<th>Graph</th>
<th>Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>1st derivative</td>
<td>$\frac{d}{dt}f(t)$</td>
<td><img src="1st_derivative_graph.png" alt="1st derivative graph" /></td>
<td>$sf(s) - f(0-)$</td>
</tr>
<tr>
<td>2nd derivative</td>
<td>$\frac{d^2}{dt^2}f(t)$</td>
<td><img src="2nd_derivative_graph.png" alt="2nd derivative graph" /></td>
<td>$s^2f(s) - sf(0-) - f(0-)$</td>
</tr>
<tr>
<td>Unit impulse</td>
<td>$\delta(t)$</td>
<td><img src="unit_impulse_graph.png" alt="Unit impulse graph" /></td>
<td>$\frac{1}{a}$ $\lim_{a \to 0}$</td>
</tr>
<tr>
<td>Unit step</td>
<td>$\mu(t)$</td>
<td><img src="unit_step_graph.png" alt="Unit step graph" /></td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>Exponential decay</td>
<td>$\exp(-\alpha t)$</td>
<td><img src="exponential_decay_graph.png" alt="Exponential decay graph" /></td>
<td>$\frac{1}{s + \alpha}$</td>
</tr>
<tr>
<td>Exponential growth</td>
<td>$1 - \exp(-\alpha t)$</td>
<td><img src="exponential_growth_graph.png" alt="Exponential growth graph" /></td>
<td>$\frac{\alpha}{s(s + \alpha)}$</td>
</tr>
<tr>
<td>Sine wave</td>
<td>$\sin \omega t$</td>
<td><img src="sine_wave_graph.png" alt="Sine wave graph" /></td>
<td>$\frac{\omega}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>Phase-shifted sine wave</td>
<td>$\sin(\omega t + \phi)$</td>
<td><img src="phase_shifted_sine_wave_graph.png" alt="Phase-shifted sine wave graph" /></td>
<td>$\frac{\omega \cos \phi + s \sin \phi}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>Exponentially damped sine wave</td>
<td>$\exp(-\alpha t) \sin \omega t$</td>
<td><img src="exponentially_damped_sine_wave_graph.png" alt="Exponentially damped sine wave graph" /></td>
<td>$\frac{\omega}{(s + \alpha)^2 + \omega^2}$</td>
</tr>
<tr>
<td>Ramp with exponential decay</td>
<td>$t \exp(-\alpha t)$</td>
<td><img src="ramp_with_exponential_decay_graph.png" alt="Ramp with exponential decay graph" /></td>
<td>$\frac{1}{(s + \alpha)^2}$</td>
</tr>
</tbody>
</table>

$^a$ Initial conditions are at $t = 0-$, just prior to $t = 0$. 

---
**Definition of element transfer function**

\[ G(s) = \frac{f_0(s)}{f_i(s)} \quad [4.9] \]

and \( f_0(s) = G(s)f_i(s) \); this means the transfer function of the output signal is simply the product of the element transfer function and the transfer function of the input signal. Because of this simple relationship the transfer function technique lends itself to the study of the dynamics of multi-element systems and block diagram representation (Figure 4.2).

**Figure 4.2** Transfer function representation.

From eqns [4.8] and [4.9] the transfer function for a first-order element is:

\[ G(s) = \frac{\Delta T(s)}{\Delta T_p(s)} = \frac{1}{1 + \tau s} \quad [4.10] \]

The above transfer function only relates changes in sensor temperature to changes in fluid temperature. The overall relationship between changes in sensor output signal \( O \) and fluid temperature is:

\[ \frac{\Delta O(s)}{\Delta T(s)} = \frac{\Delta O}{\Delta T} = \frac{\Delta T}{\Delta T_0} \quad [4.11] \]

where \( \Delta O/\Delta T \) is the **steady-state sensitivity** of the temperature sensor. For an ideal element \( \Delta O/\Delta T \) will be equal to the slope \( K \) of the **ideal straight line**. For non-linear elements, subject to small temperature fluctuations, we can take \( \Delta O/\Delta T = dO/dT \), the derivative being evaluated at the steady-state temperature \( T(0-) \) around which the fluctuations are taking place. Thus for a copper–constantan thermocouple measuring small fluctuations in temperature around 100 °C, \( \Delta E/\Delta T \) is found by evaluating \( dE/dT \) at 100 °C (see Section 2.1) to give \( \Delta E/\Delta T = 35 \mu V °C^{-1} \). Thus if the time constant of the thermocouple is 10 s the overall dynamic relationship between changes in e.m.f. and fluid temperature is:

\[ \frac{\Delta E(s)}{\Delta T_p(s)} = 35 \times \frac{1}{1 + 10s} \quad [4.12] \]

In the general case of an element with static characteristics given by eqn [2.9] and dynamic characteristics defined by \( G(s) \), the effect of small, rapid changes in \( \Delta I \) is evaluated using Figure 4.3, in which steady-state sensitivity \( (\partial O/\partial I)_h = K + K_2I_h + (dN/dI)_h \) and \( I_h \) is the steady-state value of \( I \) around which the fluctuations are
4.1 TRANSFER FUNCTION $G(S)$ FOR TYPICAL SYSTEM ELEMENTS

Table 4.2 Analogous first-order elements.

<table>
<thead>
<tr>
<th>Fluidic</th>
<th>Electrical</th>
<th>Mechanical</th>
</tr>
</thead>
<tbody>
<tr>
<td>Density $\rho$</td>
<td>Charge $q = CV$, current $i = \frac{dV}{dt}$</td>
<td>Displacement $x = \frac{F}{k}$, $k$ N m$^{-1}$ = spring stiffness</td>
</tr>
<tr>
<td>Cross-sectional area $A_F$</td>
<td>RC $\frac{dV}{dt} + V = V_{IN}$</td>
<td>$\tau_M \frac{dF}{dt} + F = F_{IN}$, $\tau_M = \frac{\lambda}{k}$</td>
</tr>
<tr>
<td>Fluid resistance $R_F$</td>
<td>$V_{IN} - V = iR$</td>
<td>$F_{IN} - F = \frac{\lambda}{k} \frac{dx}{dt}$, $\lambda$ N s m$^{-1}$ = damping constant</td>
</tr>
<tr>
<td>$h_{IN}$</td>
<td>Charge $q = CV$, current $i = \frac{dV}{dt}$</td>
<td>Displacement $x = \frac{F}{k}$, $k$ N m$^{-1}$ = spring stiffness</td>
</tr>
<tr>
<td>$Q$</td>
<td>RC $\frac{dV}{dt} + V = V_{IN}$</td>
<td>$\tau_M \frac{dF}{dt} + F = F_{IN}$, $\tau_M = \frac{\lambda}{k}$</td>
</tr>
</tbody>
</table>

Taking place. Table 4.2 shows analogous fluidic, electrical and mechanical elements, which are also described by a first-order transfer function $G(s) = 1/(1 + \tau s)$. All four elements are characterised by ‘resistance’ and ‘capacitance’ as illustrated in the table. Temperature, pressure, voltage and force are analogous ‘driving’ or effort variables; heat flow rate, volume flow rate, current and velocity are analogous ‘driven’ or flow variables. These analogies are discussed further in Section 5.2.
4.1.2 Second-order elements

The elastic sensor shown in Figure 4.4, which converts a force input $F$ into a displacement output $x$, is a good example of a second-order element. The diagram is a conceptual model of the element, which incorporates a mass $m$, a spring of stiffness $k$, and a damper of constant $\lambda$. The system is initially at rest at time $t = 0)$ so that the initial velocity $\dot{x}(0^-) = 0$ and the initial acceleration $\ddot{x}(0^-) = 0$. The initial input force $F(0^-)$ is balanced by the spring force at the initial displacement $x(0^-)$, i.e.

$$F(0^-) = kx(0^-) \quad [4.13]$$

If the input force is suddenly increased at $t = 0$, then the element is no longer in a steady state and its dynamic behaviour is described by Newton’s second law, i.e.

resultant force = mass $\times$ acceleration

i.e.

$$F - kx - \lambda \ddot{x} = m\ddot{x} \quad [4.14]$$

and

$$m\ddot{x} + \lambda \dot{x} + kx = F$$

Defining $\Delta F$ and $\Delta x$ to be the deviations in $F$ and $x$ from initial steady-state conditions:

$$\Delta F = F - F(0^-), \quad \Delta x = x - x(0^-) \quad [4.15]$$

the differential equation now becomes:

$$m\ddot{x} + \lambda \dot{x} + kx(0^-) + k\Delta x = F(0^-) + \Delta F$$

which, using [4.13], reduces to:

$$m\ddot{x} + \lambda \dot{x} + k\Delta x = \Delta F$$

i.e.

$$\frac{m}{k} \frac{d^2\Delta x}{dt^2} + \frac{\lambda}{k} \frac{d\Delta x}{dt} + \Delta x = \frac{1}{k} \Delta F$$

[4.16]

This is a second-order linear differential equation in which $\Delta x$ and its derivatives are multiplied by constant coefficients and the highest derivative present is $d^2\Delta x/dt^2$. If we define

undamped natural frequency $\omega_n = \sqrt{\frac{k}{m}} \text{ rad/s}$
4.1 TRANSFER FUNCTION $G(S)$ FOR TYPICAL SYSTEM ELEMENTS

and

\[ \text{damping ratio } \xi = \frac{\lambda}{2\sqrt{km}} \]  

then \( m/k = 1/\omega_n^2 \), \( \lambda/k = 2\xi/\omega_n \), and \([4.16]\) can be expressed in the standard form:

\[
\frac{1}{\omega_n^2} \frac{d^2 \Delta x}{dt^2} + \frac{2\xi}{\omega_n} \frac{d \Delta x}{dt} + \Delta x = \frac{1}{k} \Delta F
\]  

\[ [4.18]\]

In order to find the transfer function for the element we require the Laplace transform of eqn \([4.18]\). Using Table 4.1 we have:

\[
\frac{1}{\omega_n} \left[ s^2 \Delta \tilde{x}(s) - s \Delta x(0-) - \Delta \tilde{x}(0-) \right] + \frac{2\xi}{\omega_n} \left[ s \Delta \tilde{x}(s) - \Delta \tilde{x}(0-) \right] + \Delta \tilde{x}(s)
\]

\[ = \frac{1}{k} \Delta \tilde{F}(s) \]  

\[ [4.19]\]

Since \( \Delta \tilde{x}(0-) = \Delta \tilde{x}(0-) = 0 \) and \( \Delta x(0-) = 0 \) by definition, \([4.19]\) reduces to:

\[
\left[ \frac{1}{\omega_n} s^2 + \frac{2\xi}{\omega_n} s + 1 \right] \Delta \tilde{x}(s) = \frac{1}{k} \Delta \tilde{F}(s)
\]  

\[ [4.20]\]

Thus

\[
\frac{\Delta \tilde{x}(s)}{\Delta \tilde{F}(s)} = \frac{1}{k} G(s)
\]

\[ \text{Transfer function for a second-order element} \]

\[ G(s) = \frac{1}{\frac{1}{\omega_n^2} s^2 + \frac{2\xi}{\omega_n} s + 1} \]  

\[ [4.21]\]

Figure 4.5 shows an analogous electrical element, a series \( L-C-R \) circuit.

Comparing eqns \([4.14]\) and \([4.22]\) we see that \( q \) is analogous to \( x \), \( V \) is analogous to \( F \), and \( L \), \( R \) and \( 1/C \) are analogous to \( m \), \( \lambda \) and \( k \) respectively (see Table 5.1). The electrical circuit is also described by the above second-order transfer function with \( \omega_n = 1/\sqrt{LC} \) and \( \xi = (R/2)\sqrt{C/L} \).

Figure 4.5  
Series \( L-C-R \) circuit.

\[
V = iR + \frac{q}{C} + L \frac{di}{dt}
\]

where \( i = \frac{dq}{dt} \) (\( q \) = charge on the capacitance)

\[
\text{thus } L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{1}{C} q = V
\]  

\[ [4.22]\]
4.2 Identification of the dynamics of an element

In order to identify the transfer function \( G(s) \) of an element, standard input signals should be used. The two most commonly used standard signals are step and sine wave. This section examines the response of first- and second-order elements to step and sine wave inputs.

4.2.1 Step response of first- and second-order elements

From Table 4.1 we see that the Laplace transform of a step of unit height \( u(t) \) is \( \frac{1}{s} \). Thus if a first-order element with \( G(s) = \frac{1}{1 + \tau s} \) is subject to a unit step input signal, the Laplace transform of the element output signal is:

\[
f_o(s) = G(s) f_i(s) = \frac{1}{1 + \tau s} \cdot \frac{1}{s} = \frac{1}{(1 + \tau s)s} \quad [4.23]
\]

Expressing [4.23] in partial fractions, we have:

\[
f_o(s) = \frac{1}{(1 + \tau s)s} = \frac{A}{1 + \tau s} + \frac{B}{s}
\]

Equating coefficients of constants gives \( B = 1 \), and equating coefficients of \( s \) gives \( 0 = A + B\tau \), i.e. \( A = -\tau \). Thus:

\[
f_o(s) = \frac{1}{s} - \frac{\tau}{(1 + \tau s)} = \frac{1}{s} - \frac{1}{(s + 1/\tau)} \quad [4.24]
\]

Using Table 4.1 in reverse, i.e. finding a time signal \( f(t) \) corresponding to a transform \( f(s) \), we have:

\[
f_o(t) = u(t) - \exp\left(-\frac{t}{\tau}\right)
\]

and since \( u(t) = 1 \) for \( t > 0 \):

\[
f_o(t) = 1 - \exp\left(-\frac{t}{\tau}\right) \quad [4.25]
\]

Response of first-order element to unit step

The form of the response is shown in Figure 4.6.

As an example of the use of eqn [4.25], consider the temperature sensor of Section 4.1.1. Initially the temperature of the sensor is equal to that of the fluid, i.e. \( T(0^-) = T_F(0^-) = 25 \) °C, say. If \( T_F \) is suddenly raised to 100 °C, then this represents a step change \( \Delta T_F \) of height 75 °C. The corresponding change in sensor temperature is given by \( \Delta T = 75(1 - e^{-\tau}) \) and the actual temperature \( T \) of the sensor at time \( t \) is given by:

\[
T(t) = 25 + 75(1 - e^{-\tau}) \quad [4.26]
\]

Thus at time \( t = \tau \), \( T = 25 + (75 \times 0.63) = 72.3 \) °C. By measuring the time taken for \( T \) to rise to 72.3 °C we can find the time constant \( \tau \) of the element.
If the second-order element with transfer function

\[
G(s) = \frac{1}{\frac{1}{\omega_n^2} s^2 + \frac{2\xi}{\omega_n} s + 1}
\]

is subject to a unit step input signal, then the Laplace transform of the element output signal is:

\[
f_o(s) = \frac{1}{\left(\frac{1}{\omega_n^2} s^2 + \frac{2\xi}{\omega_n} s + 1\right)s}
\]

Expressing [4.27] in partial fractions we have:

\[
f_o(s) = \frac{As + B}{\left(\frac{1}{\omega_n^2} s^2 + \frac{2\xi}{\omega_n} s + 1\right)s} + \frac{C}{s}
\]

where \(A = -1/\omega_n^2\), \(B = -2\xi/\omega_n\), \(C = 1\). This gives:

\[
f_o(s) = \frac{1}{s} - \frac{(s + 2\xi\omega_n)}{s^2 + 2\xi\omega_ns + \omega_n^2}
\]

\[
= \frac{1}{s} - \frac{(s + 2\xi\omega_n)}{(s + \xi\omega_n)\omega_n(1 - \xi^2)}
\]

\[
= \frac{1}{s} - \frac{(s + \xi\omega_n)}{(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)} - \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_n^2(1 - \xi^2)}
\]

[4.28]

There are three cases to consider, depending on whether \(\xi\) is greater than 1, equal to 1, or less than 1. For example, if \(\xi = 1\) (critical damping) then:

\[
f_o(s) = \frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}
\]

[4.29]
Using Table 4.1 we have:

\[
f_o(t) = 1 - e^{-\omega n \sqrt{1 - \xi^2}}(1 + \omega_n t)
\]  

[4.30]

Using standard tables it can be shown that if \( \xi < 1 \) (underdamping) then:

\[
f_o(t) = 1 - e^{-\omega n \sqrt{1 - \xi^2}} \left[ \cos \omega_n \sqrt{(1 - \xi^2)} t + \frac{\xi}{\sqrt{1 - \xi^2}} \sin \omega_n \sqrt{(1 - \xi^2)} t \right]
\]  

[4.31]

and if \( \xi > 1 \) (overdamping) then:

\[
f_o(t) = 1 - e^{-\omega n \sqrt{1 - \xi^2}} \left[ \cosh \omega_n \sqrt{(\xi^2 - 1)} t + \frac{\xi}{\sqrt{\xi^2 - 1}} \sinh \omega_n \sqrt{(\xi^2 - 1)} t \right]
\]  

[4.32]

We consider the case of the underdamped response with \( \xi < 1 \), given by eqn [4.31]. Here the **damped angular frequency** of the oscillations is given by:

\[
\omega_d = \omega_n \sqrt{1 - \xi^2}
\]

where \( \omega_n \) is the natural or undamped angular frequency. The corresponding period \( T_d \) of the damped oscillations is given by:

\[
T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1 - \xi^2}}
\]

The time \( T_p \) at which the first oscillation peak occurs is correspondingly given by:

\[
T_p = T_d / 2 = \frac{\pi}{\omega_n \sqrt{1 - \xi^2}}
\]

The **settling time** \( T_s \) taken for the response to settle out at the steady-state value is governed by the exponential decay term \( e^{-\omega n \sqrt{1 - \xi^2}} \). When \( t = 5/\xi \omega_n \), i.e. \( \xi \omega_n t = 5 \), then \( e^{-5} = 0.0067 \). The time for the response to settle out approximately within 1% is therefore:

\[
T_s = 5/\xi \omega_n
\]

From Figure 4.7, we see that \( T_s \) is minimum when \( \xi = 0.7 \). The response \( f_o(t) \) has a maximum value \( f_o^m \) at the peak of the first oscillation. The difference \( (f_o^m - 1) \) between maximum and steady-state values of \( f_o(t) \), for \( \xi < 1 \), is termed the **maximum overshoot** and is given by:

\[
\exp \left[ -\frac{-\pi \xi}{\sqrt{1 - \xi^2}} \right]
\]

and depends only on \( \xi \). Thus for \( \xi < 1 \), an estimate of \( \xi \) can be found from measurement of maximum overshoot and then, knowing \( \xi \), \( \omega_n \) can be estimated from measurements of \( T_s, T_p \) and/or \( T_s \).
4.2 IDENTIFICATION OF THE DYNAMICS OF AN ELEMENT

The form of the responses is shown in Figure 4.7. As an example consider the step response of a force sensor with stiffness $k = 10^3$ N m$^{-1}$, mass $m = 10^{-1}$ kg and damping constant $\lambda = 10$ N s m$^{-1}$. The steady-state sensitivity $K = 1/k = 10^{-3}$ m N$^{-1}$, natural frequency $\omega_n = \sqrt{k/m} = 10^2$ rad s$^{-1}$ and damping coefficient $\xi = \lambda/2\sqrt{k/m} = 0.5$.

Initially at time $t = 0^-$, a steady force $F(0-) = 10$ N causes a steady displacement of $(1/103) \times 10$ metre, i.e. 10 mm. Suppose that at $t = 0$ the force is suddenly increased from 10 to 12 N, i.e. there is a step change $\Delta F$ of 2 N. The resulting change $\Delta x(t)$ in displacement is found using

$$\Delta x(t) = \text{steady-state sensitivity} \times \text{step height} \times \text{unit step response} f_s(t) \quad [4.33]$$

i.e.

$$\Delta x(t) = \frac{1}{10^3} \times 2 \times [1 - e^{-50t} (\cos 86.6t + 0.58 \sin 86.6t)] \text{ metre}$$

$$= 2[1 - e^{-50}(\cos 86.6t + 0.58 \sin 86.6t)] \text{ mm} \quad [4.34]$$

Here the damped angular frequency $\omega_d = 86.6$ rad/s and the period of the damped oscillations $T_d = 72.6$ ms. The maximum overshoot is 0.16, giving $f_s'(0) = 1.16$ so that the maximum value of $\Delta x$ is 2.32 mm. This occurs at time $T_p = 36.3$ ms. Eventually as $t$ becomes large $\Delta x$ tends to 2 mm, i.e. $x$ settles out at a new steady-state value of 12 mm with a settling time $T_s$ of 100 ms.

4.2.2 Sinusoidal response of first- and second-order elements

From Table 4.1 we see that the Laplace transform of sine wave $f(t) = \sin \omega t$, with unit amplitude and angular frequency $\omega$, is $f(s) = \omega/(s^2 + \omega^2)$. Thus if a sine wave of amplitude $I$ is input to a first-order element, then the Laplace transform of the output signal is
Expressing \( g(s) = \frac{4.35}{s^2 + \omega^2} \) in partial fractions we have

\[
f_o(s) = \frac{A}{1 + \tau s} + \frac{Bs + C}{s^2 + \omega^2}
\]

where:

\[
A = \frac{\omega \tau \omega^2}{1 + \tau^2 \omega^2}, \quad B = \frac{-\omega \tau \omega^2}{1 + \tau^2 \omega^2}, \quad C = \frac{\omega \omega^2}{1 + \tau^2 \omega^2}
\]

so that:

\[
f_o(s) = \frac{\omega \tau \omega^2}{(1 + \tau^2 \omega^2)(1 + \tau s)} + \frac{\frac{\omega \omega^2}{1 + \tau^2 \omega^2}}{(1 + \tau^2 \omega^2)} \left( \frac{-\omega \tau s + \omega}{s^2 + \omega^2} \right)
\]

\[
= \frac{\omega \tau \omega^2}{(1 + \tau^2 \omega^2)(1 + \tau s)} + \frac{\frac{\omega}{\sqrt{1 + \tau^2 \omega^2}}}{\sqrt{1 + \tau^2 \omega^2}} \left( \frac{-\omega \tau}{s^2 + \omega^2} \right)
\]

\[
= \frac{\omega \tau \omega^2}{(1 + \tau^2 \omega^2)(1 + \tau s)} + \frac{\frac{\omega \cos \phi + \omega \sin \phi}{\sqrt{1 + \tau^2 \omega^2}}}{s^2 + \omega^2}
\]

where:

\[
\cos \phi = \frac{1}{\sqrt{1 + \tau^2 \omega^2}}, \quad \sin \phi = \frac{-\omega \tau}{\sqrt{1 + \tau^2 \omega^2}}
\]

Using Table 4.1 we have:

\[
f_o(t) = \frac{\omega \tau \omega^2}{1 + \tau^2 \omega^2} e^{-\tau t} + \frac{\frac{\omega}{\sqrt{1 + \tau^2 \omega^2}}}{\sqrt{1 + \tau^2 \omega^2}} \frac{\omega}{\omega} \sin(\omega t + \phi)
\]

In a sine wave test experiment, we wait until the transient term has decayed to zero and measure the sinusoidal signal:

\[
f_o(t) = \frac{\frac{\omega}{\sqrt{1 + \tau^2 \omega^2}}}{\sqrt{1 + \tau^2 \omega^2}} \sin(\omega t + \phi)
\]

which remains. We see therefore that the output signal is also a sine wave of frequency \( \omega \) but with amplitude \( \omega \sqrt{1 + \tau^2 \omega^2} \), and shifted in phase by \( \phi = -\tan^{-1}(\omega \tau) \) relative to the input sine wave. These amplitude and phase results can be found directly from the transfer function \( G(s) = 1/(1 + \tau s) \) without having to use the table of transforms. If we replace \( s \) by \( j \omega \) (\( j = \sqrt{-1} \)) in \( G(s) \) we form the complex number \( G(j \omega) = 1/(1 + j \tau \omega) \). The magnitude \( |G(j \omega)| = 1/\sqrt{1 + \tau^2 \omega^2} \) of this complex number is equal to the ratio of output amplitude to input amplitude, and the angle or argument \( \arg G(j \omega) = -\tan^{-1}(\omega \tau) \) is equal to the phase difference \( \phi \) between output and input sine waves. Figure 4.8 shows amplitude ratio versus frequency and phase versus frequency graphs for a first-order element; these are known as the frequency response characteristics of the element. From the above equations we see that when \( \omega \tau = 1 \),
i.e. $\omega = 1/\tau$, the amplitude ratio $= 1/\sqrt{2}$ and phase difference $\phi = -45^\circ$. These results enable the value of $\tau$ to be found from experimental frequency response data.

The above results can be generalised to an element with steady-state sensitivity $K$ (or $\partial O/\partial I$) and transfer function $G(s)$, subject to a sinusoidal input signal $I = I \sin \omega t$ as in Figure 4.9. In the steady state we can make four statements about the output signal:

(a) $O$ is also a sine wave.
(b) The frequency of $O$ is also $\omega$.
(c) The amplitude of $O$ is $\bar{O} = K|G(j\omega)|\bar{I}$.
(d) The phase difference between $O$ and $I$ is $\phi = \arg G(j\omega)$.

Using the above rules we can quickly find the amplitude ratio and phase relations for a second-order element with:

$$G(s) = \frac{1}{\omega^2 + \frac{2\xi}{\omega_n}s + 1}$$

Here we have:

$$G(j\omega) = \frac{1}{\omega^2(j\omega)^2 + \frac{2\xi}{\omega_n}(j\omega) + 1}$$
so that

\[
Amplitude \ ratio = |G(j\omega)| = \frac{1}{\sqrt{(1 - \frac{\omega^2}{\omega_n^2})^2 + 4\xi^2\frac{\omega^2}{\omega_n^2}}}
\]

\[
Phase \ difference = \arg G(j\omega) = -\tan^{-1}\left[\frac{2\xi\omega/\omega_n}{1 - \omega^2/\omega_n^2}\right]
\]

These characteristics are shown graphically in Figure 4.10; both amplitude ratio and phase characteristics are critically dependent on the value of $\xi$. We note that for $\xi < 0.7$, $|G(j\omega)|$ has a maximum value which is greater than unity. This maximum value is given by:

\[
|G(j\omega)|_{\text{MAX}} = \frac{1}{2\xi\sqrt{1 - \xi^2}}
\]

and occurs at the resonant frequency $\omega_n$ where:

\[
\omega_n = \omega_n\sqrt{1 - 2\xi^2}, \quad \xi < 1/\sqrt{2}
\]
4.3 Dynamic errors in measurement systems

Thus by measuring $|G(j\omega)|_{\text{MAX}}$ and $\omega_r$, $\xi$ and $\omega_n$ can be found. An alternative to plotting $|G(j\omega)|$ versus $\omega$ is a graph of the number of decibels $N_{\text{dB}}$ versus $\omega$, where $N = 20 \log_{10}|G(j\omega)|$. Thus if $|G(j\omega)| = 1$, $N = 0$ dB; if $|G(j\omega)| = 10$, $N = +20$ dB; and if $|G(j\omega)| = 0.1$, $N = -20$ dB.

Dynamic errors in measurement systems

Figure 4.11 shows a complete measurement system consisting of $n$ elements. Each element $i$ has ideal steady-state and linear dynamic characteristics and can therefore be represented by a constant steady-state sensitivity $K_i$ and a transfer function $G_i(s)$.

In principle we can use eqn [4.41] to find the system output signal $\Delta O(t)$ corresponding to a time-varying input signal $\Delta I(t)$. We first find the Laplace transform $\Delta \hat{I}(s)$ of $\Delta I(t)$, then using [4.9] the Laplace transform of the output signal is $\Delta \hat{O}(s) = G(s)\Delta \hat{I}(s)$. By expressing $\Delta \hat{O}(s)$ in partial fractions, and using standard tables of Laplace transforms, we can find the corresponding time signal $\Delta O(t)$. Expressing this mathematically:

$$\Delta O(t) = \mathcal{L}^{-1}\left[G(s)\Delta \hat{I}(s)\right]$$  \[4.42\]

where $\mathcal{L}^{-1}$ denotes the inverse Laplace transform. The dynamic error $E(t)$ of the measurement system is the difference between the measured signal and the true signal, i.e. the difference between $\Delta O(t)$ and $\Delta I(t)$:

$$E(t) = \Delta O(t) - \Delta I(t)$$  \[4.43\]

Using [4.42] we have:

$$E(t) = \mathcal{L}^{-1}\left[G(s)\Delta \hat{I}(s)\right] - \Delta I(t)$$  \[4.44\]

The simple temperature measurement system (Figure 4.12), first introduced in Section 3.1, provides a good example of dynamic errors. The thermocouple has a time constant of 10 s, the amplifier has a time constant of $10^{-4}$ s (Chapter 9), and the recorder (Chapter 11) is a second-order element with $\omega_n = 200$ rad/s and $\xi = 1.0$. The overall steady-state sensitivity of the system is unity.
We can now calculate the dynamic error of the system for a step input of $+20\,\text{°C}$, i.e. $\Delta T_T(t) = 20u(t)$ and $\Delta T_T(s) = 20(1/s)$. Thus the Laplace transform of the output signal is:

$$\Delta T_M(s) = \frac{1}{s(1+10s)(1+10^{-4}s)(1+1/200s)^2} - \frac{A}{s+0.1} - \frac{B}{s+10^3} - \frac{C s + D}{s+200} \quad [4.45]$$

Using Table 4.1 and eqn [4.30],

$$\Delta T_M(t) = 20\{u(t) - A e^{-0.1t} - B e^{-10^4t} - E e^{-200t}(1+200t)\}$$

and the dynamic error:

$$E(t) = \Delta T_M(t) - \Delta T_T(t) = -20\{A e^{-0.1t} + B e^{-10^4t} - E e^{-200t}(1+200t)\} \quad [4.46]$$

where the negative sign indicates too low a reading. The $B e^{-10^4t}$ term decays to zero after about $5 \times 10^{-4} \text{s}$, and the $E e^{-200t}(1+200t)$ term decays to zero after about $25 \text{ms}$. The $A e^{-0.1t}$ term, which corresponds to the 10s time constant of the thermocouple, takes about 50 s to decay to zero and so has the greatest effect on the dynamic error.

We can use the rules developed in Section 4.2.2 to find the dynamic error of a system, with transfer function $G(s)$ subject to a sinusoidal input $\Delta I(t) = I \sin \omega t$. From Figure 4.9 we have:

$$\Delta O(t) = |G(j\omega)| \hat{I} \sin(\omega t + \phi)$$

giving

$$E(t) = \hat{I} \{ |G(j\omega)| \sin(\omega t + \phi) - \sin \omega t \} \quad [4.47]$$

where $\phi = \arg G(j\omega)$.

Suppose that the above temperature measurement system is measuring a sinusoidal temperature variation of amplitude $\tilde{T}_T = 20\,\text{°C}$ and period $T = 6 \text{s}$, i.e. angular frequency $\omega = 2\pi/T = 1.0 \text{ rad s}^{-1}$. The frequency response function $G(j\omega)$ is:

$$G(j\omega) = \frac{1}{(1+10j\omega)(1+10^{-4}j\omega)[1+10^{-2}j\omega+2.5 \times 10^{-5}(j\omega)^2]} \quad [4.48]$$

so that at $\omega = 1$

$$|G(j\omega)|_{\omega=1} = \frac{1}{\sqrt{(1+100)(1+10^{-8})(1-2.5 \times 10^{-5})^2 + 10^{-4}}} = 0.10$$

and

$$\arg G(j\omega)_{\omega=1} = -\tan^{-1}(10) - \tan^{-1}(10^{-4}) - \tan^{-1}(10^{-3}) = -85^\circ$$
We note from the above equations that the values of |G(jω)| and arg G(jω) at ω = 1 are determined mainly by the 10 s time constant; the dynamic characteristics of the other elements will only begin to affect the system performance at much higher frequencies. Since \( T_1(t) = 20 \sin t \) and \( T_2(t) = 0.1 \times 20 \sin(t - 85^\circ) \), the error is:

\[
E(t) = 20\{0.1 \sin(t - 85^\circ) - \sin t\} \quad \text{[4.50]}
\]

We note that in the case of a sine wave input, the output recording is also a sine wave, i.e. the waveform of the signal is unaltered even though there is a reduction in amplitude and a phase shift.

In practice the input signal to a measurement system is more likely to be periodic rather than a simple sine wave. A periodic signal is one that repeats itself at equal intervals of time \( T \), i.e. \( f(t) = f(t + T) = f(t + 2T) \) etc. where \( T \) is the period. One example of a periodic measurement signal is the time variation of the temperature inside a diesel engine; another is the vibration of the casing of a centrifugal compressor. In order to calculate dynamic errors for periodic signals, we need to use Fourier analysis. Any periodic signal \( f(t) \) with period \( T \) can be expressed as a series of sine and cosine waves; these have frequencies which are harmonics of the fundamental frequency \( \omega_1 = \frac{2\pi}{T} \) rad s\(^{-1}\), i.e.

\[
f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_1 t + \sum_{n=1}^{\infty} b_n \sin n\omega_1 t \quad \text{[4.51]}
\]

where

\[
a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega_1 t \, dt \quad \text{[4.52]}
\]

\[
b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega_1 t \, dt
\]

and

\[
a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, dt = \text{average value of } f(t) \text{ over } T
\]

If \( f(t) = \Delta I(t) \), where \( \Delta I(t) \) is the deviation of measurement input signal \( I(t) \) from steady-state or d.c. value \( I_0 \), then \( a_0 = 0 \). If we also assume that \( f(t) \) is odd, i.e. \( f(t) = -f(-t) \), then \( a_n = 0 \) for all \( n \), i.e. there are only sine terms present in the series. This simplifying assumption does not affect the general conclusions drawn in the following section. The system input signal is thus given by

\[
\Delta I(t) = \sum_{n=1}^{\infty} \tilde{I}_n \sin n\omega_1 t \quad \text{[4.53]}
\]

where \( \tilde{I}_n = b_n \) is the amplitude of the \( n \)th harmonic at frequency \( n\omega_1 \). In order to find \( \Delta O(t) \), let us first suppose that only the \( n \)th harmonic \( \tilde{I}_n \sin n\omega_1 t \) is input to the system. From Figure 4.9 the corresponding output signal is

\[
\hat{I}_n \vert jn\omega_1 \vert \sin(n\omega_1 t + \phi_n)
\]

where \( \phi_n = \text{arg } G(jn\omega_1) \).
We now require to use the principle of superposition, which is a basic property of linear systems (i.e. systems described by linear differential equations). This can be stated as follows:

If an input $I_1(t)$ causes an output $O_1(t)$ and an input $I_2(t)$ causes an output $O_2(t)$, then an input $I_1(t) + I_2(t)$ causes an output $O_1(t) + O_2(t)$, provided the system is linear.

This means that if the total input signal is the sum of many sine waves (equation [4.53]), then the total output signal is the sum of the responses to each sine wave, i.e.

$$\Delta O(t) = \sum_{n=1}^{\infty} I_n |G(jn\omega_1)| \sin(n\omega_1 t + \phi_n)$$

[4.54]

The dynamic error is thus

$$\Delta E(t) = \sum_{n=1}^{\infty} I_n \{|G(jn\omega_1)| \sin(n\omega_1 t + \phi_n) - \sin n\omega_1 t\}$$

[4.55]

As an example, suppose that the input to the temperature measurement system is a square wave of amplitude 20 °C and period $T = 6$ s (i.e. $\omega_1 = 2\pi T = 1$ rad s$^{-1}$). The Fourier series for the input signal is:

$$\Delta T_s(t) = \frac{80}{\pi} \left[ \sin t + \frac{1}{3} \sin 3t + \frac{1}{5} \sin 5t + \frac{1}{7} \sin 7t + \ldots \right]$$

[4.56]

Figure 4.13 shows the waveforms of the input square wave and the first four Fourier components with frequencies 1, 3, 5 and 7 rad s$^{-1}$.

Figure 4.14 shows the amplitude–frequency and phase–frequency relationships for the input temperature; these define the frequency spectrum of the signal. The spectrum consists of a number of lines at frequencies $\omega_1$, $3\omega_1$, $5\omega_1$, etc., of decreasing length to represent the smaller amplitudes of the higher harmonics. In practical cases we can terminate or truncate the series at a harmonic where the amplitude is negligible; in this case we choose $n = 7$. In order to find the output signal, i.e. the recorded waveform, we need to evaluate the magnitude and argument of $G(j\omega)$ at $\omega = 1, 3, 5$ and 7 rad s$^{-1}$. Thus

$$|G(j)| = 0.100, \quad |G(3j)| = 0.033, \quad |G(5j)| = 0.020, \quad |G(7j)| = 0.014$$

and

$$\arg G(j) = -85^\circ, \quad \arg G(3j) = -90^\circ, \quad \arg G(5j) = -92^\circ, \quad \arg G(7j) = -93^\circ$$

[4.57]

Again the above values are determined mainly by the 10 s thermocouple time constant; the highest signal frequency $\omega_7$ is still well below the natural frequency of the recorder $\omega_n = 200$. The system output signal is:

$$\Delta T_m(t) = \frac{80}{\pi} \left[ 0.100 \sin(t - 85^\circ) + 0.011 \sin(3t - 90^\circ) \
+ 0.004 \sin(5t - 92^\circ) + 0.002 \sin(7t - 93^\circ) \right]$$

[4.58]
Figure 4.13 Waveforms for input square wave and Fourier components.

Figure 4.14 shows the system frequency response characteristics, the output signal frequency spectrum and the output waveform. We note that, in the output signal, the amplitudes of the 3rd, 5th and 7th harmonics have been reduced relative to the amplitude of the fundamental. The recorded waveform has therefore a different shape from the input signal as well as being reduced in amplitude and changed in phase.
The above ideas can be extended to calculating the dynamic error for random input signals. Random signals can be represented by continuous frequency spectra (Chapter 6).

**Figure 4.14** Calculation of dynamic errors with periodic input signal.

**4.4 Techniques for dynamic compensation**

From eqn [4.55] we see that in order to have $E(t) = 0$ for a periodic signal, the following conditions must be obeyed:

$$|G(j\omega_1)| = |G(j2\omega_1)| = \ldots = |G(jn\omega_1)| = \ldots = |G(jm\omega_1)| = 1$$

$$\arg G(j\omega_1) = \arg G(j2\omega_1) = \ldots = \arg G(jn\omega_1) = \ldots = \arg G(jm\omega_1) = 0$$

[4.59]
where $m$ is the order of the highest significant harmonic. For a random signal (Chapter 6) with a continuous frequency spectrum containing frequencies between 0 and $\omega_{\text{MAX}}$, we require:

$$ |G(j\omega)| = 1 \quad \text{and} \quad \arg G(j\omega) = 0 \quad \text{for} \quad 0 < \omega \leq \omega_{\text{MAX}} $$  \[4.60\]

The above conditions represent a theoretical ideal which will be difficult to realise in practice. A more practical criterion is one which limits the variation in $|G(j\omega)|$ to a few per cent for the frequencies present in the signal. For example, the condition

$$ 0.98 < |G(j\omega)| < 1.02 \quad \text{for} \quad 0 < \omega \leq \omega_{\text{MAX}} $$  \[4.61\]

will ensure that the dynamic error is limited to $\pm 2$ per cent for a signal containing frequencies up to $\omega_{\text{MAX}}/2\pi$ Hz (Figure 4.15).

Another commonly used criterion is that of bandwidth. The bandwidth of an element or a system is the range of frequencies for which $|G(j\omega)|$ is greater than $1/\sqrt{2}$. Thus the bandwidth of the system, with frequency response shown in Figure 4.15, is $0$ to $\omega_b$ rad s$^{-1}$. The highest signal frequency $\omega_{\text{MAX}}$ must be considerably less than $\omega_b$. Since, however, there is a $30\%$ reduction in $|G(j\omega)|$ at $\omega_b$, bandwidth is not a particularly useful criterion for complete measurement systems.

Bandwidth is commonly used in specifying the frequency response of amplifiers (Chapter 9); a reduction in $|G(j\omega)|$ from 1 to $1/\sqrt{2}$ is equivalent to a decibel change of $N = 20 \log(1/\sqrt{2}) = -3.0$ dB. A first-order element has a bandwidth between 0 and $1/\tau$ rad s$^{-1}$.

If a system fails to meet the specified limits on dynamic error $E(t)$, i.e. the system transfer function $G(s)$ does not satisfy a condition such as [4.61], then the first step is to identify which elements in the system dominate the dynamic behaviour. In the temperature measurement system of the previous section, the dynamic error is almost entirely due to the $10\,\text{s}$ time constant of the thermocouple.

Having identified the dominant elements in the system, the most obvious method of improving dynamic response is that of inherent design. In the case of a first-order temperature sensor with $\tau = MC/UA$, $\tau$ can be minimised by minimising the mass/area ratio $M/A$ – for example by using a thermistor in the form of a thin flake. In the case of a second-order force sensor with $\omega_n = \sqrt{k/m}$, $\omega_n$ can be maximised by maximising $k/m$, i.e. by using high stiffness $k$ and low mass $m$. Increasing $k$, however, reduces the steady-state sensitivity $K = 1/k$. 

![Figure 4.15 Percentage limits and bandwidth.](image-url)
From second-order step and frequency response graphs we see that the optimum value of damping ratio $\xi$ is around 0.7. This value ensures minimum settling time for the step response and $|G(j\omega)|$ closest to unity for the frequency response.

Another possible method is that of open-loop dynamic compensation (Figure 4.16). Given an uncompensated element or system $G_{U}(s)$, a compensating element $G_{C}(s)$ is introduced into the system, such that the overall transfer function $G(s) = G_{U}(s)G_{C}(s)$ satisfies the required condition (for example eqn [4.61]). Thus if a lead/lag circuit (Figure 9.12) is used with a thermocouple (Figure 4.16), the overall time constant is reduced to $\tau_2$, so that $|G(j\omega)|$ is close to unity over a wider range of frequencies. The main problem with this method is that $\tau$ can change with heat transfer coefficient $U$, thus reducing the effectiveness of the compensation (Chapter 14).

Another method is to incorporate the element to be compensated into a closed-loop system with high-gain negative feedback. An example of this is the constant temperature anemometer system for measuring fluid velocity fluctuations (Section 14.3). Another example is the closed-loop accelerometer shown in schematic and block diagram form in Figure 4.17.

Figure 4.16 Open-loop dynamic compensation.

Figure 4.17 Schematic and block diagram of closed-loop accelerometer.
The applied acceleration $a$ produces an inertia force $ma$ on the seismic mass $m$ (Chapter 8). This is balanced by the force of the permanent magnet on the current feedback coil. Any imbalance of forces is detected by the elastic force element to produce a displacement which is detected by a potentiometric displacement sensor (Chapter 5). The potentiometer output voltage is amplified, giving a current output which is fed to the feedback coil through a standard resistor to give the output voltage.

Analysis of the block diagram shows that the overall system transfer function is:

$$\Delta \tilde{P}(s) = \frac{mR}{K_F} \left\{ \frac{1}{K_J K_D K_F} \frac{K}{\omega_n^2 s^2 + \frac{2\xi}{\omega_n} \frac{k s}{\omega_n} + \left(1 + \frac{1}{K_J K_D K_F}\right)} \right\}$$ [4.62]

If $K_F$ is made large so that $K_J K_D K_F / k \gg 1$ (Chapter 3), then the system transfer function can be expressed in the form:

$$\Delta \tilde{P}(s) = \frac{K_s}{\omega_n^2 s^2 + \frac{2\xi_s}{\omega_n} s + 1}$$

where

- system steady-state sensitivity $K_s = \frac{mR}{K_F}$
- system natural frequency $\omega_n = \frac{\sqrt{K_J K_D K_F}}{k}$
- system damping ratio $\xi_s = \xi \sqrt{\frac{k}{K_J K_D K_F}}$ [4.63]

We see that the system natural frequency $\omega_n$ is now much greater than that of the elastic force element itself. The system damping ratio $\xi_s$ is much less than $\xi$, but by making $\xi$ large a value of $\xi_s = 0.7$ can be obtained. Furthermore the system steady-state sensitivity depends only on $m$, $K_F$ and $R$, which can be made constant to a high degree.

Conclusion

The dynamic characteristics of typical measurement system elements were initially discussed; in particular the transfer functions of first- and second-order elements were derived. The response of both first- and second-order elements to step and sine wave inputs was then studied. A general description of the dynamic error of a complete measurement system was then developed and applied to a temperature measurement system subject to step, sine wave and periodic input signals. Finally methods of dynamic compensation, which reduce dynamic error, were explained.
Problems

4.1 A temperature measurement system consists of linear elements and has an overall steady-state sensitivity of unity. The dynamics of the system are determined by the first-order transfer function of the sensing element. At time \( t = 0 \), the sensing element is suddenly transferred from air at 20 °C to boiling water. One minute later the element is suddenly transferred back to air. Using the data given below, calculate the system dynamic error at the following times: \( t = 10, 20, 50, 120 \) and \( 300 \) s.

Sensor data
- Mass = \( 5 \times 10^{-2} \) kg
- Surface area = \( 10^{-3} \) m²
- Specific heat = \( 0.2 \) J kg\(^{-1}\) °C\(^{-1}\)
- Heat transfer coefficient for air = \( 0.2 \) W m\(^{-2}\) °C\(^{-1}\)
- Heat transfer coefficient for water = \( 1.0 \) W m\(^{-2}\) °C\(^{-1}\)

4.2 A force sensor has a mass of 0.5 kg, stiffness of \( 2 \times 10^2 \) N m\(^{-1}\) and a damping constant of 6.0 N s m\(^{-1}\).

(a) Calculate the steady-state sensitivity, natural frequency and damping ratio for the sensor.
(b) Calculate the displacement of the sensor for a steady input force of 2 N.
(c) If the input force is suddenly increased from 2 to 3 N, derive an expression for the resulting displacement of the sensor.

4.3 A force measurement system consists of linear elements and has an overall steady-state sensitivity of unity. The dynamics of the system are determined by the second-order transfer function of the sensing element, which has a natural frequency \( \omega_n = 40 \) rad s\(^{-1}\) and a damping ratio \( \xi = 0.1 \). Calculate the system dynamic error corresponding to the periodic input force signal:

\[
F(t) = 50(\sin 10t + \frac{1}{3} \sin 30t + \frac{1}{5} \sin 50t)
\]

4.4 An uncompensated thermocouple has a time constant of 10 s in a fast-moving liquid.

(a) Calculate the bandwidth of the thermocouple frequency response.
(b) Find the range of frequencies for which the amplitude ratio of the uncompensated thermocouple is flat within ±5%.
(c) A lead/lag circuit with transfer function \( G(s) = (1 + 10s)/(1 + s) \) is used to compensate for thermocouple dynamics. Calculate the range of frequencies for which the amplitude ratio of the compensated system is flat within ±5%.
(d) The velocity of the liquid is reduced, causing the thermocouple time constant to increase to 20 s. By sketching |\( G(j\omega) \)| explain why the effectiveness of the above compensation is reduced.

4.5 An elastic force sensor has an effective seismic mass of 0.1 kg, a spring stiffness of 10 N m\(^{-1}\) and a damping constant of 14 N s m\(^{-1}\).

(a) Calculate the following quantities:
   (i) sensor natural frequency
   (ii) sensor damping ratio
   (iii) transfer function relating displacement and force.

(b) The above sensor is incorporated into a closed-loop, force balance accelerometer. The following components are also present:
- Potentiometer displacement sensor: sensitivity 1.0 V m\(^{-1}\)
- Amplifier: voltage input, current output, sensitivity 40 A V\(^{-1}\)
- Coil and magnet: current input, force output, sensitivity 25 N A\(^{-1}\)
- Resistor: 250 Ω.
(i) Draw a block diagram of the accelerometer.
(ii) Calculate the overall accelerometer transfer function.
(iii) Explain why the dynamic performance of the accelerometer is superior to that of the elastic sensor.

4.6 A load cell consists of an elastic cantilever and a displacement transducer. The cantilever has a stiffness of $10^3 \text{ N m}^{-1}$, a mass of 0.5 kg and a damping constant of 2 N s m$^{-1}$. The displacement transducer has a steady-state sensitivity of 10 V m$^{-1}$.

(a) A package of mass 0.5 kg is suddenly dropped onto the load cell. Use eqn [4.31] to derive a numerical equation describing the corresponding time variation of the output voltage ($g = 9.81 \text{ m s}^{-2}$).
(b) The load cell is used to weigh packages moving along a conveyor belt at the rate of 60 per minute. Use the equation derived in (a) to explain why the load cell is unsuitable for this application. Explain what modifications to the load cell are necessary.

4.7 A force measurement system consisting of a piezoelectric crystal, charge amplifier and recorder is shown in Figure Prob. 7.

(a) Calculate the system dynamic error corresponding to the force input signal:

$$F(t) = 50(\sin 10t + \frac{1}{3} \sin 30t + \frac{1}{5} \sin 50t)$$

(b) Explain briefly the system modifications necessary to reduce the error in (a) (hint: see Figure 8.22).

4.8 A thermocouple is used to measure the temperature inside a vessel, which is part of a high-speed batch process. At time $t = 0$, with the vessel at an initial temperature of 50 °C, the vessel is instantaneously filled with gas at 150 °C. One minute later, instantaneously the gas is removed and the vessel is filled with liquid at 50 °C. The thermocouple can be regarded as having linear steady-state characteristics and first-order dynamics.

(a) Use the data given below to sketch a graph of how the thermocouple e.m.f. changes with time. The axes of the graph should have suitable scales and the answer should include supporting numerical calculations.
(b) Comment on whether the thermocouple is suitable for this application.
(c) What modifications should be made?

**Data**

- Thermocouple sensitivity = 40 \(\mu\text{V} \text{ °C}^{-1}\)
- Thermocouple mass = $5 \times 10^{-2}$ kg
- Thermocouple specific heat = 0.2 J kg$^{-1}$ °C$^{-1}$
- Thermocouple surface area = $10^{-3}$ m$^2$
- Heat transfer coefficient for gas = 0.2 W m$^{-2}$ °C$^{-1}$
- Heat transfer coefficient for liquid = 1.0 W m$^{-2}$ °C$^{-1}$

Response of first-order element, with unit sensitivity, to unit step:

$$F^0(t) = 1 - \exp \left( \frac{-t}{\tau} \right)$$
A temperature measurement system for a gas reactor consists of linear elements and has an overall steady-state sensitivity of unity. The temperature sensor has a time constant of 5.0 s; an ideal low-pass filter with a cut-off frequency of 0.05 Hz is also present. The input temperature signal is periodic with period 63 s and can be approximated by the Fourier series:

\[ T(t) = 10(sin \omega_0 t + \frac{1}{2} sin 2\omega_0 t + \frac{1}{3} sin 3\omega_0 t + \frac{1}{4} sin 4\omega_0 t) \]

where \( \omega_0 \) is the angular frequency of the fundamental component.

(a) Calculate expressions for the time response of:
   (i) the system output signal
   (ii) the system dynamic error.

(b) Explain what modifications are necessary to the system to minimise the dynamic error in (a).

Note
An ideal low-pass filter has a gain of one and zero phase shift up to the cut-off frequency. The gain is zero above the cut-off frequency.
In our discussion of measurement systems no consideration has yet been given to the effects of loading. One important effect is that of inter-element loading where a given element in the system may modify the characteristics of the previous element (for example by drawing current). In turn, the characteristics of this element may be modified by the following element in the system. Inter-element loading is normally an electrical loading effect which is described in the first section of this chapter using Thévenin and Norton equivalent circuits. The second section begins by discussing the analogies between electrical and non-electrical variables. This means that mechanical and thermal systems can be described by equivalent circuits and sensing elements by two-port networks. Two-port networks are then used to describe process loading; here the introduction of the sensing element into the process or system being measured causes the value of the measured variable to change. Finally two-port networks are used to describe bilateral transducers which use reversible physical effects.

5.1 Electrical loading

We have so far represented measurement systems as blocks connected by single lines where the transfer of information and energy is in terms of one variable only. Thus in the temperature measurement system of Figure 3.2 the information transfer between elements is in terms of voltage only. No allowance can therefore be made for the amplifier drawing current from the thermocouple and the indicator drawing current from the amplifier. In order to describe both voltage and current behaviour at the connection of two elements, we need to represent each element by equivalent circuits characterised by two terminals. The connection is then shown by two lines.

5.1.1 Thévenin equivalent circuit

Thévenin’s theorem states that any network consisting of linear impedances and voltage sources can be replaced by an equivalent circuit consisting of a voltage source $E_{\text{Th}}$ and a series impedance $Z_{\text{Th}}$ (Figure 5.1). The source $E_{\text{Th}}$ is equal to the open circuit voltage of the network across the output terminals, and $Z_{\text{Th}}$ is the impedance looking back into these terminals with all voltage sources reduced to zero and replaced by their internal impedances. Thus connecting a load $Z_L$ across the output...
terminals of the network is equivalent to connecting $Z_L$ across the Thévenin circuit. The current $i$ in $Z_L$ is then simply given by:

$$i = \frac{E_{Th}}{Z_{Th} + Z_L} \quad [5.1]$$

and the voltage $V_L$ across the load by:

$$V_L = iZ_L = E_{Th} \frac{Z_L}{Z_{Th} + Z_L} \quad [5.2]$$

From eqn [5.2] we see that if $Z_L \gg Z_{Th}$, then $V_L = E_{Th}$; i.e. in order to get maximum voltage transfer from the network to the load, the load impedance should be far greater than the Thévenin impedance for the network. In order to get maximum power transfer from network to load, the load impedance should be equal to the network impedance; i.e. $Z_L = Z_{Th}$.[1] (An example of the calculation of $E_{Th}$ and $Z_{Th}$ for a potentiometer displacement transducer is given in the following section and for a deflection bridge in Section 9.1.)

We can now discuss the Thévenin equivalent circuit for the temperature measurement system of Figure 3.2. The thermocouple may be represented by $Z_{Th} = 20 \, \Omega$ (resistive) and $E_{Th} = 40T \, \mu \text{V}$, where $T$ is the measurement junction temperature, if non-linear and reference junction temperature effects are ignored. The amplifier acts both as a load for the thermocouple and as a voltage source for the indicator. Figure 5.2 shows a general equivalent circuit for an amplifier with two pairs of terminals. Using typical amplifier data (Chapter 9), we have input impedance $Z_{IN} = R_{IN} = 2 \times 10^6 \, \Omega$, closed-loop voltage gain $A = 10^3$, and output impedance $Z_{OUT} = R_{OUT} = 75 \, \Omega$. The indicator is a resistive load of $10^4 \, \Omega$. The complete equivalent circuit for the system is shown in Figure 5.3, and using eqn [5.2] we have:

$$V_{IN} = 40 \times 10^{-6}T \left( \frac{2 \times 10^6}{2 \times 10^6 + 20} \right)$$

$$V_L = 1000 \left( \frac{10^4}{10^4 + 75} \right) \quad [5.3]$$

If the indicator scale is drawn so that a change of $1 \, \text{V}$ in $V_L$ causes a change in deflection of $25 \, ^\circ \text{C}$, then the measured temperature $T_M = 25V_L$. This gives:
5.1 ELECTRICAL LOADING

\[ T_M = \left( \frac{2 \times 10^6}{2 \times 10^6 + 20} \right) \left( \frac{10^4}{10^4 + 75} \right) = 0.9925T \]  

\[ [5.4] \]

i.e. we have to introduce the factor \( Z_L / (Z_{Th} + Z_L) \) at every interconnection of two elements to allow for loading. The loading error is \(-0.0075T\); this is in addition to the steady-state error due to element imperfections calculated in Chapter 3.

The loading error in the above example is small, but if care is not taken it can be very large. Suppose a pH glass electrode (Chapter 8), with sensitivity 59 mV per pH, i.e. \( E_{Th} = 59 \) pH mV and \( Z_{Th} = R_{Th} = 10^9 \) Ω, is directly connected to an indicator with \( Z_L = R_I = 10^4 \) Ω and a scale of sensitivity \( \frac{1}{59} \) pH/mV. The measured pH is:

\[ \text{pH}_M = 59 \text{pH} \left( \frac{10^4}{10^4 + 10^9} \right) \left( \frac{1}{59} \right) = 10^{-5} \text{pH} \]  

\[ [5.5] \]

i.e. there will be effectively a zero indication for any non-zero value.

This problem is solved by connecting the electrode to the indicator via a buffer amplifier. This is characterised by large \( Z_{IN} \), small \( Z_{OUT} \) and unity gain \( A = 1 \). For example, an operational amplifier with a field effect transistor (FET) input stage connected as a voltage follower (Figure 9.9) would have \( Z_{IN} = 10^{12} \) Ω, \( Z_{OUT} = 10 \) Ω. The indicated pH value for the modified system (Figure 5.4) is:

\[ \text{pH}_M = \frac{10^{12}}{10^{12} + 10^9} \times \frac{10^4}{10^4 + 10^9} \text{pH} \]  

\[ [5.6] \]

and the loading error is now \(-0.002\) pH, which is negligible.
An example of a.c. loading effects is given in Figure 5.5, which shows the equivalent circuit for a variable reluctance tachogenerator connected to a recorder. The Thévenin voltage $E_{Th}$ for the tachogenerator is a.c. with an amplitude $\hat{E}$ and angular frequency $\omega$, both proportional to the mechanical angular velocity $\omega_r$ (Section 8.4). In this example, $\hat{E} = (5.0 \times 10^{-3}) \omega_r$ volts and $\omega = 6\omega_r$ rad s$^{-1}$. The Thévenin impedance $Z_{Th}$ for the tachogenerator is an inductance and resistance in series (coil surrounding magnet), i.e. $Z_{Th} = R_{Th} + j\omega L_{Th}$. Thus if $\omega_r = 10^3$ rad s$^{-1}$:

$$\hat{E} = 5 \text{ V}, \quad \omega = 6 \times 10^3 \text{ rad s}^{-1}$$

and

$$Z_{Th} = 1.5 + 6.0j \text{ k}\Omega$$

so that the amplitude of the recorded voltage is

$$V_L = \frac{\hat{E}}{|Z_{Th} + R_L|} = 5 \frac{10}{\sqrt{(11.5)^2 + (6.0)^2}} = 3.85 \text{ V}$$

[5.7]

If the recorder scale sensitivity is set at $1/(5 \times 10^{-3})$ rad s$^{-1}$ V$^{-1}$, then the recorded angular velocity is 770 rad s$^{-1}$. This error can be removed either by increasing the recorder impedance, or by changing the recorder sensitivity to allow for loading effects. A better alternative is to replace the recorder by a counter which measures the frequency rather than the amplitude of the tachogenerator signal (Section 10.3).

### 5.1.2 Example of Thévenin equivalent circuit calculation: Potentiometric displacement sensor

Figure 5.6 shows a schematic diagram of a potentiometric sensor for measuring displacements $d$. The resistance of the potentiometer varies linearly with displacement.
Thus if \( x = d/d_P \) is the fractional displacement, the corresponding resistance is \( R_P x \), where \( R_P \, \Omega \) is the total resistance of the potentiometer. The Thévenin voltage \( E_{Th} \) is the open circuit voltage across the output terminals \( AB \). The ratio between \( E_{Th} \) and supply voltage \( V_S \) is equal to the ratio of fractional resistance \( R_P x \) to total resistance \( R_P \); that is

\[
\frac{E_{Th}}{V_S} = \frac{R_P x}{R_P}, \quad \text{giving} \quad E_{Th} = V_S x \tag{5.8}
\]

The Thévenin impedance \( Z_{Th} \) is found by setting supply voltage \( V_S = 0 \), replacing the supply by its internal impedance (assumed to be zero), and calculating the impedance looking back into the terminals \( AB \) as shown in Figure 5.7. Thus:

\[
\frac{1}{R_{Th}} = \frac{1}{R_P x} + \frac{1}{R_P (1 - x)}
\]

giving

\[
R_{Th} = R_P x (1 - x) \tag{5.9}
\]

Thus the effect of connecting a resistive load \( R_L \) (recorder or indicator) across the terminals \( AB \) is equivalent to connecting \( R_L \) across the Thévenin circuit. The load voltage is thus:

\[
V_L = E_{Th} \frac{R_L}{R_{Th} + R_L} = V_S x \frac{R_L}{R_P x (1 - x) + R_L}
\]

i.e.

\[
V_L = V_S x \frac{1}{(R_P/R_L) x (1 - x) + 1} \tag{5.10}
\]

The relationship between \( V_L \) and \( x \) is non-linear, the amount of non-linearity depending on the ratio \( R_P/R_L \) (Figure 5.8). Thus the effect of loading a linear potentiometric sensor is to introduce a non-linear error into the system given by:

\[
N(x) = E_{Th} - V_L = V_S x \left\{ 1 - \frac{1}{(R_P/R_L) x (1 - x) + 1} \right\} \tag{5.11}
\]

i.e.

\[
N(x) = V_S \left\{ \frac{x^2 (1 - x) (R_P/R_L)}{1 + (R_P/R_L) x (1 - x)} \right\}
\]
which reduces to:

\[ N(x) = V_S \frac{R_p}{R_L} (x^2 - x^3) \]

if \( R_p/R_L \ll 1 \) (the usual situation). \( N(x) \) has a maximum value of \( \hat{N} = \frac{4}{27} V_S (R_p/R_L) \) when \( x = \frac{2}{3} \), corresponding to \( dN/dx = 0 \) and negative \( d^2N/dx^2 \). Expressing \( \hat{N} \) as a percentage of full-scale deflection or span \( V_S \) volts gives:

\[ \hat{N} = \frac{400}{27} \frac{R_p}{R_L} \% = 15 \frac{R_p}{R_L} \% \quad [5.12] \]

Non-linearity, sensitivity and maximum power requirements are used to specify the values of \( R_p \) and \( V_S \) for a given application. Suppose that a 10 cm range potentiometer is to be connected to a 10 kΩ recorder. If the maximum non-linearity must not exceed 2%, then we require \( 15R_p/R_L \leq 2 \), i.e. \( R_p \leq \frac{20}{15} \times 10^3 \) Ω; thus a 1 kΩ potentiometer would be suitable.

Since sensitivity \( dV_p/dx = V_S \), the greater \( V_S \) the higher the sensitivity, but we must satisfy the requirement that power dissipation \( V_S^2/R_p \) should not exceed maximum value \( \hat{W} \) watts. If \( \hat{W} = 0.1 \) W we require \( V_S \leq \sqrt{0.1 \times 10^3} \), i.e. \( V_S \leq 10 \) V; if \( V_S = 10 \) V, then corresponding sensitivity = 1.0 V cm⁻¹.

### 5.1.3 Norton equivalent circuit

Norton’s theorem states that any network consisting of linear impedances and voltage sources can be replaced by an equivalent circuit consisting of a current source \( i_N \) in parallel with an impedance \( Z_N \) (Figure 5.9). \( Z_N \) is the impedance looking back into the output terminals with all voltage sources reduced to zero and replaced by their internal impedances, and \( i_N \) is the current which flows when the terminals are short circuited. Connecting a load \( Z_L \) across the output terminals of the network is equivalent to connecting \( Z_L \) across the Norton circuit. The voltage \( V_L \) across the load is given by \( V_L = i_N Z \), where \( 1/Z = 1/Z_N + 1/Z_L \), giving:
5.1 ELECTRICAL LOADING

From eqn [5.13] we note that if $Z_L \ll Z_N$, then $V_L \rightarrow i_N Z_L$; i.e. in order to develop the maximum current through the load, the load impedance should be far smaller than the Norton impedance for the network.

A common example of a current source is an electronic differential pressure transmitter giving an output current signal, range 4 to 20 mA, proportional to an input differential pressure, typical range 0 to $2 \times 10^4$ Pa (Section 9.4). Figure 5.10 shows a typical equivalent circuit for the transmitter connected to a recorder via a cable. Using eqn [5.13], the voltage across the total load $R_C + R_R$ of recorder and cable is

$$V_L = i_N \frac{Z_N \cdot Z_L}{Z_N + Z_L} \quad [5.13]$$

and the ratio $V_R/V_L = R_R/(R_C + R_R)$, giving the recorder voltage:

$$V_R = i_N R_R \frac{R_N}{R_C + R_R + R_R} \quad [5.15]$$

Using the data given, we have $V_R = 0.9995i_N R_R$, so that the recorded voltage deviates from the desired range of 1 to 5 V by only 0.05%.

A second example of a current generator is provided by a piezoelectric crystal acting as a force sensor. If a force $F$ is applied to any crystal, then the atoms of the crystal undergo a small displacement $x$ proportional to $F$. For a piezoelectric material the crystal acquires a charge $q$ proportional to $x$, i.e. $q = Kx$. The crystal can therefore be regarded as a Norton current source of magnitude $i_N = dq/dt = K(dx/dt)$,
where \( \frac{dx}{dt} \) is the velocity of the atomic deformations. This effect is discussed more fully in Section 8.7, where we see that the crystal acts as a capacitance \( C_N \) in parallel with the current source \( i_N \). Figure 5.11 shows the equivalent circuit and typical component values for a crystal connected via a capacitive cable \( C_C \) to a recorder acting as a resistive load \( R_L \). The voltage \( V_L \) across the load is given by \( i_N Z \), where \( Z \) is the impedance of \( C_N \), \( C_C \) and \( R_L \) in parallel. Since

\[
\frac{1}{Z} = C_N s + C_C s + \frac{1}{R_L}
\]

\[
Z = \frac{R_L}{1 + R_L(C_N + C_C)s}
\]

where \( s \) denotes the Laplace operator. The transfer function relating dynamic changes in source current and recorder voltage is thus:

\[
\frac{\Delta V_L(s)}{\Delta i_N(s)} = \frac{R_L}{1 + R_L(C_N + C_C)s}
\]

[5.16]

Thus the effect of electrical loading in this example is to introduce a first-order transfer function into the force measurement system; this will affect dynamic accuracy.

5.2 Two-port networks

5.2.1 Generalised effort and flow variables

We have seen in the previous section how electrical loading effects can be described using a pair of variables, voltage and current. Voltage is an example of an across or effort variable \( y \); current is an example of a through or flow variable \( x \). An effort variable drives a flow variable through an impedance. Other examples of effort/flow pairs are force/velocity, torque/angular velocity, pressure difference/volume flow rate and temperature difference/heat flow rate.\(^2\) Each \( y-x \) pair has the following properties:

(a) The product \( yx \) represents power in watts.
(b) The ratio \( y/x \) represents impedance.

The only exception is the thermal variables where the product has the dimensions of watts and temperature. Table 5.1 lists the effort/flow pairs for different forms of energy and for each pair defines the related quantities of impedance, stiffness, compliance.
Table 5.1 Flow/effort variables and related quantities.

<table>
<thead>
<tr>
<th>Variables</th>
<th>Flow</th>
<th>Effort</th>
<th>Impedance</th>
<th>Stiffness</th>
<th>Compliance</th>
<th>Inertance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{d}{dt} \dot{x}$</td>
<td>$\int \dot{x} , dt$</td>
<td>$\dot{x}$</td>
<td>$y$</td>
<td>$\int \dot{x} , dt$</td>
<td>$\frac{\dot{x}}{d\dot{x}/dt}$</td>
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<td>acceleration</td>
<td>displacement</td>
<td>velocity</td>
<td>force</td>
<td>force</td>
<td>displacement</td>
</tr>
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<td>displacement</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>damping constant</td>
<td>stiffness</td>
</tr>
<tr>
<td>Mechanical-rotation</td>
<td>angular acceleration</td>
<td>angular displacement</td>
<td>angular velocity</td>
<td>torque</td>
<td>torque</td>
<td>angular disp.</td>
</tr>
<tr>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>ang. velocity</td>
<td>angular disp.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>damping constant</td>
<td>mechanical stiffness</td>
</tr>
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<td>Electrical</td>
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<td>charge</td>
<td>current</td>
<td>voltage</td>
<td>voltage</td>
<td>charge</td>
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<td>current</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>resistance</td>
<td></td>
</tr>
<tr>
<td>Fluidic</td>
<td>volume</td>
<td>volume flow rate</td>
<td>pressure</td>
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<td></td>
<td></td>
<td>vol. flow rate</td>
<td>fluidic capacitance</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>fluidic resistance</td>
<td></td>
</tr>
<tr>
<td>Thermal</td>
<td>heat</td>
<td>heat flow rate</td>
<td>temperature</td>
<td>heat</td>
<td>temperature</td>
<td>current</td>
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<td></td>
<td></td>
<td></td>
<td></td>
<td>thermal resistance</td>
<td></td>
</tr>
</tbody>
</table>

Source: adapted from Finkelstein and Watts, 1971[21].
and inerance. Thus we see that the concept of impedance is applicable to mechanical, fluidic and thermal systems as well as electrical. For a mechanical system, mass is analogous to electrical inductance, damping constant is analogous to electrical resistance, and 1/stiffness is analogous to electrical capacitance. For a thermal system, thermal resistance is analogous to electrical resistance, and thermal capacitance is analogous to electrical capacitance. This means we can generalise the electrical equivalent circuits of Thévenin and Norton to non-electrical systems.

Figure 5.12(a) shows a parallel mechanical system consisting of a mass \( m \), spring stiffness \( k \) and damper constant \( \lambda \). Figure 5.12(b) shows a series electrical circuit consisting of an inductance \( L \), a capacitance \( C \) and a resistance \( R \). Since mechanical impedance is the ratio of force/velocity, the mechanical impedance transfer function is

\[
Z_M(s) = \frac{\Delta P}{\Delta \dot{x}} = ms + \lambda + \frac{k}{s} \tag{5.17}
\]

The impedance transfer function for the electrical circuit is

\[
Z_E(s) = \frac{\Delta V}{\Delta i} = Ls + R + \frac{1}{Cs} \tag{5.18}
\]

We see that these have a similar form, with \( m \) corresponding to \( L \), \( \lambda \) corresponding to \( R \), and \( k \) corresponding to \( 1/C \). Thus the parallel mechanical system can be represented by an equivalent circuit consisting of an inductive element \( m \), a resistive element \( \lambda \) and a capacitive element \( 1/k \) in series (Figure 5.12(c)).

Figure 5.13(a) shows a thermal system consisting of a body at temperature \( T \) immersed in a fluid at temperature \( T_f \). The body has a mass \( M \), specific heat \( C_H \) and surface area \( A \); \( U \) is the heat transfer coefficient between the body and the fluid. Figure 5.13(b) shows a series electrical circuit with resistance \( R \), capacitance \( C \), input voltage \( V_{IN} \) and output voltage \( V_{OUT} \). The differential equation for the thermal system is:
5.2 TWO-PORT NETWORKS

5.2.2 Two-port networks

We saw in Section 5.1 that the electrical output of a sensing element such as a thermocouple or piezoelectric crystal can be represented by a Thévenin or Norton equivalent circuit. The sensor has therefore two output terminals which allow both voltage and current flow to be specified; this is referred to as an electrical output port. The sensing element will have a mechanical, thermal or fluidic input; we saw in the previous section that mechanical and thermal systems can be represented by equivalent circuits which show the relation between the corresponding effort and flow variables. Thus the input to a mechanical or thermal sensor can be represented by two input terminals which allow both the effort and flow variables to be specified; this is either a mechanical or a thermal input port. Thus a sensing element can be represented by a two-port or four-terminal network. Figure 5.14(a) shows a two-port representation of a mechanical sensor with input mechanical port and output electrical port; Figure 5.15(a) shows the two-port representation of a thermal sensor. Figures 5.14(b) and (c) show the detailed two-port networks for a range of mechanical sensing elements. Figure 5.14(b) shows the equivalent circuit for sensing elements with a Thévenin equivalent circuit at the electrical output port. $Z_M$ is the input mechanical impedance; $E_{Th}$ and $Z_{Th}$ are the Thévenin voltage and impedance. For a displacement sensor $E_{Th}$ is proportional to displacement $x$, i.e.
LOADING EFFECTS AND TWO-PORT NETWORKS

Figure 5.14 Mechanical sensing elements as two-port networks:
(a) Overall two-port representation
(b) Equivalent circuit with Thévenin output
(c) Equivalent circuit with Norton output.

\[ E_{Th} = K_x \dot{x} \]  
where \( K_x \) is the sensitivity and \( x = \int \dot{x} \, dt \).

For a potentiometer displacement sensor (Section 5.1.2) \( K_x = V_S \) (supply voltage), and for a linear variable differential transformer (LVDT, Section 8.3.2) \( K_x \) is the slope of the linear portion of the a.c. voltage versus displacement characteristics. For a velocity sensor, \( E_{Th} \) is proportional to velocity \( \dot{x} \), i.e.

Figure 5.15 Thermal sensing elements as two-port networks:
(a) Overall two-port representation
(b) Equivalent circuit with Thévenin output
(c) Equivalent circuit with Norton output.

\[ E_{Th} = K_T \theta \]  
where \( K_T \) is the sensitivity and \( \theta = \frac{d}{dt} \) (pyroelectric), \( \dot{i} = i_N = \frac{V}{Z_N} \).
5.2 TWO-PORT NETWORKS

\[ E_{th} = K_V \dot{x} \] \[5.22\]

For an electromagnetic linear velocity sensor (Section 12.5.1) \( K_V = B l \) where \( B \) is the applied magnetic field and \( l \) the length of the conductor. In the case of an electromagnetic angular velocity sensor or tachogenerator (Section 8.4) we have \( E_{th} = K_\phi \omega \), where \( \omega \) is the angular velocity and \( K_\phi = dN/d\theta \) the rate of change of flux \( N \) with angle \( \theta \). Figure 5.14(c) shows the equivalent circuit for sensing elements with a Norton equivalent circuit \( i_N, Z_N \) at the electrical output port. For a piezoelectric sensor \( i_N \) is proportional to velocity \( \dot{x} \) (Section 8.7), i.e.

\[ i_N = K \dot{x} \] \[5.23\]

where \( K = d k \), \( d \) is the charge sensitivity to force and \( k \) is the stiffness of the crystal.

Figures 5.15(b) and (c) show the detailed two-port networks for two examples of thermal sensing elements. Figure 5.15(b) shows the equivalent circuit for a sensing element with a Thévenin equivalent circuit \( E_{th}, Z_{th} \) at the electrical output port; \( Z_H \) is the thermal input impedance. For a thermocouple temperature sensor (Section 8.5) with the reference junction at 0 °C we have \( E_{th} = E_{T,0} \) where \( E_{T,0} \) is the e.m.f. at the measured junction at \( T \) °C and is given by the power series

\[ E_{T,0} = a_1 T + a_2 T^2 + a_3 T^3 + \ldots \] \[5.24\]

Figure 5.15(c) shows the equivalent circuit for a sensing element with a Norton equivalent circuit \( i_N, Z_N \) at the electrical output port. For a pyroelectric detector (Section 15.5.1) \( i_N \) is proportional to rate of change of temperature \( dT/dt \), i.e.

\[ i_N = K \frac{dT}{dt} \] \[5.25\]

where \( K = A (dP/dT) \), \( A \) is the area of the electrodes and \( dP/dT \) is the slope of the polarisation temperature characteristics.

5.2.3 Process loading

Having introduced the concepts of equivalent circuits and two-port networks for mechanical and thermal systems, we can now use these concepts to study examples of how a primary sensing element can ‘load’ the process or element being measured.

Figure 5.16 shows a mechanical system or ‘process’ represented by a mass, spring and damper. The force \( F \) applied to the ‘process’ is being measured by a force sensor, consisting of an elastic element in conjunction with a potentiometric displacement sensor. The elastic force sensor can also be represented by a mass spring and damper (Section 4.1.2). Under steady-state conditions when both velocity \( \dot{x} = 0 \) and acceleration \( \ddot{x} = 0 \), we have the following force balance equations:

**process** \[ F = k_P x + F_S \] \[5.26\]

**sensor** \[ F_S = k_S x \]

showing that the relationship between the measured force \( F_S \) and the true force \( F \) is:

\[ F_S = \frac{k_S}{k_S + k_P} F = \frac{1}{1 + k_P/k_S} F \] \[5.27\]
We see that in order to minimise the loading error in the steady state the sensor stiffness $k_S$ should be very much greater than the process stiffness $k_P$.

Under unsteady conditions when $\dot{x}$ and $\ddot{x}$ are non-zero, Newton’s second law gives the following differential equations:

**process** \[ F - k_P x - \lambda_P \dot{x} - F_S = m_P \ddot{x} \] \[ F \] \[ \dot{x} \]

**sensor** \[ F_S - k_S x - \lambda_S \dot{x} = m_S \ddot{x} \]

i.e.

\[ \frac{d\dot{x}}{dt} = m_P \ddot{x} + \lambda_P \dot{x} + k_P \int \dot{x} \, dt = F - F_S \] \[ \frac{d\ddot{x}}{dt} = m_S \ddot{x} + \lambda_S \dot{x} + k_S \int \dot{x} \, dt = F_S \]

Using the analogues given earlier, the sensor can be represented by $F_S$ driving $\dot{x}$ through the mechanical $L$, $C$, $R$ circuit $m_S$, $1/k_S$, $\lambda_S$; and the process can be represented by $F - F_S$ driving $\dot{x}$ through the mechanical $L$, $C$, $R$ circuit $m_P$, $1/k_P$, $\lambda_P$. If $\Delta x$, $\Delta F$ and $\Delta F_S$ are deviations from initial steady conditions, then the Laplace transforms of eqns [5.29] are:

\[ \begin{bmatrix} m_P s + \lambda_P + \frac{k_P}{s} & \Delta \ddot{x} = \Delta F - \Delta F_S \\ \Delta \ddot{x} = \Delta F_S & m_S s + \lambda_S + \frac{k_S}{s} \end{bmatrix} \]

Using Table 5.1 we can define mechanical impedance transfer functions by $Z_{sp}(s) = \Delta \ddot{x}(s)/\Delta \ddot{x}(s)$, so that:

**process impedance** \[ Z_{sp}(s) = m_P s + \lambda_P + \frac{k_P}{s} \] \[ s \]

**sensor impedance** \[ Z_{sp}(s) = m_S s + \lambda_S + \frac{k_S}{s} \] \[ s \]
From [5.30] and [5.31] the relationship between measured and actual dynamic changes in force is:

\[ \Delta \tilde{F}_s(s) = \frac{Z_{MS}}{Z_{MS} + Z_{MP}} \Delta F(s) \]  

Thus in order to minimise dynamic loading effects, sensor impedance \( Z_{MS} \) should be very much greater than process impedance \( Z_{MP} \). Figure 5.17 shows the equivalent circuit for the system: process, force sensor and recorder.

Figure 5.18 shows a hot body, i.e. a thermal ‘process’ whose temperature \( T_p \) is being measured by a thermocouple sensor. Under unsteady conditions, heat flow rate considerations give the following differential equations (Section 4.1):

\[
\begin{align*}
\text{process} & \quad m_p C_p \frac{dT_p}{dt} = W_p - W_S, \quad W_p = U_p A_p (T_p - T_F) \\
\text{sensor} & \quad m_S C_S \frac{dT_S}{dt} = W_S, \quad W_S = U_S A_S (T_p - T_S)
\end{align*}
\]  

\[ [5.33] \]
Figure 5.19 Equivalent circuit for thermal system showing thermocouple as a two-port network.

\[ q = dF \]
i.e. this is a conversion of mechanical energy to electrical energy and the device acts as a receiver or sensor. In the inverse effect a voltage \( V \) applied to the crystal produces a mechanical deformation \( x \), proportional to \( V \), according to:

\[
    x = dV
\]

This is a conversion of electrical energy to mechanical energy and the device acts as a transmitter or sender. The detailed equivalent circuits for a piezoelectric transmitter and receiver are given in Section 16.2.1.

Another reversible physical effect is the electromagnetic effect. In the direct effect, a conductor of length \( l \) moving with velocity \( \dot{x} \) perpendicular to a magnetic field \( B \) has a voltage

\[
    E = Bl\dot{x}
\]

induced across the ends of the conductor. This is a conversion of mechanical to electrical energy and the device acts as a receiver or sensor. In the inverse effect a conductor of length \( l \) carrying a current \( i \) in a transverse magnetic field \( B \) experiences a force

\[
    F = Bli
\]

This is a conversion of electrical to mechanical energy and the device acts as a transmitter or sender. Figure 5.21 gives the detailed equivalent circuits for an electromagnetic transmitter and receiver. At the electrical ports the applied voltage drives a current through the electrical impedance \( L, R \); at the mechanical ports the applied force drives a velocity through the mechanical impedance \( m, \frac{1}{k}, \lambda \). The transmitter can be used as a voltage indicator (Section 11.2) and the receiver as a velocity sensor (Sections 5.1.1 and 8.4).
Conclusion

We have seen how the use of equivalent circuits and two-port networks has enabled both inter-element and process loading effects to be described.

References


Problems

5.1 A glass pH electrode with a sensitivity of 59 mV pH$^{-1}$ and a resistance of 10$^9$ Ω is used to measure pH in the range 0 to 15. The electrode is to be connected to a recorder of input range 0 to 100 mV and resistance 100 Ω using a buffer amplifier of unity gain and output resistance 100 Ω.

(a) Calculate the input impedance of the amplifier, and the sensitivity of the recorder scale necessary to obtain an accurate recording of pH.

(b) The resistance of the electrode increases to 2 × 10$^9$ Ω due to chemical action. Calculate the resulting measurement error in the above system, as a percentage of full scale, for a true pH of 7.

5.2 The motion of a hydraulic ram is to be recorded using a potentiometer displacement sensor connected to a recorder. The potentiometer is 25 cm long and has linear resistance displacement characteristics. A set of potentiometers with maximum power rating of 5 W and resistance values ranging from 250 to 2500 Ω in 250 Ω steps is available. The recorder has a resistance of 5000 Ω and the non-linear error of the system must not exceed 2% of full scale. Find:

(a) the maximum potentiometer sensitivity that can be obtained;

(b) the required potentiometer resistance and supply voltage in order to achieve maximum sensitivity.

5.3 An electronic differential transmitter gives a current output of 4 to 20 mA linearly related to a differential pressure input of 0 to 10$^4$ Pa. The Norton impedance of the transmitter is 10$^5$ Ω. The transmitter is connected to an indicator of impedance 250 Ω via a cable of total resistance 500 Ω. The indicator gives a reading between 0 and 10$^4$ Pa for an input voltage between 1 and 5 V. Calculate the system measurement error, due to loading, for an input pressure of 5 × 10$^3$ Pa.

5.4 A sensor with mass 0.1 kg, stiffness 10$^3$ N m$^{-1}$ and damping constant 10 N s m$^{-1}$ is used to measure the force on a mechanical structure of mass 5 kg, stiffness 10$^2$ N m$^{-1}$ and damping constant 20 N s m$^{-2}$. Find the transfer function relating measured and actual changes in force.
**Basic problems**

5.5 A linear thermocouple with a sensitivity of 0.04 mV/°C and resistance of 100 Ω is connected to a load with a resistance of 1 kΩ. Find the voltage across the load for a temperature of 250 °C.

5.6 A potentiometer displacement sensor has a supply voltage of 15 V and a resistance of 50 kΩ. The fractional displacement of the wiper is 0.3. Find the Thévenin voltage and resistance for the circuit.

5.7 A potentiometer has a supply voltage of 10 V, a resistance of 10 kΩ and a length of 10 cm. A recorder of resistance 10 kΩ is connected across the potentiometer. Calculate the Thévenin equivalent circuit for the sensor and the recorder voltage for each of the following displacements:
   (a) 2 cm
   (b) 5 cm
   (c) 8 cm.

5.8 A potentiometer has a total length of 10 cm and a resistance of 100 Ω.
   (a) Calculate the supply voltage so that power dissipation = 1 W.
   (b) Draw the Thévenin equivalent circuit for 7 cm displacement.
   (c) The potentiometer is connected to a recorder with a resistance $R_L$. Find $R_L$ such that the recorder voltage is 5% less than the open circuit voltage at 7 cm displacement.

5.9 A pressure transducer consists of a Bourdon tube elastic element connected to a potentiometer displacement sensor. The input range of the Bourdon tube is 0 to $10^4$ Pa and the output range is 0 to 1 cm. The potentiometer has a length of 1 cm, a resistance of 10 kΩ and a supply voltage of 10 V. If the input pressure is $5 \times 10^3$ Pa, calculate:
   (a) the displacement of the potentiometer wiper (assume a linear Bourdon tube)
   (b) the open circuit transducer output voltage
   (c) the voltage indicated by a voltmeter of resistance 10 kΩ connected across the potentiometer.
6 Signals and Noise in Measurement Systems

6.1 Introduction

In Chapter 4 we studied the dynamic response of measurement systems to step, sine wave and square wave input signals. These signals are examples of deterministic signals: a deterministic signal is one whose value at any future time can be exactly predicted. Thus if we record these signals for an observation period $T_O$ (Figure 6.1), the future behaviour of the signal, once the observation period is over, is known exactly. The future behaviour of real processes, such as chemical reactors, blast furnaces and aircraft, will depend on unknown factors such as the type of feedstock, reliability of equipment, changes in throughput and atmospheric conditions, and cannot be known in advance. This means that the future value of measured variables, such as reactor temperature, flow in a pipe and aircraft speed, cannot be exactly predicted. Thus in real measurement applications the input signal to the measurement system is not deterministic but random. If a random signal is recorded for an observation period $T_O$ (Figure 6.1) the behaviour of the signal, once the observation period is over, is not known exactly. However, five statistical quantities – mean, standard deviation, probability density function, power spectral density and autocorrelation function – are used to estimate the behaviour of random signals. These are defined and explained in Section 6.2.

Random variations in the measured input variable produce corresponding random variations in the electrical output signals of elements in the system. More precisely the Thévenin voltage and Norton current signals defined in Chapter 5 vary randomly with time and will be referred to as the measurement signal. Examples are random fluctuations in the millivolt output of a thermocouple, random fluctuations in the current output of a differential pressure transmitter, and random fluctuations in the amplitude and frequency of the a.c. output voltage of a variable reluctance tacho-generator. However, unwanted electrical signals may also be present in the measurement circuit. These may be due to sources inside the measurement circuit or caused by coupling to sources outside the circuit. The magnitude of the unwanted signal may be comparable to or larger than that of the measurement signal itself, resulting in a measurement error for the overall system. This is an example of the interfering inputs discussed in Chapter 2. The unwanted signal then may be either random, e.g. signals caused by the random motion of electrons, or deterministic, e.g. sinusoidal signals at 50 Hz caused by power cables. Unwanted random signals are usually referred to as noise signals and unwanted deterministic signals as interference signals.
Sections 6.3 and 6.4 discuss the sources of noise and interference signals and how they affect the measurement circuit. Section 6.5 examines ways of reducing the effects of noise and interference.

### 6.2 Statistical representation of random signals

Figure 6.2 shows a recording of a section of a random signal obtained during an observation period $T_O$. Since the signal is random we cannot write down a continuous algebraic equation $y(t)$ for the signal voltage $y$ at time $t$. We can, however, write down the values $y_1$ to $y_N$ of $N$ samples taken at equal intervals $\Delta T$ during $T_O$. The first sample $y_1$ is taken at $t = \Delta T$, the second $y_2$ is taken at $t = 2\Delta T$, and the $i$th $y_i$ is taken at $t = i\Delta T$, where $i = 1, \ldots, N$. The sampling interval $\Delta T = T_O/N$ must satisfy the Nyquist sampling theorem, which is explained in Section 10.1. We can now use these samples to calculate statistical quantities for the observed section of the signal. These observed statistical quantities will provide a good estimate of the future behaviour of the signal, once the observation period is over, provided:

![Figure 6.2 Sampling of a random signal.](image-url)
6.2 STATISTICAL REPRESENTATION OF RANDOM SIGNALS

(a) \( T_o \) is sufficiently long, i.e. \( N \) is sufficiently large;
(b) the signal is stationary, i.e. long-term statistical quantities do not change with time.

6.2.1 Mean \( \bar{y} \)

For a signal defined in terms of a continuous function \( y(t) \) in the interval 0 to \( T_o \), the mean is given by:

Mean for continuous signal

\[
\bar{y} = \frac{1}{T_o} \int_0^{T_o} y(t) \, dt 
\]  

[6.1]

If, however, the signal is represented by the set of sampled values \( y_i \) we have

Mean for sampled signal

\[
\bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i 
\]  

[6.2]

6.2.2 Standard deviation \( \sigma \)

This is a measure of the average spread or deviation of the signal from the mean value \( \bar{y} \). In the continuous case:

Standard deviation for continuous signal

\[
\sigma^2 = \frac{1}{T_o} \int_0^{T_o} [y(t) - \bar{y}]^2 \, dt 
\]  

[6.3]

and in the sampled case:

Standard deviation for sampled signal

\[
\sigma^2 = \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})^2 
\]  

[6.4]

In the special case that \( \bar{y} = 0 \), the standard deviation \( \sigma \) is equal to the root mean square (r.m.s.) value \( y_{rms} \), where:

\[
y_{rms} = \sqrt{\frac{1}{T_o} \int_0^{T_o} y^2 \, dt} 
\]  

[6.5]

or

\[
y_{rms} = \sqrt{\frac{1}{N} \sum_{i=1}^{N} y_i^2} 
\]  

[6.6]
6.2.3 Probability density function \( p(y) \)

This is a function of signal value \( y \) and is a measure of the probability that the signal will have a certain range of values. Figure 6.3 shows the set of sample values \( y_i \) and the \( y \) axis divided into \( m \) sections each of width \( \Delta y \). We can then count the number of samples occurring within each section, i.e. \( n_1 \) in section 1, \( n_2 \) in section 2, \( n_j \) in section \( j \), etc., where \( j = 1, \ldots, m \). The probability \( P_j \) of the signal occurring in the \( j \)th section is thus:

\[
P_j = \frac{\text{number of times sample occurs in the } j\text{th section}}{\text{total number of samples}} \quad [6.7]
\]

\[
P_j = \frac{n_j}{N}, \quad j = 1, \ldots, m
\]

The cumulative probability \( C_j \) is the total probability that the signal will occur in the first \( j \) sections and is given by:
Cumulative probability

\[ C_j = P_1 + P_2 + \ldots + P_j \]
\[ = \frac{1}{N} (n_1 + n_2 + \ldots + n_j) \] [6.8]

Figure 6.3 shows the corresponding forms of \( P_j \) and \( C_j \). The final value of \( C_j \) when \( j = m \) is:

\[ C_m = \frac{1}{N} (n_1 + n_2 + \ldots + n_n) \]
\[ = \frac{1}{N} \times N = 1 \quad (\text{since } n_1 + n_2 + \ldots + n_n = N) \] [6.9]

i.e. the total probability of finding the signal in all \( m \) sections is unity. In the limit as the interval \( \Delta y \) tends to zero the discrete cumulative probability \( C_j \) tends to a continuous function (Figure 6.3). This is the cumulative probability distribution function (c.d.f.) \( P(y) \), which is defined by:

Cumulative probability distribution function

\[ P(y) = \lim_{\Delta y \to 0} C_j \] [6.10]

The probability density function (p.d.f.) \( p(y) \) is more commonly used and is the derivative of \( P(y) \), i.e.

Probability density function

\[ p(y) = \frac{dP}{dy} \] [6.11]

Thus the probability \( P_{y_1, y_2} \) that the signal will lie between \( y \) and \( y + \Delta y \) is given by:

\[ P_{y, y+\Delta y} = \Delta P = p(y) \Delta y \] [6.12]

i.e. by the area of a strip of height \( p(y) \) and width \( \Delta y \). Similarly the probability \( P_{y_1, y_2} \) that the signal will lie between \( y_1 \) and \( y_2 \) is given by:

\[ P_{y_1, y_2} = \int_{y_1}^{y_2} p(y) \, dy \] [6.13]

i.e. the area under the probability density curve between \( y_1 \) and \( y_2 \). The total area under the probability density curve is equal to unity, corresponding to the total probability of the signal having any value of \( y \). The normal probability density function (Section 2.3), i.e.

\[ p(y) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left[ -\frac{(y - \bar{y})^2}{2\sigma^2} \right] \] [6.14]

usually provides an adequate description of the amplitude distribution of random noise signals.
6.2.4 Power spectral density $\phi(\omega)$

If we record a random signal for several observation periods, each of length $T_o$ (Figure 6.4), the waveform will be different for each period. However, the average signal power will be approximately the same for each observation period. This means that signal power is a stationary quantity which can be used to quantify random signals. In Section 4.3 we saw that a periodic signal can be expressed as a Fourier series, i.e. a sum of sine and cosine waves with frequencies which are harmonics of the fundamental frequency. The power in a periodic signal is therefore distributed amongst these harmonic frequencies. A random signal is not periodic and cannot be represented by Fourier series but does contain a large number of closely spaced frequencies. Power spectral density is a stationary quantity which is a measure of how the power in a random signal is distributed amongst these different frequencies.

In order to explain the meaning of $\phi(\omega)$ we approximate the random signal by a periodic signal (Figure 6.4) in which the waveform recorded during the first observation period $T_o$ is exactly repeated during each subsequent observation period. This periodic approximation is valid:

(a) in the limit that the observation period, i.e. the signal period $T_o$, becomes large,
(b) provided we use it to calculate the power distribution in the signal.

![Figure 6.4 Power spectrum and power spectral density.](image-url)
The Fourier series for a voltage signal with period $T_o$ is

$$y(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_1 t + \sum_{n=1}^{\infty} b_n \sin n\omega_1 t$$  \[4.51a\]

where the fundamental frequency $\omega_1 = \frac{2\pi}{T_o}$, $a_0$ = mean $\bar{y}$, and

**Fourier series coefficients for sampled signal**

$$a_n = \frac{2}{N} \sum_{i=1}^{N} y_i \cos(n\omega_1 \Delta T) = \frac{2}{N} \sum_{i=1}^{N} y_i \cos \left( \frac{2\pi n i}{N} \right)$$ \[6.15\]

$$b_n = \frac{2}{N} \sum_{i=1}^{N} y_i \sin(n\omega_1 \Delta T) = \frac{2}{N} \sum_{i=1}^{N} y_i \sin \left( \frac{2\pi n i}{N} \right)$$

Equations [6.15] use the set of sample values $y_i$ and are equivalent to eqns [4.52] for the Fourier coefficients of a continuous signal. If the $n$th harmonic $a_n \cos n\omega_1 t$ is applied across a $1 \, \Omega$ resistor, the instantaneous power in the resistor at time $t$ is $a_n^2 \cos^2 n\omega_1 t$ watts, and the average power over period $T_o$ is:

$$a_n^2 \frac{1}{T_o} \int_0^{T_o} \cos^2 n\omega_1 t \, dt = \frac{a_n^2}{2}$$ \[6.16\]

Similarly the average power in a $1 \, \Omega$ resistor due to $b_n \sin n\omega_1 t$ is $b_n^2/2$, so that the power due to the $n$th harmonic at frequency $n\omega_1$ is:

$$w_n = \frac{1}{2}(a_n^2 + b_n^2)$$ \[6.17\]

Figure 6.4 shows that the relationship between $w_n$ and $\omega$ is a series of lines at the harmonic frequencies $n\omega_1$. This is referred to as the power spectrum of the signal and is terminated at $\omega_{\text{MAX}}$, the harmonic frequency beyond which $w_n$ becomes negligible. The **cumulative power** $W_n$ is the total power in a $1 \, \Omega$ resistor due to the first $n$ harmonics and the d.c. component $a_0$, i.e.

$$W_n = w_0 + w_1 + w_2 + \ldots + w_n$$ \[6.18\]

The diagram shows the relation between $W_n$ and $\omega$; it is in the form of a staircase, each step having width $\omega_n$. At $\omega = \omega_{\text{MAX}}$, $W_n = W_{\text{TOT}}$, the total power in the signal. However, in the limit that signal period $T_o \to \infty$, $\omega_1 \to 0$ and $W_n$ becomes a continuous function of $\omega$. This is the **cumulative power function** (c.p.f.) $W(\omega)$, which is defined by:

$$W(\omega) = \lim_{\omega_n \to \omega} W_n$$ \[6.19\]

The power spectral density (PSD) $\phi(\omega)$ is more commonly used and is the derivative of $W(\omega)$ (Figure 6.4), i.e.
Power spectral density

\[
\phi(\omega) = \frac{dW}{d\omega} \tag{6.20}
\]

The PSD \( \phi \) has units \( \text{watt s rad}^{-1} \) or \( \text{W/Hz} \). Thus the power produced in a 1 \( \Omega \) resistor due to frequencies between \( \omega \) and \( \omega + \Delta \omega \) is:

\[
W_{\omega,\omega+\Delta\omega} = \Delta W = \phi(\omega)\Delta\omega \text{ watts} \tag{6.21}
\]

i.e. the area of a strip of height \( \phi(\omega) \) and width \( \Delta \omega \). Similarly the power due to frequencies between \( \omega_1 \) and \( \omega_2 \) is:

\[
W_{\omega_1,\omega_2} = \int_{\omega_1}^{\omega_2} \phi(\omega) \, d\omega \text{ watts} \tag{6.22}
\]

i.e. the area under the power spectral density curve between \( \omega_1 \) and \( \omega_2 \). The total area under the PSD curve is the total power in the signal, i.e.

\[
W_{\text{TOT}} = \int_{0}^{\omega_{\text{MAX}}} \phi(\omega) \, d\omega \text{ watts} \tag{6.23}
\]

Internal noise sources in electrical circuits can often be regarded as white noise, which has a uniform PSD over an infinite range of frequencies, i.e.

\[
\phi(\omega) = A, \quad 0 \leq \omega \leq \infty \tag{6.24}
\]

Another useful representation for both noise and measurement signals is a power spectral density which is constant up to a cut-off frequency \( \omega_c \) and zero for higher frequencies (band limited white noise):

\[
\phi(\omega) = \begin{cases} 
A, & 0 \leq \omega \leq \omega_c \\
0, & \omega > \omega_c 
\end{cases} \tag{6.25}
\]

i.e. \( W_{\text{TOT}} = A\omega_c \).

The maximum frequency \( \omega_{\text{MAX}} \) for a measurement signal depends on the nature of the measured variable. Thus the vibration displacement of part of a machine may contain frequencies up to many kHz, whereas the temperature variations in a chemical reactor may only contain frequencies up to 0.01 Hz. In order to accurately measure random fluctuations in a measured variable, the transfer function \( G(s) \) for the measurement system must satisfy the conditions of Section 4.4, i.e. \( |G(j\omega)| = 1 \) and \( \arg G(j\omega) = 0 \) for all \( \omega \) up to \( \omega_{\text{MAX}} \).

### 6.2.5 Autocorrelation function

Figure 6.5 shows a block diagram of a simple correlator. The input signal \( y(t) \) is passed through a variable time delay unit to give a delayed signal \( y(t-\beta) \). The signals \( y(t) \) and \( y(t-\beta) \) are multiplied together to give the product waveform \( y(t)y(t-\beta) \). This product waveform is passed through an averager and the average value \( \overline{y(t)y(t-\beta)} \) is displayed on a meter: this is the autocorrelation coefficient \( R_{yy} \). If the time delay \( \beta \) is altered, the shape of the product waveform changes, causing the meter reading \( R_{yy} \) to change. The relationship between \( R_{yy} \) and time delay \( \beta \) is the autocorrelation function \( R_{yy}(\beta) \) of the signal. We note that \( R_{yy}(\beta) \) has a maximum value
$R_y(0)$ when $\beta = 0$; this is because the corresponding product waveform is $y^2(t)$, which is always positive and has a maximum average value.

If the signal is defined by a continuous function $y(t)$ in the interval $0$ to $T_0$, then $R_y(\beta)$ can be evaluated using:

$$R_y(\beta) = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} y(t)y(t-\beta)\,dt \quad [6.26]$$

Thus, if $y(t) = b \sin(\omega_1 t + \phi)$,

$$R_y(\beta) = b^2 \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} \sin(\omega_1 t + \phi)\sin[\omega_1(t-\beta) + \phi] \, dt$$

$$= b^2 \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} \frac{1}{2} \{ \cos \omega_1 \beta - \cos[\omega_1(2t-\beta) + 2\phi] \} \, dt$$

$$= b^2 \frac{\cos \omega_1 \beta}{2} \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} dt - b^2 \frac{\cos \omega_1(2T_0 - \beta) + 2\phi}{2} \lim_{T_0 \to \infty} \frac{1}{T_0} \int_0^{T_0} \cos[\omega_1(2t-\beta) + 2\phi] \, dt$$

$$= b^2 \frac{\cos \omega_1 \beta}{2} \quad [6.27]$$

since the average value of $\cos[\omega_1(2t-\beta) + 2\phi]$ is zero. Thus the autocorrelation function of a sinusoidal signal is a cosine function of the same frequency, but the phase information $\phi$ in the sine wave is lost. The autocorrelation function of any periodic signal has the same period as the signal itself (see Problem 6.2).
A random signal is usually characterised by a set of \( N \) sample values \( y_i \). Since information is only available at discrete time intervals, the time delay \( \beta \) is normally an integer multiple of the sampling interval \( \Delta T \), i.e.

\[
\beta = m \Delta T, \quad m = 0, 1, 2 \ldots
\]  

[6.28]

In this case the autocorrelation function of the signal is found by evaluating the set of autocorrelation coefficients \( R_{yy}(m\Delta T) \), which are given by:

\[
R_{yy}(m\Delta T) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{i=N} y_i y_{i-m}
\]  

[6.29]

where \( y_i \) is the sample value at time \( i\Delta T \) and \( y_{i-m} \) the value at time \( (i-m)\Delta T \) \((m \) sampling intervals earlier). Figure 6.5 shows the evaluation of \( R_{yy}(3\Delta T) \) for a sampled waveform; this involves calculating the products \( y_1y_{-2}, y_2y_{-1}, y_3y_0 \ldots y_Ny_{N-3} \), summing, and dividing by \( N \).

The autocorrelation function of a random signal can also be found from the power spectral density \( \phi(\omega) \). To illustrate this we first consider a periodic signal which is the sum of three harmonics:

\[
y(t) = b_1 \sin \omega_1 t + b_2 \sin 2\omega_1 t + b_3 \sin 3\omega_1 t
\]

Using eqn [6.27] the autocorrelation function is:

\[
R_{yy}(\beta) = \frac{b_1^2}{2} \cos \omega_1 \beta + \frac{b_2^2}{2} \cos 2\omega_1 \beta + \frac{b_3^2}{2} \cos 3\omega_1 \beta
\]

which has period \( 2\pi/\omega_1 \) (Figure 6.6). Using eqn [6.17], the power spectrum of the signal consists of three lines at frequencies \( \omega_1, 2\omega_1 \) and \( 3\omega_1 \) with heights \( b_1^2/2, b_2^2/2 \) and \( b_3^2/2 \) respectively. Thus the power spectrum can be obtained from the autocorrelation function by Fourier analysis. Similarly the autocorrelation function can be obtained from the power spectrum by adding the harmonics together as in Fourier synthesis. For random signals, \( R_{yy}(\beta) \) and \( \phi(\omega) \) are related by the Fourier transform or Wiener–Khinchin relations:

![Figure 6.6 Relationships between power spectrum and autocorrelation function for periodic and random signals.](image)
6.3 Effects of Noise and Interference on Measurement Circuits

\[ R_{yy}(\beta) = \int_{0}^{\infty} \phi(\omega) \cos \omega \beta \, d\omega \]  
\[ \phi(\omega) = \frac{2}{\pi} \int_{0}^{\infty} R_{yy}(\beta) \cos \omega \beta \, d\beta \]  

[6.30]

Thus for a signal with \( \phi(\omega) \) constant up to \( \omega_C \) and zero for higher frequencies we have:

\[ R_{yy}(\beta) = \int_{0}^{\omega_C} A \cos \omega \beta \, d\omega = A \left[ \frac{\sin \omega_C \beta}{\beta} \right]_{0}^{\omega_C} = A \frac{\sin \omega_C \beta}{\beta} \]  

[6.31]

The form of both functions is shown in Figure 6.6; we see that \( R_{yy}(\beta) \) has its first zero crossings at \( \beta = \pm \pi/\omega_C \), i.e. the width of the central ‘spike’ is \( 2\pi/\omega_C \). Thus a rapidly varying random signal has a high value of \( \omega_C \), i.e. a broad power spectrum but a narrow autocorrelation function, that falls off sharply as \( \beta \) is increased. A slowly varying random signal, however, has a low value of \( \omega_C \), i.e. a narrow power spectrum but a broad autocorrelation function that falls slowly as \( \beta \) is increased.

6.2.6 Summary

In order to specify a random signal we need to know:

\[ \begin{align*} 
\text{either} & \quad \text{probability density function} \quad \text{to specify amplitude} \\
\text{or} & \quad \text{mean and standard deviation} \quad \text{behaviour} \\
\end{align*} \]

and

\[ \begin{align*} 
\text{either} & \quad \text{power spectral density} \quad \text{to specify frequency/time} \\
\text{or} & \quad \text{autocorrelation function} \quad \text{behaviour} \\
\end{align*} \]

Important relations between these different quantities can be derived by considering \( R_{yy}(0) \), the autocorrelation coefficient at zero time delay. From eqns [6.26] and [6.30] we have

\[ R_{yy}(0) = \lim_{T_o \to \infty} \frac{1}{T_o} \int_{0}^{T_o} y^2 \, dt = \int_{0}^{\infty} \phi(\omega) \, d\omega \]  

[6.32]

From eqn [6.5], the first expression is equal to \( y_{\text{RMS}}^2 \) in the limit of infinitely long observation time \( T_o \). From equation [6.23] the second expression is equal to \( W_{\text{TOT}} \), the total power produced by the signal in a 1 \( \Omega \) resistor. Thus:

\[ R_{yy}(0) = y_{\text{RMS}}^2 = W_{\text{TOT}} \]  

[6.33]

6.3 Effects of noise and interference on measurement circuits

In Section 5.1 we saw that the interconnection of two measurement system elements, e.g. a thermocouple and an amplifier, or a differential pressure transmitter and a recorder, could be represented by an equivalent circuit in which either a Thévenin voltage
source or a Norton current source is connected to a load. In industrial installations, source and load may be typically 100 metres apart and noise and/or interference voltages may also be present.

Figure 6.7(a) shows a voltage transmission system subject to series mode interference; here a noise or interference voltage $V_{SM}$ is in series with the measurement signal voltage $E_{Th}$. The current $i$ through the load is:

$$i = \frac{E_{Th} + V_{SM}}{Z_{Th} + R_C + Z_L}$$

and the corresponding voltage across the load is:

$$V_L = \frac{Z_L}{Z_{Th} + R_C + Z_L} (E_{Th} + V_{SM})$$ [6.34]

Normally we make $Z_L \gg R_C + Z_{Th}$ to obtain maximum voltage transfer to the load (Section 5.1.1); under these conditions eqn [6.34] becomes:

$$V_L = E_{Th} + V_{SM}$$ [6.35]

This means that with a voltage transmission system all of $V_{SM}$ is across the load; this affects the next element in the system and possibly results in a system measurement error. We define signal-to-noise or signal to interference ratio $S/N$ in decibels by:
**Signal-to-noise ratio**

\[
\frac{S}{N} = 20 \log_{10} \left( \frac{E_{Th}}{V_{SM}} \right) = 10 \log_{10} \left( \frac{W_S}{W_N} \right) \text{ dB} \quad [6.36]
\]

where \(E_{Th}\) and \(V_{SM}\) are the r.m.s. values of the voltages, and \(W_S\) and \(W_N\) are the corresponding total signal and noise powers. Thus if \(E_{Th} = 1\) V and \(V_{SM} = 0.1\) V, \(S/N = +20\) dB.

Figure 6.7(b) shows a **current transmission system** subject to the same series mode interference voltage \(V_{SM}\). The Norton source current \(i_n\) divides into two parts, one part through the source impedance \(Z_N\), the other part through \(Z_L\). Using the current divider rule, the current through the load due to the source is:

\[
i = i_n \frac{Z_N}{Z_N + R_C + Z_L}
\]

In addition there is an interference current

\[
i_{SM} = \frac{V_{SM}}{Z_N + R_C + Z_L}
\]

through the load due to the interference voltage. The total voltage across the load is therefore:

\[
V_L = iZ_L + i_{SM}Z_L = i_n Z_L \cdot \frac{Z_N}{Z_N + R_C + Z_L} + V_{SM} \cdot \frac{Z_L}{Z_N + R_C + Z_L} \quad [6.37]
\]

Normally we make \(R_C + Z_L \ll Z_N\) to obtain maximum current transfer to the load (Section 5.1.3); under these conditions eqn [6.37] becomes:

\[
V_L = i_n Z_L + \frac{Z_L}{Z_N} V_{SM} \quad [6.38]
\]

Since \(Z_L/Z_N \ll 1\), this means that with a current transmission system only a small fraction of \(V_{SM}\) is across the load. Thus a current transmission system has far greater inherent immunity to series mode interference than a voltage transmission system.

In a thermocouple temperature measurement system, therefore, it may be better to convert the thermocouple millivolt e.m.f. into a current signal (Section 9.4.1) prior to transmission, rather than transmit the e.m.f. directly.

Figure 6.7(c) shows a **voltage transmission system** subject to **common mode interference** in which the potentials of both sides of the signal circuit are raised by \(V_{CM}\) relative to a common earth plane. If, as above, \(Z_L \gg R_C + Z_{Th}\), then current \(i \to 0\) so that the potential drops \(iR_C/2\), etc., can be neglected. Under these conditions:

Potential at \(B = V_{CM}\)
Potential at \(A = V_{CM} + E_{Th}\)

and

\[
V_L = V_B - V_A = E_{Th}
\]

This means that the voltage across the load is unaffected by \(V_{CM}\); there is, however, the possibility of conversion of a common mode voltage to series mode.
6.4 Noise sources and coupling mechanisms

6.4.1 Internal noise sources

The random, temperature-induced motion of electrons and other charge carriers in resistors and semiconductors gives rise to a corresponding random voltage which is called thermal or Johnson noise. This has a power spectral density which is uniform over an infinite range of frequencies (white noise) but proportional to the absolute temperature $\theta$ K of the conductor, i.e.

$$\phi = 4Rk\theta \text{ W/Hz}$$  \[6.39\]

where $R$Ω is the resistance of the conductor and $k$ is the Boltzmann constant = $1.4 \times 10^{-23}$ J K$^{-1}$. From eqn [6.22] the total thermal noise power between frequencies $f_1$ and $f_2$ is:

$$W = \int_{f_1}^{f_2} 4Rk\theta \, df = 4Rk\theta(f_2 - f_1) \text{ W}$$  \[6.40\]

and from [6.33] the corresponding r.m.s. voltage is:

$$V_{\text{RMS}} = \sqrt{W} = \sqrt{4Rk\theta(f_2 - f_1)} \text{ V}$$  \[6.41\]

Thus if $R = 10^6$ Ω, $f_2 - f_1 = 10^6$ Hz and $\theta = 300$ K, $V_{\text{RMS}} = 130$ µV and is therefore comparable with low-level measurement signals such as the output from a strain gauge bridge.

A similar type of noise is called shot noise; this occurs in transistors and is due to random fluctuations in the rate at which carriers diffuse across a junction. This is again characterised by a uniform power spectral density over a wide range of frequencies.

6.4.2 External noise and interference sources[1]

The most common sources of external interference are nearby a.c. power circuits which usually operate at 240 V, 50 Hz. These can produce corresponding sinusoidal interference signals in the measurement circuit, referred to as mains pick-up or hum. Power distribution lines and heavy rotating machines such as turbines and generators can cause serious interference.

D.C. power circuits are less likely to cause interference because d.c. voltages are not coupled capacitively and inductively to the measurement circuit.

However, switching often occurs in both a.c. and d.c. power circuits when equipment such as motors and turbines is being taken off line or brought back on line. This causes sudden large changes in power, i.e. steps and pulses, which can produce corresponding transients in the measurement circuit.

The air in the vicinity of high voltage power circuits can become ionised and a corona discharge results. Corona discharge from d.c. circuits can result in random noise in the measurement circuit and that from a.c. circuits results in sinusoidal interference at the power frequency or its second harmonic.

Fluorescent lighting is another common interference source; arcing occurs twice per cycle so that most of the interference is at twice the power frequency.
Radio-frequency transmitters, welding equipment and electric arc furnaces can produce **r.f. interference** at frequencies of several MHz.

### 6.4.3 Coupling mechanisms to external sources

**Inductive coupling**

Figure 6.8(a) shows inductive or **electromagnetic coupling** between the measurement circuit and a nearby power circuit. If the circuits are sufficiently close together, then there may be a significant mutual inductance $M$ between them. This means that an alternating current $i$ in the power circuit induces a series mode interference voltage in the measurement circuit of magnitude

$$V_{SM} = M \frac{di}{dt} \tag{6.42}$$

Thus if $M = 1 \mu H$ and $\frac{di}{dt} = 10^3$ A s$^{-1}$ (possible in a 1 horsepower motor) then $V_{SM} = 1$ mV, which can be comparable with the measurement signal. The mutual inductance $M$ depends on the geometry of the two circuits, namely on the overlapping length and separation, but is distributed over the entire length of the circuits rather than the ‘lumped’ equivalent value shown in Figure 6.8. Inductive coupling will occur even if the measurement circuit is completely isolated from earth.
Capacitive coupling\textsuperscript{[1,2]}

Another important coupling mechanism is \textbf{capacitive} or \textbf{electrostatic coupling}, which is illustrated in Figure 6.8(b). The diagram shows the measurement circuit close to an a.c. power cable which is at a potential of 240 V (r.m.s.) relative to the earth plane. The power cable, earth plane and signal leads are all conductors, so that there may be some capacitance between the power cable and the signal leads and between the signal leads and the earth plane. These capacitances will be distributed over the entire length of the measurement circuit, but are represented by ‘lumped’ equivalents. \(C_1\) and \(C_2\) are the capacitances between the power cable and signal leads, and \(C_{1E}\) and \(C_{2E}\) the capacitances between the signal lead and the earth plane; all four capacitances will be proportional to the length of the measurement circuit, which could be tens of metres in an industrial installation. Ignoring the measurement signal voltage \(E_{Th}\) for the moment, the potentials at \(B\) and \(E\) are determined by the potential dividers \(ABC\) and \(DEF\):

\[
V_B = 240 \left[ \frac{1/(j\omega C_{1E})}{1/(j\omega C_{1E}) + 1/(j\omega C_1)} \right] = 240 \frac{C_1}{C_1 + C_{1E}} \tag{6.43}
\]

\[
V_E = 240 \left[ \frac{1/(j\omega C_{2E})}{1/(j\omega C_{2E}) + 1/(j\omega C_2)} \right] = 240 \frac{C_2}{C_2 + C_{2E}} \tag{6.44}
\]

Thus we have a common mode interference voltage \(V_{CM} = V_E\) and a series mode interference voltage:

\[
V_{SM} = V_B - V_E = 240 \left( \frac{C_1}{C_1 + C_{1E}} - \frac{C_2}{C_2 + C_{2E}} \right) \tag{6.44}
\]

Thus series mode interference is zero only if there is perfect balance between the coupling capacitances, i.e. \(C_1 = C_2\) and \(C_{1E} = C_{2E}\); in practice small imbalances are present due to slightly different distances between each signal lead and the power cable/earth plane.

\textbf{Multiple earths}\textsuperscript{[1]}

The above explanation assumes an earth plane having a potential of 0 volts at every point on its surface. Heavy electrical equipment can, however, cause currents to flow through the earth, causing different potentials at different points. If the measurement circuit is completely isolated from the earth plane there is no problem. In practice, however, there may be a leakage path connecting the signal source to one earth point and another leakage path connecting the recorder or indicator to a different earth point, some distance away. If the two earth points are at different potentials, then common and series mode interference voltages are produced in the measurement circuit.

Figure 6.8(c) illustrates the general problem of \textbf{multiple earths}. The measurement signal source \(E_{Th}\) is connected via a resistive cable to a receiver represented by a resistive load \(R_L\). Provided \(R_L > R_c + R_{Th}\), the current flow in \(PQRS\) is negligible and \(V_L = E_{Th}\), provided the circuit is completely isolated from earth. However, leakage paths \(Z_{SE}\) and \(Z_{RE}\) exist between source/source earth and receiver/receiver earth. If
$V_E$ is the difference in potential between source earth and receiver earth, then a current $i_E$ flows in the circuit $UPST$, given by:

$$i_E = \frac{V_E}{Z_E + Z_{SE} + (R_C/2) + Z_{RE}} \quad [6.45]$$

Thus

- Potential at $P = V_E - i_E(Z_E + Z_{SE})$
- Potential at $Q, R = V_E - i_E(Z_E + Z_{SE}) + E_{Th}$
- Potential at $S = i_E Z_{RE}$

Thus there is a common mode interference voltage:

$$V_{CM} = V_S = \frac{Z_{RE}}{Z_E + Z_{SE} + (R_C/2) + Z_{RE}} \quad [6.46]$$

To find the series mode interference voltage we need to calculate the voltage across $R_L$:

$$V_L = V_R - V_S = V_E - i_E(Z_E + Z_{SE} + Z_{RE}) + E_{Th} = E_{Th} + i_E R_C/2$$

i.e. there is a series mode interference voltage:

$$V_{SM} = V_E = \frac{R_C/2}{Z_E + Z_{SE} + (R_C/2) + Z_{RE}} \quad [6.47]$$

Ideally we require both $Z_{SE}$ and $Z_{RE}$ to be as large as possible in order to minimise $i_E$ and $V_{SM}$; this, however, is not always possible in an industrial application. A common example is a thermocouple installation where in order to achieve as good a speed of response as possible the tip of the thermocouple touches the thermowell or sheath (Section 14.2). The thermowell is itself bolted to a metal vessel or pipe which is in turn connected to one point in the earth plane. Thus $Z_{SE}$ will be very small, say $Z_{SE} = 10 \Omega$ (resistive), so that the receiver must be isolated from earth to minimise $V_{SM}$. Taking $Z_E = 1 \Omega$, $R_C/2 = 10 \Omega$, $V_E = 1 \text{ V}$ and $Z_{RE} = 10^6 \Omega$, we have:

$$V_{SM} = 1 \times \frac{10}{1 + 10 + 10 + 10^6} \text{ V} = 10 \mu\text{V}$$

The worst case is when the receiver is also directly connected to earth, i.e. $Z_{RE} = 0$, giving $V_{SM} = 0.48 \text{ V}$. Thus if the measurement circuit must be connected to earth, the connection must be made at one point only.

### 6.5 Methods of reducing effects of noise and interference

#### 6.5.1 Physical separation

Since mutual inductances and coupling capacitances between measurement and power circuits are inversely proportional to the distance between them, this distance should be as large as possible.
6.5.2 Electromagnetic shielding

The simplest way of reducing the effects of inductive coupling to an external interference source is shown in Figure 6.9. The two conductors $A$ and $B$ of the measurement circuit are twisted into loops of approximately equal area. This arrangement is commonly known as **twisted pairs** and is explained in ref. [2]. The magnitude of the interference voltage induced in a given loop is proportional to the area of the loop and the rate of change of the external magnetic field. The sign of the induced voltage depends on the orientation of conductors $A$ and $B$. Thus if a voltage $V_{XY}$ is induced in the $j$th loop between points $X$ and $Y$, then an opposing voltage $V_{YZ}$ is induced in the $(j+1)$th loop between $Y$ and $Z$. In the ideal case of both loops having the same area and experiencing the same magnetic fields, $|V_{XY}| = |V_{YZ}|$, i.e. there is a zero resultant induced voltage between $X$ and $Z$. This process is repeated for the whole length of the twisted pair, giving a reduced overall interference voltage.

6.5.3 Electrostatic screening and shielding

The best method of avoiding the problem of capacitive coupling to a power circuit (Section 6.4.3) is to enclose the entire measurement circuit in an earthed metal screen or shield. Figure 6.10(a) shows the ideal arrangement; the screen is connected directly to earth at a single point, at either the source or the receiver. There is no direct connection between the screen and the measurement circuit, only high impedance leakage paths via the small (screen/measurement circuit) capacitances $C_{SM}$. The screen provides a low impedance path to earth for the interfering currents $i$; the currents through $C_{SM}$ and $C_E$ are small, thus reducing series and common mode interference.

The above ideal of the measurement circuit completely insulated from the screen and the screen earthed at one point only may be difficult to achieve in practice for the following reasons:

(a) The signal source may be directly connected to a local earth point via the structure on which it is mounted; an example is the thermocouple installation mentioned in Section 6.4.3.

(b) The receiver may be directly connected to a local earth; an example is in a computer-based system where the receiver must be directly connected to the computer earth.

(c) There may be indirect connections via leakage impedances.

Figure 6.10(b) illustrates the general problem. The measurement circuit PQRS is connected to the screen (impedance $Z_S$) via source/screen impedance $Z_{SS}$ and receiver/screen impedance $Z_{RS}$. The screen is connected to earth point $U$ at the source end via $Z_{SG}$ and to earth point $T$ at the receiver end via $Z_{ST}$. The measurement circuit can be affected by interference voltages from both $V_e$ (potential difference between $U$ and $T$) and nearby power circuits.
Analysis of circuits UXYT and XPSY using Kirchhoff’s laws gives:

UXYT \[ V_E = i_1 Z_E + i_1 Z_{SU} + (i_1 - i_2) Z_S + i_1 Z_{ST} \] [6.48]

XPSY \[ 0 = -(i_1 - i_2) Z_S + i_1 Z_{SS} + i_2 R_c/2 + i_1 Z_{RS} \] [6.49]

Solution of these equations gives:

\[ i_1 = \frac{(Z_S + Z_{SS} + R_c/2 + Z_{RS}) V_E}{(Z_E + Z_{SU} + Z_S + Z_{ST})(Z_S + Z_{SS} + R_c/2 + Z_{RS}) - Z_S^2} \] [6.50]

\[ i_2 = \frac{Z_S V_E}{(Z_E + Z_{SU} + Z_S + Z_{ST})(Z_S + Z_{SS} + R_c/2 + Z_{RS}) - Z_S^2} \] [6.51]

The series mode interference voltage in the measurement circuit PQRS is the voltage drop across PS, i.e.
The common mode interference voltage is the voltage drop across ST, i.e.

\[ V_{CM} = V_{SF} + V_{YT} = i_z Z_{RS} + i_z Z_{ST} \]  \[ 6.53 \]

To minimise \( V_{SM} \) and \( V_{CM} \), we want the product term:

\[ (Z_{SU} + Z_{ST}) (Z_{SS} + Z_{RS}) \]  \[ 6.54 \]

However, we cannot have both \( Z_{SU} \) and \( Z_{ST} \) large; either \( Z_{ST} \) or \( Z_{SU} \) must be small, otherwise the screen will not be earthed and there is therefore no low impedance path to earth for the capacitively coupled interference currents. Condition (6.54) therefore reduces to the two conditions:

\[ Z_{SU} (Z_{SS} + Z_{RS}) = \text{HIGH; } Z_{ST} = \text{LOW} \]  \[ 6.55 \]

OR

\[ Z_{ST} (Z_{SS} + Z_{RS}) = \text{HIGH; } Z_{SU} = \text{LOW} \]

which are satisfied if:

\[ Z_{SU} = \text{HIGH AND } (Z_{SS} \text{ OR } Z_{RS} \text{ OR both } = \text{HIGH}), \text{ } Z_{ST} = \text{LOW} \]  \[ 6.56 \]

OR

\[ Z_{ST} = \text{HIGH AND } (Z_{SS} \text{ OR } Z_{RS} \text{ OR both } = \text{HIGH}), \text{ } Z_{SU} = \text{LOW} \]

As mentioned above, in many practical situations it may be impossible to have the measurement circuit completely isolated from the screen, i.e. both \( Z_{SS} \) and \( Z_{RS} \) high. In this situation, possible confusion is avoided if \( Z_{SU} \) and \( Z_{SS} \) are both high, or \( Z_{ST} \) and \( Z_{RS} \) are both high.

In this situation the conditions become:

\[ Z_{SU} = \text{HIGH AND } Z_{SS} = \text{HIGH, } Z_{RS} = \text{LOW AND } Z_{ST} = \text{LOW} \] (Isolated source)

OR

\[ Z_{ST} = \text{HIGH AND } Z_{RS} = \text{HIGH, } Z_{SS} = \text{LOW AND } Z_{SU} = \text{LOW} \] (Isolated receiver)

### 6.5.4 Use of differential amplifiers

Common mode interference voltages can be successfully rejected by the use of a differential amplifier (Figure 6.11 and Section 9.2). An ideal differential amplifier has an output:

\[ V_{OUT} = \frac{R_F}{R_1} (V_2 - V_1) = \frac{R_F}{R_1} E_{th} \]  \[ 6.58 \]
6.5 METHODS OF REDUCING EFFECTS OF NOISE AND INTERFERENCE

6.5.5 Filtering

A filter is an element which transmits a certain range (or ranges) of frequencies and rejects all other frequencies. An analogue filter is an electrical network, consisting usually of resistors, capacitors and operational amplifiers, which conditions continuous signals. A digital filter is usually a digital computer programmed to process sampled values of a signal (Chapter 10). Provided that the power spectrum of the measurement signal occupies a different frequency range from that of the noise or interference signal, then filtering improves the signal-to-noise ratio.

Figures 6.12(a)–(d) show the use of low pass, high pass, band pass and band stop filters in rejecting noise. In all cases the filter transmits the measurement signal but rejects the noise signal, which occupies a different frequency range. The diagrams show the amplitude ratio $|G(j\omega)|$ for each filter and the power spectral densities $\phi(\omega)$ for signal and noise. In order to transmit the measurement signal without distortion the transfer function $G(s)$ of the filter must, ideally, satisfy the conditions of Section 4.4, i.e. that $|G(j\omega)| = 1$ and $\arg G(j\omega) = 0$ for all the frequencies present in the measurement signal spectrum. Analogue filtering can be implemented at the signal conditioning stage; the a.c. amplifier of Figure 9.12 is an example of a band pass filter. Digital filtering can be implemented at the signal processing stage (Chapter 10).

If, however, measurement signal and noise spectra overlap, filtering has limited value. Figure 6.13(a) shows a measurement signal affected by wide band noise; a low-pass filter with bandwidth matched to the signal spectrum removes as much of the noise as possible, but the noise inside the filter bandwidth still remains. Figure 6.13(b)
Figure 6.12 Use of filtering to reject noise.

shows the measurement signal affected by 50 Hz interference; a narrow band-stop filter centred on 50 Hz rejects the interference but also rejects measurement signal frequencies around 50 Hz and causes amplitude and phase distortion over a wider range of frequencies.

6.5.6 Modulation

The problem of Figure 6.13(b) can be solved by modulating the measurement signal onto a higher frequency carrier signal, e.g. a 5 kHz sine wave as shown in Figure 6.13(c). The simplest form of modulation is amplitude modulation; this involves the multiplication of measurement and carrier signals and is discussed in detail in Section 9.3.

Modulation causes the spectrum of the measurement signal to be shifted to around 5 kHz (Figure 6.13(d)). If the 50 Hz interference is added after modulation, i.e. during transmission from sensor/modulator to a remote amplifier/demodulator, the interference spectrum is not shifted. The interference can then be easily rejected by an a.c. amplifier, i.e. a band pass filter with bandwidth matched to the spectrum of the amplitude modulated signal. Modulation, however, does not help the problem
6.5 METHODS OF REDUCING EFFECTS OF NOISE AND INTERFERENCE

6.5.7 Averaging

Signal averaging can be used to recover a repetitive measurement signal affected by random noise, even if the signal r.m.s. value is much less than that of the noise.[3] The process is shown in Figure 6.14.

Suppose that \( T \) is the time for each complete cycle of the repetitive signal; \( p \) sections of the noise-affected signal, each of duration \( T \), are fed into the averager. \( N \) samples from each section are taken and stored, giving \( pN \) samples in total; typically...
we may have \( p = 50 \), \( N = 100 \). The sampling is exactly synchronised: i.e. if the \( i \)th sample of the 1st section is taken at time \( t_0 \), the \( i \)th sample of the 2nd section is taken at \( t_0 + T \), the \( i \)th sample of the 3rd at \( t_0 + 2T \), and so on. Corresponding samples from each section are then averaged: e.g. the first sampled values from each of the \( p \) sections are added together and divided by \( p \). Thus the average value of the \( i \)th sample is:

\[
y_{iAV} = \frac{1}{p} (y_{1i} + y_{2i} + \ldots + y_{pi}), \quad i = 1, \ldots, N
\]

Each of these \( N \) average sample values are then displayed at the appropriate time to give the averaged signal. Corresponding sample values \( A, B, C, \ldots \) of the signal component are approximately equal, so that the average value \( Z \) has a similar magnitude. Corresponding sample values \( A', B', C', \ldots \) of the noise component are very different, some positive and some negative, so that the average value \( Z' \) is reduced in magnitude. Averaging therefore maintains the r.m.s. value of the measurement signal while reducing the r.m.s. value of the random noise.

This improvement in signal-to-noise ratio can be readily calculated for random noise with a normal probability density function. Suppose we have \( p \) normal signals \( y_1(t) \) to \( y_p(t) \), with standard deviations \( \sigma_1 \) to \( \sigma_p \), respectively; then from Section 2.3 the average signal

\[
y_{AV}(t) = \frac{1}{p} [y_1(t) + y_2(t) + \ldots + y_p(t)]
\]

is also normal with standard deviation

\[
\sigma_{AV} = \sqrt{\frac{\sigma_1^2}{p^2} + \frac{\sigma_2^2}{p^2} + \ldots + \frac{\sigma_p^2}{p^2}}
\]

If \( \sigma_1 = \sigma_2 = \sigma_p = \sigma \), then
6.5 METHODS OF REDUCING EFFECTS OF NOISE AND INTERFERENCE

6.5.8 Autocorrelation

Autocorrelation can be used to detect the presence of a sinusoidal or any periodic signal buried in random noise. The actual waveform of the measurement signal cannot be recovered because phase information is lost in correlation, but we can measure the amplitude and period of the signal from the autocorrelation function (ACF) of the noise-affected signal. The ACF for the (signal + noise) is the sum of the signal ACF and noise ACF, i.e.

\[ R_{yy}^{SN}(\beta) = R_{yy}^{S}(\beta) + R_{yy}^{N}(\beta) \]  

Thus using eqns [6.27] and [6.31] the ACF for a sinusoidal signal affected by band-limited white noise is:

\[ R_{yy}^{SN}(\beta) = \frac{b^2}{2} \cos \omega_1 \beta + A \frac{\sin \omega_1 \beta}{\beta} \]  

The form of \( R_{yy}^{SN}(\beta) \) is shown in Figure 6.15; at large values of \( \beta \) the \( A(\sin \omega_1 \beta) / \beta \) term due to the noise decays to zero, leaving the \((b^2/2) \cos \omega_1 \beta \) term due to the signal. Thus the amplitude \( b \) and period \( 2\pi / \omega_1 \) of the original signal can be found from the amplitude and period of the autocorrelation function at large values of time delay. This method can be used in the vortex flowmeter (Chapter 12) to measure the vortex frequency in the presence of random flow turbulence.

---

**Figure 6.15**

Autocorrelation detection of periodic signal buried in noise.
Conclusion

The chapter began by defining random signals and deterministic signals and explained that in many practical situations the wanted signal may be random. Unwanted signals may also be present in the measurement circuit; these can be classified as either interference (deterministic) or noise (random).

The chapter then explained how random signals can be quantified using the following statistical functions: mean, standard deviation, probability density function, power spectral density and autocorrelation function.

The effects of noise and interference voltage on measurement circuits using both voltage and current transmission were then discussed. In the following section internal noise and external interference sources were discussed and the mechanisms whereby external sources are coupled to the measurement circuit were explained. The chapter concluded by explaining methods of reducing the effects of noise and interference, including electromagnetic shielding, electrostatic screening, filtering, modulation, averaging and autocorrelation.

References


Problems

6.1 Table Prob. 1 gives 50 sample values of a random signal.
(a) Estimate the mean and standard deviation of the signal.
(b) Using an interval $\Delta y = 0.5 \text{ V}$, draw the $P_j$ and $C_j$ discrete probability distributions for the signal.

Table Prob. 1.

<table>
<thead>
<tr>
<th>$y$</th>
<th>$P_j$</th>
<th>$C_j$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−0.59</td>
<td>1.02</td>
<td>−0.25</td>
</tr>
<tr>
<td>−1.00</td>
<td>1.36</td>
<td>0.03</td>
</tr>
<tr>
<td>0.65</td>
<td>0.11</td>
<td>0.99</td>
</tr>
<tr>
<td>2.15</td>
<td>0.91</td>
<td>0.89</td>
</tr>
<tr>
<td>0.42</td>
<td>0.05</td>
<td>0.26</td>
</tr>
<tr>
<td>−0.63</td>
<td>−0.06</td>
<td>−0.61</td>
</tr>
</tbody>
</table>

6.2 Two complete periods of a square wave can be represented by the following 20 sample values:

$+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1$,

$+1+1+1+1+1+1+1+1$,

$+1+1+1+1+1+1+1+1$,

$+1+1+1+1+1+1+1+1$,

$+1+1+1+1+1+1+1+1$,

$+1+1+1+1+1+1+1+1$,

$+1+1+1+1+1+1+1+1$.

Find the autocorrelation function of the signal over one complete period by evaluating the coefficients $R_{yy}(m\Delta T)$ for $m = 0, 1, 2, \ldots, 10$. 
6.3 A sinusoidal signal of amplitude 1.4 mV and frequency 5 kHz is ‘buried’ in Gaussian noise with zero mean value. The noise has a uniform power spectral density of 100 pW Hz$^{-1}$ up to a cut-off frequency of 1 MHz.

(a) Find the total power, r.m.s. value and standard deviation for the noise signal.
(b) What is the signal-to-noise ratio in dB?
(c) Sketch the autocorrelation function for the combined signal and noise.
(d) The combined signal is passed through a band-pass filter with centre frequency 5 kHz and bandwidth 1 kHz. What improvement in signal-to-noise ratio is obtained?
(e) The filtered signal is then passed through a signal averager which averages corresponding samples of 100 sections of signal. What further improvement in signal-to-noise ratio is obtained?

6.4 A thermocouple giving a 10 mV d.c. output voltage is connected to a high impedance digital voltmeter some distance away. A difference in potential exists between earth at the thermocouple and earth at the voltmeter. Using the equivalent circuit given in Figure Prob. 4.

(a) Calculate the r.m.s. values of the series mode and common mode interference voltages at the voltmeter input.
(b) If the digital voltmeter has a common mode rejection ratio of 100 dB, find the minimum and maximum possible measured voltages.

Figure Prob. 4.

6.5 A sinusoidal signal is transmitted over a noisy transmission link to a remote correlator acting as a receiver. Figure Prob. 5 shows a typical autocorrelation function. Use the figure to estimate the following quantities:

(a) Signal power
(b) Noise power
(c) Signal-to-noise ratio in decibels
(d) Signal amplitude
(e) Signal frequency
(f) Noise standard deviation (assume zero mean).

Hint: use eqns [6.27] and [6.33].
Figure Prob. 5.

$R_{ys}(\beta) \text{ mW}$

Time delay $\beta \text{ ms}$

Graph showing the variation of $R_{ys}(\beta)$ with time delay $\beta$. The graph displays periodic oscillations with varying amplitude.
In Chapters 3 and 4 we defined the accuracy of a measurement system and explained how measurement error can be calculated, under both steady-state and dynamic conditions. Reliability is another important characteristic of a measurement system; it is no good having an accurate measurement system which is constantly failing and requiring repair. The first section of this chapter deals with the reliability of measurement systems, first explaining the fundamental principles of reliability and the reliability of practical systems, then failure rate data, and finally examining ways of improving reliability. The following section examines the problems of how to choose the most appropriate measurement system, for a given application, from several competing possibilities. Initially a specification for the required application can be drawn up: this will be a list of important parameters such as accuracy, reliability, etc., each with a desired value. This can then be compared with the manufacturer’s specification for each of the competing measurement systems and the system with the closest specification is chosen. Even if all the required information is available this procedure is far from satisfactory because it takes no account of the relative importance of each parameter. A better method, explained in the final section, is to choose the system with minimum total lifetime operating cost.

7.1 Reliability of measurement systems

7.1.1 Fundamental principles of reliability

Probability \( P \)

If a large number of random, independent trials are made, then the probability of a particular event occurring is given by the ratio:

\[
P = \frac{\text{number of occurrences of the event}}{\text{total number of trials}} \quad \quad [7.1]
\]
in the limit that the total number of trials tends to infinity. Thus the probability of a tossed coin showing heads tends to the theoretical value of $\frac{3}{2}$ over a large number of trials.

**Reliability $R(t)$**

The reliability of a measurement element or system can be defined as: ‘the probability that the element or system will operate to an agreed level of performance, for a specified period, subject to specified environmental conditions’. In the case of a measurement system ‘agreed level of performance’ could mean an accuracy of $\pm 1.5\%$. If the system is giving a measurement error outside these limits, then it is considered to have failed, even though it is otherwise working normally. The importance of environmental conditions on the reliability of measurement systems will be discussed more fully later. Reliability decreases with time; a measurement system that has just been checked and calibrated should have a reliability of 1 when first placed in service. Six months later, the reliability may be only 0.5 as the probability of survival decreases.

**Unreliability $F(t)$**

This is ‘the probability that the element or system will fail to operate to an agreed level of performance, for a specified period, subject to specified environmental conditions’. Since the equipment has either failed or not failed the sum of reliability and unreliability must be unity, i.e.

$$R(t) + F(t) = 1 \quad [7.2]$$

Unreliability also depends on time; a system that has just been checked and calibrated should have an unreliability of zero when first placed in service, increasing to, say, 0.5 after six months.

### 7.1.2 Practical reliability definitions

Since $R(t)$ and $F(t)$ are dependent on time, it is useful to have measures of reliability which are independent of time. We will consider two cases: in the first the items are non-repairable and in the second the items are repairable.

**Non-repairable items**

Suppose that $N$ individual items of a given non-repairable component are placed in service and the times at which failures occur are recorded during a test interval $T$. We further assume that all the $N$ items fail during $T$ and that the $i$th failure occurs at time $T_i$, i.e. $T_i$ is the survival time or up time for the $i$th failure. The total up time for $N$ failures is therefore $\sum_{i=1}^{N} T_i$ and the mean time to failure is given by

$$\text{Mean time to fail} = \frac{\text{total up time}}{\text{number of failures}}$$
7.1 RELIABILITY OF MEASUREMENT SYSTEMS

i.e.

\[ \text{MTTF} = \frac{1}{N} \sum_{i=1}^{\lfloor N \rfloor} T_i \]  \[7.3\]

The mean failure rate \( \bar{\lambda} \) is correspondingly given by:

\[ \text{Mean failure rate} = \frac{\text{number of failures}}{\text{total up time}} \]

i.e.

\[ \bar{\lambda} = \frac{N}{\sum_{i=1}^{\lfloor N \rfloor} T_i} \]  \[7.4\]

i.e. mean failure rate is the reciprocal of MTTF.

There are \( N \) survivors at time \( t = 0 \), \( N - i \) at time \( t = T_i \), decreasing to zero at time \( t = T \); Figure 7.1(a) shows how the probability of survival, i.e. reliability, \( R_i = (N - i)/N \) decreases from \( R_i = 1 \) at \( t = 0 \), to \( R_i = 0 \) at \( t = T \). The \( i \)th rectangle has height \( 1/N \), length \( T_i \) and area \( T_i/N \). Therefore from eqn [7.3] we have:

\[ \text{MTTF} = \text{total area under the graph} \]

In the limit that \( N \to \infty \), the discrete reliability function \( R_i \) becomes the continuous function \( R(t) \). The area under \( R(t) \) is \( \int_0^T R(t) \, dt \) so that we have in general:

\[ \text{MTTF} = \int_0^\infty R(t) \, dt \]  \[7.5\]

The upper limit of \( t = \infty \) corresponds to \( N \) being infinite.

Repairable items

Figure 7.1(b) shows the failure pattern for \( N \) items of a repairable element observed over a test interval \( T \). The down time \( T_{dj} \) associated with the \( j \)th failure is the total time that elapses between the occurrence of the failure and the repaired item being put back into normal operation. The total down time for \( N_F \) failures is therefore \( \sum_{j=1}^{N_F} T_{dj} \) and the mean down time is given by

\[ \text{Mean down time} = \frac{\text{total down time}}{\text{number of failures}} \]

i.e.

\[ \text{MDT} = \frac{1}{N_F} \sum_{j=1}^{N_F} T_{dj} \]  \[7.6\]
Figure 7.1
Failure patterns:
(a) Non-repairable items
(b) Repairable items.
7.1 RELIABILITY OF MEASUREMENT SYSTEMS

The **total up time** can be found by subtracting the total down time from $NT$, i.e.:

$$\text{Total up time} = NT - \sum_{j}^{N} T_{Dj}$$

$$= NT - N_{F} \text{ MDT}$$

The mean up time or the **mean time between failures** (MTBF) is therefore given by:

$$\text{Mean time between failures} = \frac{\text{total up time}}{\text{number of failures}}$$

i.e.

$$\text{MTBF} = \frac{NT - N_{F} \text{ MDT}}{N_{F}} \quad [7.7]$$

The **mean failure rate** $\lambda$ is correspondingly given by:

$$\text{Mean failure rate} = \frac{\text{number of failures}}{\text{total up time}}$$

i.e.

$$\lambda = \frac{N_{F}}{NT - N_{F} \text{ MDT}} \quad [7.8]$$

Again mean failure rate is the reciprocal of MTBF.

Thus if 150 faults are recorded for 200 transducers over 1.5 years with a mean down time of 0.002 years, then the observed MTBF is 1.998 years and the mean failure rate is 0.5005 yr$^{-1}$.

The **availability** $A$ of the element is the fraction of the total test interval over which it is performing within specification, i.e. up; thus we have:

$$\text{Availability} = \frac{\text{total up time}}{\text{test interval}}$$

$$= \frac{\text{total up time}}{\text{total up time} + \text{total down time}}$$

$$= \frac{N_{F} \times \text{MTBF}}{N_{F} \times \text{MTBF} + N_{F} \times \text{MDT}}$$

i.e.

$$A = \frac{\text{MTBF}}{\text{MTBF} + \text{MDT}} \quad [7.9]$$

Using the above data of MTBF = 1.998 years and MDT = 0.002 years gives $A = 0.999$. 
Unavailability $U$ is similarly defined as the fraction of the total test interval over which it is not performing to specification, i.e. failed or down; thus we have:

$$\text{Unavailability} = \frac{\text{total down time}}{\text{test interval}}$$

giving:

$$U = \frac{\text{MDT}}{\text{MTBF} + \text{MDT}} \quad [7.10]$$

It follows from eqns [7.9] and [7.10] that:

$$A + U = 1 \quad [7.11]$$

### 7.1.3 Instantaneous failure rate and its relation to reliability

We assume to begin with that $n$ items of an element survive up to time $t = \xi$ and that $\Delta n$ items fail during the small time interval $\Delta \xi$ between $\xi$ and $\xi + \Delta \xi$. The probability of failure during interval $\Delta \xi$ (given survival to time $\xi$) is therefore equal to $\Delta n/n$. Assuming no repair during $\Delta \xi$ the corresponding instantaneous failure rate or hazard rate at time $\xi$ is, from eqn [7.8], given by:

$$\lambda(\xi) = \frac{\Delta n}{n \Delta \xi} = \frac{\text{failure probability}}{\Delta \xi} \quad [7.12]$$

The unconditional probability $\Delta F$ that an item fails during the interval $\Delta \xi$ is:

$$\Delta F = \text{probability that item survives up to time } \xi$$

AND

$$\text{probability that item fails between } \xi \text{ and } \xi + \Delta \xi \text{ (given survival to } \xi).$$

The first probability is given by $R(\xi)$ and from eqn [7.12] the second probability is $\lambda(\xi) \Delta \xi$. The combined probability $\Delta F$ is the product of these probabilities:

$$\Delta F = R(\xi) \lambda(\xi) \Delta \xi$$

i.e.

$$\frac{\Delta F}{\Delta \xi} = R(\xi) \lambda(\xi)$$

Thus in the limit that $\Delta \xi \to 0$, we have:

$$\frac{dF}{d\xi} = R(\xi) \lambda(\xi) \quad [7.13]$$

Also since $F(\xi) = 1 - R(\xi)$, $dF/d\xi = -(dR/d\xi)$, giving:

$$\frac{dR}{d\xi} = R(\xi) \lambda(\xi)$$
7.1 RELIABILITY OF MEASUREMENT SYSTEMS

7.1.4 Typical forms of failure rate function

In the previous section instantaneous failure rate or hazard rate $\lambda(t)$ was defined. Figure 7.2 shows the most general form of $\lambda(t)$ throughout the lifetime of an element. This is the so-called bathtub curve and consists of three distinct phases: early failure, useful life and wear-out failure. The early failure region is characterised by $\lambda(t)$ decreasing with time. When items are new, especially if the element is of a new design, early failures can occur due to design faults, poor quality components, manufacturing faults, installation errors, operator and maintenance errors; the latter may be due to unfamiliarity with the product. The hazard rate falls as design faults are rectified, weak components are removed, and the user becomes familiar with installing, operating and maintaining the element. The useful life region is characterised by a low,
constant failure rate. Here all weak components have been removed: design, manufacture, installation, operating and maintenance errors have been rectified so that failure is due to a variety of unpredictable lower-level causes. The wear-out region is characterised by $\lambda(t)$ increasing with time as individual items approach the end of the design life for the product; long-life components which make up the element are now wearing out.

Many measurement elements have a useful life region lasting many years, so that a constant failure rate model is often a good approximation. Here we have:

$$\lambda(t) = \lambda(\xi) = \lambda = \text{constant}$$

so that:

$$R(t) = \exp \left[ -\lambda \int_0^t \xi \right] = \exp(-\lambda t)$$

Thus a constant failure or hazard rate gives rise to an exponential reliability time variation or distribution as shown in Figure 7.3.

### 7.1.5 Reliability of systems

**Series systems**

We saw in Chapter 1 that a complete measurement system consists of several elements usually in series or cascade. Figure 7.4 shows a series system of $m$ elements with individual reliabilities $R_1, R_2, \ldots, R_n, \ldots, R_m$ respectively. The system will only survive if every element survives; if one element fails then the system fails.

![Figure 7.4 Reliability of series system.](image)
Assuming that the reliability of each element is independent of the reliability of the other elements, then the probability that the system survives is the probability that element 1 survives and the probability that 2 survives and the probability that 3 survives, etc. The system reliability $R_{\text{SYST}}$ is therefore the product of the individual element reliabilities, i.e.

$$R_{\text{SYST}} = R_1 R_2 \ldots R_i \ldots R_m$$  \[7.18\]

If we further assume that each of the elements can be described by a constant failure rate $\lambda$ (Section 7.1.4), and if $\lambda_i$ is the failure rate of the $i$th element, then $R_i$ is given by the exponential relation (eqn [7.17]):

$$R_i = e^{-\lambda_i t}$$  \[7.19\]

Thus

$$R_{\text{SYST}} = e^{-\lambda_1 t} e^{-\lambda_2 t} \ldots e^{-\lambda_i t} \ldots e^{-\lambda_m t}$$  \[7.20\]

so that if $\lambda_{\text{SYST}}$ is the overall system failure rate:

$$R_{\text{SYST}} = e^{-\lambda_{\text{SYST}} t} = e^{-\lambda_1 t - \lambda_2 t + \ldots + \lambda_i t + \ldots + \lambda_m t}$$  \[7.21\]

and

$$\lambda_{\text{SYST}} = \lambda_1 + \lambda_2 + \ldots + \lambda_i + \ldots + \lambda_m$$  \[7.22\]

This means that the overall failure rate for a series system is the sum of the individual element or component failure rates. Equations [7.18] and [7.22] show the importance of keeping the number of elements in a series system to a minimum; if this is done the system failure rate will be minimum and the reliability maximum. A measurement system consisting of a thermocouple ($\lambda_1 = 1.1$), a millivolt to current converter ($\lambda_2 = 0.1$) and a recorder ($\lambda_3 = 0.1$) in series will therefore have a failure rate $\lambda_{\text{SYST}} = 1.3$.

**Parallel systems**

Figure 7.5 shows an overall system consisting of $n$ individual elements or systems in parallel with individual unreliabilities $F_1, F_2, \ldots, F_j, \ldots, F_n$ respectively. Only one individual element or system is necessary to meet the functional requirements placed on the overall system. The remaining elements or systems increase the reliability of the overall system; this is termed redundancy. The overall system will only fail if every element/system fails; if one element/system survives the overall system survives. Assuming that the reliability of each element/system is independent of the reliability of the other elements, then the probability that the overall system fails is the probability that element/system 1 fails and the probability that 2 fails and the probability that 3 fails, etc. The overall system unreliability $F_{\text{SYST}}$ is therefore the product of the individual element/system unreliabilities, i.e.

$$F_{\text{SYST}} = F_1 F_2 \ldots F_j \ldots F_n$$  \[7.23\]
Comparing eqns [7.18] and [7.23] we see that, for series systems, system reliability is the product of element reliabilities, whereas for parallel systems system unreliability is the product of element unreliabilities. Often the individual elements/systems are identical, so that $F_1 = F_2 = \ldots = F_i = \ldots = F_n = F$, giving:

$$F_{\text{SYST}} = F^n$$  \[7.24\]

The temperature measurement system of the previous section has a failure rate $\lambda_{\text{SYST}} = 1.3 \text{ yr}^{-1}$; the corresponding unreliability $F$ is given by

$$F = 1 - e^{-\lambda_{\text{SYST}} t}$$

Thus if $t = 0.5 \text{ year}$ then $F = 0.478$. Figure 7.6 shows a redundant system consisting of three single temperature measurement systems in parallel. The overall system unreliability is therefore:

$$F_{\text{OVERALL}} = F^3 = 0.109$$

so that the probability of a failure with the overall system is less than a quarter of that of a single system.

The above parallel system, while reliable, is expensive. Since the thermocouple failure rate is 11 times greater than the converter and recorder failure rates, a more
7.1 RELIABILITY OF MEASUREMENT SYSTEMS

Cost-effective redundant system would have three thermocouples in parallel and only one converter and recorder. One possible system is shown in Figure 7.7. The three thermocouple e.m.f.’s $E_1$, $E_2$, and $E_3$ are input into a middle value selector element. The selector output signal is that input e.m.f. which is neither the lowest nor the highest; thus if $E_1 = 5.0$ mV, $E_2 = 5.2$ mV and $E_3 = 5.1$ mV, the output signal is $E_3$. If, however, thermocouple 3 fails so that $E_3 = 0$ mV, the selector output signal is $E_1$. The reliability of this system can be analysed by replacing the three thermocouples by a single element of unreliability

$$F_i = (1 - e^{-\lambda t})^3 = (1 - e^{-0.55})^3 = 0.076$$

or reliability $R_1 = 1 - 0.076 = 0.924$. The reliability of the other elements with $\lambda = 0.1$ is:

$$R_2 = R_3 = R_4 = e^{-\lambda t} = e^{-0.05} = 0.951$$

Using [7.18], the overall system reliability is:

$$R_{\text{OVERALL}} = R_1R_2R_3R_4 = 0.924(0.951)^3 = 0.795$$

i.e. $F_{\text{OVERALL}} = 0.205$. This is almost twice the unreliability of the parallel system but less than half that of a single system.

7.1.6 Failure rate data and models

A distinction must now be made between components and elements. A component is defined as a ‘non-repairable’ device, i.e. when it fails it is removed and thrown away. Examples are a resistor or an integrated circuit. An element, however, is a repairable part of a system, which is usually made up of several components. Examples are a pressure sensor, a temperature transmitter and a recorder.

Failure rate data for both components and elements can be found experimentally by direct measurements of the frequency of failure of a number of items of a given type. Equation [7.4] can be used to find the observed failure rate of non-repairable components and eqn [7.8] to find $\hat{\lambda}$ for repairable elements.

Table 7.1 gives observed average failure rates for typical instruments. These data have been taken from the UK data bank operated by the Systems Reliability Service (SRS). The table specifies the environment in which each type of instrument is located. For an element located in the process fluid, the environment is determined by the nature of the fluid, e.g. temperature, corrosion properties, presence of dirt or solid particles.
For an element located in the atmosphere, the environment is determined by atmospheric conditions, e.g. temperature, humidity, salinity, presence of dust. The failure rate of a given type of element will depend on the environment in which it is located: an iron–copper thermocouple will have a higher failure rate in a corrosive acid than in water.

Table 7.2 shows the observed failure rates for given types of elements at three works A, B, C which process different materials and fluids and have different background environments.[2] The observed failure rates can be regarded as the product of a base failure rate $\lambda_B$ and an environmental correction factor $\pi_E$:

$$\lambda_{OBS} = \pi_E \times \lambda_B$$  \[7.25\]

Here $\lambda_B$ corresponds to the best environmental conditions and $\pi_E$ has values 1, 2, 3 or 4, the highest figure corresponding to the worst environment.

The failure rate of elements can alternatively be calculated from the failure rate data/models for the basic components which make up the element.

Table 7.3 shows the calculation of the overall failure rate, from basic component data, for an electronic square root extractor module.[3] The module gives an output voltage signal in the range 1–5 V, proportional to the square root of the input signal with range 4–20 mA; this type of module is commonly used in fluid flow rate control systems. The module is made up from basic electronic components of various types, all connected in series. Several components of each type are present.
eqn [7.22] the failure rate $\lambda$ of a module containing $m$ different component types in series with failure rates $\lambda_1, \lambda_2, \ldots, \lambda_i, \ldots, \lambda_m$, and one of each type present is:

$$\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_i + \ldots + \lambda_m$$  \[7.26\]

If there are multiple components of each type, all connected in series, then the module failure rate is given by:

$$\lambda = N_i \lambda_1 + N_2 \lambda_2 + \ldots + N_i \lambda_i + \ldots + N_m \lambda_m$$  \[7.27\]

where $N_1, N_2, \ldots, N_i, \ldots, N_m$ are the quantities of each component type. The failure rate of each component type is calculated using the model equation:

$$\lambda_i = (F_1 + F_2 + F_3) \times K_1 \times K_2 \times K_3 \times K_4$$  \[7.28\]

Table 7.3 gives values of $F_1, F_2, F_3, K_1, K_2, K_3, K_4$ and failure rate $\lambda_i$ for each component. Each failure rate is then multiplied by the appropriate quantity $N_i$ and the $N_i \lambda_i$ values are added together to give a total module failure rate of $3.01 \times 10^{-6}$ per hour.
Table 7.3 Calculation of overall failure rate for electronic square root module.\(^a\)

<table>
<thead>
<tr>
<th>Component</th>
<th>Failure rates per 10(^{10}) hours</th>
<th>(F_1)</th>
<th>(F_2)</th>
<th>(F_3)</th>
<th>(K_1)</th>
<th>(K_2)</th>
<th>(K_3)</th>
<th>(K_4)</th>
<th>Rate(^b)</th>
<th>Qty</th>
<th>Value</th>
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<td>Carbon film</td>
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<td>0</td>
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<td>1</td>
<td>1</td>
<td>100</td>
<td>17</td>
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<td>33 nF &lt; (C) ≤ 1 (\mu) F</td>
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<td>3.32 × 10(^{5}) hours</td>
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</table>

\(^a\) Data sources:
1: Electronic Reliability Data, National Centre of Systems Reliability, Application Code 2, 25C
3: Component supplier’s information.

\(^b\) Rate = (\(F_1 + F_2 + F_3\)) × \(K_1\) × \(K_2\) × \(K_3\) × \(K_4\).

Source: After Hellyer\(^{10}\).
7.1 RELIABILITY OF MEASUREMENT SYSTEMS

7.1.7 Design and maintenance for reliability

Design for reliability

The following general principles should be observed.

**Element selection.** Only elements with well-established failure rate data/models should be used. Furthermore some technologies are inherently more reliable than others. Thus an inductive LVDT displacement sensor (Chapter 8) is inherently more reliable than a resistive potentiometer; the latter involves a contact sliding over a wire track, which will eventually become worn. A vortex flowmeter (Chapter 12) involves no moving parts and is therefore likely to be more reliable than a turbine flowmeter which incorporates a rotor assembly.

**Environment.** The environment in which the element is to be located should first be defined and the element should consist of components and elements which are capable of withstanding that environment. Thus the diaphragm of a differential pressure transmitter on a sulphuric acid duty should be made from a special alloy, e.g. Hastelloy C, which is resistant to corrosion.

**Minimum complexity.** We saw above that, for a series system, the system failure rate is the sum of the individual component/element failure rates. Thus the number of components/elements in the system should be the minimum required for the system to perform its function.

**Redundancy.** We also saw that the use of several identical elements/systems connected in parallel increases the reliability of the overall system. Redundancy should be considered in situations where either the complete system or certain elements of the system have too high a failure rate.

**Diversity.** In practice faults can occur which cause either more than one element in a given system, or a given element in each of several identical systems, to fail simultaneously. These are referred to as common mode failures and can be caused by incorrect design, defective materials and components, faults in the manufacturing process, or incorrect installation. One common example is an electronic system where several of the constituent circuits share a common electrical power supply; failure of the power supply causes all of the circuits to fail. This problem can be solved using diversity; here a given function is carried out by two systems in parallel, but each system is made up of different elements with different operating principles. One example is a temperature measurement system made up of two subsystems in parallel, one electronic and one pneumatic.

Maintenance

The mean down time, MDT, for a number of items of a repairable element has been defined as the mean time between the occurrence of the failure and the repaired element being put back into normal operation. It is important that MDT is as small as possible in order to minimise the financial loss caused by the element being out of action.
There are two main types of maintenance strategy used with measurement system elements. Breakdown maintenance simply involves repairing or replacing the element when it fails. Here MDT or mean repair time, $T_R$, is the sum of the times taken for a number of different activities. These include realisation that a fault has occurred, access to the equipment, fault diagnosis, assembly of repair equipment, components and personnel, active repair/replacement and finally checkout. Preventive maintenance is the servicing of equipment and/or replacement of components at regular fixed intervals; the corresponding maintenance frequency is $m$ times per year. Here MDT or mean maintenance time, $T_M$, is the sum of times for access, service/replacement and checkout activities and therefore should be significantly less than mean repair time with breakdown maintenance.

### 7.2 Choice of measurement systems

The methods to be used and problems involved in choosing the most appropriate measurement system for a given application can be illustrated by a specific example. The example used will be the choice of the best system to measure the volume flow rate of a clean liquid hydrocarbon, range 0 to 100 $\text{m}^3\text{h}^{-1}$, in a 0.15 m (6 inch) diameter pipe. The measured value of flow rate must be presented to the observer in the form of a continuous trend on a chart recorder. The first step is to draw up a specification for the required flow measurement system. This will be a list of all important parameters for the complete system such as measurement error, reliability and cost, each with a desired value or range of values. The first two columns of Table 7.4 are an example of such a ‘job specification’. As explained in Chapter 3, system measurement error in the steady state can be quantified in terms of the mean $\bar{E}$ and standard deviation $\sigma_E$ of the error probability distribution $p(E)$. These quantities depend on the imperfections, e.g. non-linearity and repeatability, of every element in the system. System failure rate $\lambda$ and repair time $T_R$ were defined in Section 7.1. Initial cost $C_I$ is the cost of purchase, delivery, installation and commissioning of the complete system. $C_R$ is the average cost of materials for each repair.

#### Table 7.4 Comparison table for selection of flow measurement system.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Job specification</th>
<th>System 1 Orifice plate</th>
<th>System 2 Vortex</th>
<th>System 3 Turbine</th>
<th>System 4 Electromagnetic</th>
</tr>
</thead>
<tbody>
<tr>
<td>Measurement error</td>
<td>$\bar{E} \leq 0.25$</td>
<td>0.2</td>
<td>0.1</td>
<td>0.03</td>
<td></td>
</tr>
<tr>
<td>(at 50 $\text{m}^3\text{h}^{-1}$)</td>
<td>$\sigma_E \leq 0.8$</td>
<td>0.7</td>
<td>0.3</td>
<td>0.1</td>
<td></td>
</tr>
<tr>
<td>Initial cost</td>
<td>$C_I \leq £4000$</td>
<td>3500</td>
<td>3000</td>
<td>4200</td>
<td>Not technically feasible</td>
</tr>
<tr>
<td>Annual failure rate</td>
<td>$\lambda \leq 2.0$</td>
<td>1.8</td>
<td>1.0</td>
<td>2.0</td>
<td></td>
</tr>
<tr>
<td></td>
<td>failures yr$^{-1}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average repair time</td>
<td>$T_R \leq 8$ h</td>
<td>6</td>
<td>5</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>Material repair cost</td>
<td>$C_R \leq £200$</td>
<td>100</td>
<td>100</td>
<td>300</td>
<td></td>
</tr>
</tbody>
</table>
In this example the choice could be between four competing systems based on the orifice plate, vortex, turbine and electromagnetic primary sensing elements. The principles and characteristics of all four elements are discussed in Chapter 12. The configuration of the four systems could be:

2. Vortex element – frequency to voltage converter – recorder.
3. Turbine element – frequency to voltage converter – recorder.

The next step is to decide whether all the competing systems are technically feasible. The electromagnetic device will not work with electrically non-conducting fluids such as hydrocarbons, so that system 4 is technically unsuitable. Systems 1, 2 and 3 are feasible and the specification for each competing system must then be written down to see whether it satisfies the job specification. Table 7.4 gives possible specifications for the orifice plate, vortex and turbine systems. This data is entirely fictitious and is given only to illustrate the problems of choice. In practice a prospective user may not have all the information in Table 7.4 at his disposal. The manufacturer will be able to give estimates of initial cost $C_i$ and the limits of measurement error, e.g. ±2% of full scale, for the orifice plate system at 50 m$^3$/h. He will not, however, be able to give values of mean error $\bar{E}$, failure rates and repair times. The last two quantities will depend on the environment of the user’s plants and the maintenance strategy used. This information may be available within the user’s company if adequate maintenance records have been kept. From Table 7.4 we see that the turbine flowmeter system 3 does not satisfy the job specification; both initial cost and material repair cost are outside the limits set. This would appear to rule out system 3, leaving 1 and 2. Both these systems satisfy the job specification but the vortex system 2 is cheaper, more accurate and more reliable than the orifice plate system 1. Thus, based on a straightforward comparison of job and system specification, the vortex measurement system 2 would appear to be the best choice for this application.

Under certain circumstances, however, the above conclusion could be entirely wrong. The turbine system is more expensive and less reliable than the vortex system, but is three times more accurate. We must now ask how much this increased accuracy is worth. Suppose the market value of the hydrocarbon is £100 m$^{-3}$. A measurement error of one standard deviation in the turbine system, where $\sigma_E = 0.1$ m$^3$/h, represents a potential cash loss of £10 per hour or approximately £80 000 per annum. A corresponding error in the vortex system, where $\sigma_E = 0.3$ m$^3$/h, represents an approximate potential loss of £240 000 per annum. The difference between these two figures far outweighs the extra initial and maintenance costs, so that the turbine system 3 is the best choice in this case. We can conclude, therefore, that in order to choose the correct system for a given application, the financial value of each parameter in the job specification must be taken into account. In a costing application of this type, a digital printout of flow rate is more suitable than an analogue trend record.

### 7.3 Total lifetime operating cost

The total lifetime operating costing (TLOC) of a measurement system is the total cost penalty, incurred by the user, during the lifetime of the system. The TLOC is given by
TLOC = initial cost of system (purchase, delivery, installation and commissioning) 
+ cost of failures and maintenance over lifetime of the system 
+ cost of measurement error over lifetime of the system [7.29] 
and therefore takes account of the financial value of each parameter in the job 
specification. The best system for a given application is then the one with minimum 
TLOC. This method also enables the user to decide whether a measurement system 
is necessary at all. If no system is installed, TLOC may still be very large because 
no measurement implies a large measurement error. A measurement system should 
be purchased if it produces a significant reduction in TLOC.

Using eqn [7.29], we can derive an algebraic expression for TLOC using the 
parameters listed in Table 7.4. The initial cost of the system is £C_I. If the system life-
time is T years and average failure rate \( \lambda \) faults yr \(^{-1} \), then the total number of faults 
is \( \lambda T \). Since the average repair time is \( T_R \) hours then the total ‘downtime’ due to repair 
is \( \lambda T T_R \) hours. The total lifetime cost of failures is the sum of the repair cost (materials and labour) and the process cost, i.e. the cost of lost production and efficiency 
while the measurement system is withdrawn for repair. If we define 

\[
\begin{align*}
\£C_R &= \text{average materials cost per repair} \\
\£C_L &= \text{repair labour cost per hour} \\
\£C_P &= \text{process cost per hour}
\end{align*}
\]
then the total repair cost is \( (C_R \lambda T + C_L T_R \lambda T) \) and the total process cost is \( C_P T_R \lambda T \), giving:

Total lifetime cost of failures = \( [C_R + (C_L + C_P) T_R] \lambda T \) \[7.30\] 
The above costs only apply to breakdown maintenance; many users also practise 
preventive maintenance in order to reduce failure rates. Suppose preventive mainten-
ance is carried out on a measurement system \( m \) times yr \(^{-1} \), the average maintenance 
time is \( T_M \) hours and the materials cost per service is \( £C_M \). The total number of 
services is \( mT \) and the total time taken for preventive maintenance is \( mT T_M \) hours. 
Usually preventive maintenance of measurement systems is carried out at a time when 
the process or plant itself is shut down for repair and maintenance. This means that 
no process costs are incurred during preventive maintenance, giving:

Total lifetime maintenance cost = \( (C_M + C_L T_M) mT \) \[7.31\] 
The last term in eqn [7.29] involves the total lifetime cost of measurement error. 
In order to evaluate this we first need to evaluate the cost penalty function \( C(E) \) 
(\£ yr \(^{-1} \)) associated with a given steady-state measurement system error \( E \). The form 
of \( C(E) \) depends on the economics of the process on which the measurement is being 
made. A good example is temperature measurement in a chemical reactor where a 
degradation reaction is taking place. Here two reactions occur simultaneously: 
the feedstock A is converted into a desired product B but B is also degraded to an 
undesired product C. The rates of both reactions increase sharply with temperature, 
the rate of B to C being more temperature sensitive than the rate of A to B.

There is an optimum temperature at which the yield of B is maximum. If the 
reactor is operated at either above or below this optimum temperature, then the yield 
of B is sharply reduced and a cost penalty is incurred. This situation will occur if 
there is a measurement system error \( E \) between measured and true values of reactor 
temperature: the system tells the operator that the reactor is at optimum temperature
when it is not. Figure 7.8(a) shows the form of the cost penalty function $C(E)$ in this case. We see that when $E = 0$, $C(E) = 0$ corresponding to optimum temperature; but $C(E)$ increases rapidly with positive and negative values of $E$ as the yield of $B$ decreases.

In Figure 7.8(b) $C(E)$ represents the cost penalty, due to imperfect flow measurement, incurred by a customer receiving a fluid by pipeline from a manufacturer. If $E$ is positive, the customer is charged for more fluid than he actually receives and so is penalised, i.e. $C(E)$ is positive. If $E$ is negative, the customer is charged for less fluid than was actually received and $C(E)$ is negative. Figure 7.8(c) refers to a non-critical liquid level measurement in a vessel. The vessel should be about half full, but plant problems will occur if the vessel is emptied or completely filled. Thus a cost penalty is incurred only if there is gross measurement error, i.e. the measurement system shows the vessel to be half full when almost empty.

We saw in Chapter 3 that the exact value of measurement error $E$, for a given measurement system at a given time, cannot be found. We can, however, find the probability that the system will have a certain error. This is quantified using a normal probability density function $p(E)$, with mean $\bar{E}$ and standard deviation $\sigma_E$ (Table 3.1). The probability of getting a measurement error between $\bar{E}$ and $\bar{E} + dE$ is $p(E) \, dE$; the corresponding cost penalty is $C(E)p(E) \, dE$ per year, or $TC(E)p(E) \, dE$ throughout the system lifetime. The total lifetime cost of measurement error is then found by integrating the above expression over all possible values of $E$, i.e.

$$\text{Total lifetime cost of measurement error} = T \int_{-\infty}^{\infty} C(E)p(E) \, dE \quad [7.32]$$

The integral has a finite value, because the value of $p(E)$ becomes negligible for $|E|$ greater than 3 or 4 standard deviations; it can be evaluated numerically using values of the normalised Gaussian distribution.

From eqns [7.29]–[7.32] we have

$$T\text{LOC} = C_I + [C_k + (C_L + C_M) T_k] \lambda T + (C_M + C_H T_H) mT$$

$$+ T \int_{-\infty}^{\infty} C(E)p(E) \, dE \quad [7.33]$$
The relative importance of the terms in the above equation will depend on the application. In the chemical reactor \( C(E) \) and \( C_p \) will be the major factors so that accuracy and reliability will be far more important than initial cost. In the tank level application, measurement error is unimportant and minimum TLOC will be obtained by the best trade-off between initial cost, reliability, and maintainability.

Conclusion

The first section of this chapter discussed the reliability of measurement systems. The fundamental principles and practical definitions of reliability were first explained and the relationship between reliability and instantaneous failure rate was derived. The typical variation in instantaneous failure rate throughout the lifetime of an element was then discussed and the reliability of series and parallel systems examined. The section concluded by looking at failure rate data and models and general strategies in design and maintenance for reliability.

The second section dealt with the problem of choice of measurement systems approached by comparing the job specification with those of the competing systems. This method does not take account of the financial value of each item in the specification. A better method, discussed in the final section, is to choose the systems with minimum lifetime operating cost.

References


Problems

7.1 A batch of 100 identical thermocouples were tested over a 12-week period. Twenty failures were recorded and the corresponding down times in hours were as follows:

5, 6, 7, 8, 4, 7, 8, 10, 5, 4, 8, 5, 4, 5, 6, 5, 4, 9, 8, 6
Calculate:

(a) Mean down time
(b) Mean time between failures
(e) Mean failure rate
(d) Availability.

A flow measurement system consists of an orifice plate \( \lambda = 0.75 \), differential pressure transmitter \( \lambda = 1.0 \), square root extractor \( \lambda = 0.1 \) and recorder \( \lambda = 0.1 \). Calculate the probability of losing the flow measurement after 0.5 year for the following:

(a) A single flow measurement system
(b) Three identical flow measurement systems in parallel
(c) A system with three orifice plates, three differential pressure transmitters and a middle value selector relay \( \lambda = 0.1 \). The selected transmitter output is passed to a single square root extractor and recorder.

Annual failure rate data are given in brackets; assume that all systems were initially checked and found to be working correctly.

Use the data given in Table Prob. 3 to decide which level measurement system should be purchased. Assume a breakdown maintenance only strategy is practised, each system has the same measurement error, and there is a 10-year total lifetime.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>System 1</th>
<th>System 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial cost</td>
<td>£</td>
<td>1000</td>
</tr>
<tr>
<td>Materials cost per repair</td>
<td>£</td>
<td>20</td>
</tr>
<tr>
<td>Labour cost per hour</td>
<td>£</td>
<td>10</td>
</tr>
<tr>
<td>Process cost per hour</td>
<td>£</td>
<td>100</td>
</tr>
<tr>
<td>Repair time</td>
<td>h</td>
<td>8</td>
</tr>
<tr>
<td>Annual failure rate</td>
<td>yr(^{-1})</td>
<td>2.0</td>
</tr>
</tbody>
</table>