A new element for analyzing large deformation of thin Naghdi shell model. Part 1: Elastic

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**A B S T R A C T**

One of the best approaches for modeling large deformation of shells is the Cosserat surface. However, the finite-element implementation of this model suffers from membrane and shear locking, especially for very thin shells. The basic assumption of this theory is that the mid-surface of the shell is regarded as a Cosserat surface with one inextensible director. In this paper, it is shown that by constraining the director vector normal to the mid-surface, besides very good and accurate results, shear locking is also eliminated. This constraint is in fact a limiting analysis of the Cosserat theory in which Kirchhoff’s hypothesis is enforced. Numerical solution is performed using nine-node isoparametric element. The principal of virtual work is used to obtain the weak form of the governing differential equations and the material and geometric stiffness matrices are derived through a linearization process. The validity and the accuracy of the method are illustrated by numerical examples.

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1. Introduction

Interesting in the large deformation response of plates and shells grew as a direct consequence of the use of thinner sections as a mean of achieving material economy. Analytical and numerical analysis of shell structures have been carried out for a very long time. Yang et al. [1,2] presented a survey of the development of the shell theory starting from the origination of the curved shell finite elements in the mid-1960s.

In finite-element approximation of shell models two distinct classes of shell elements emerge [3]:

1. Degenerated shell elements based on three dimensional continuum theory.
2. Shell elements founded on the classical shell theory.

The degenerate solid approach is described firstly in the paper of Ahmad et al. [4]. The works of Ramm [5], Bathe and Dvorkin [6], and Liu et al. [7] among many others, are representative of such an approach.

The second afore-mentioned methodology represents a return to the origins of classical shell theory, which has its modern point of departure in the pioneering work of Cosserat that further elaborated upon by a number of authors (Naghdi [8], Antman [9], Simo et al. [10–12]). The basic assumption of this theory is that the mid-surface of the shell is regarded as a Cosserat surface with one inextensible director. Typically this approach yields an exact analytical definition of the initial geometry of the shells and the stress and strain are represented in a curvilinear coordinate system.

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In the many of above-mentioned works, shear deformations in the direction of the thickness are taken into account. In the finite-element models, taking the transverse shear strain into account, for thin shells, causes an undesirable numerical effect, the so-called shear locking phenomenon. Consequently, as the thickness of the plate and shell becomes extremely thin, the shear strain energy predicted by the finite-element analysis can vary unreasonably, even though the average value of the shear strain over the area approaches zero.

Membrane locking phenomenon occurs due to different orders of magnitude for the membrane and bending strains when the shell is thin. In Cosserat theory, since director and displacement shape functions are different, membrane locking may also arise from inconsistency between the director and displacement shape functions. Therefore, the matching field approach and high-order shape functions need to be used to avoid this problem [3].

According to the well known Kirchhoff’s hypothesis, straight lines perpendicular to the mid-surface remain perpendicular to the deformed mid-surface. This hypothesis yields satisfactory results only when the thickness approaches zero and the deformation is not large. This hypothesis can lead to numerical difficulties, if used for large deformations. However, it will be shown in this paper that by employing Cosserat’s surface and constraining the director vector to be normal to the mid-surface, very good results can be obtained for large deformation of thin plates and shells. This constraint is in fact a limiting analysis of the Cosserat theory in which Kirchhoff’s hypothesis is enforced, hence the shear strains in the direction of the shell’s thickness are ignored and no shear locking can occur. Also by employing this idea, it does not require interpolating the director vector separately. Therefore, there is no concern about consistency between displacement and director shape functions and the speed of solution increases because only displacement field needs to be interpolated. In the other word advantages of Cosserat surface and Kirchhoff theory are collected together.

Numerical solution is performed using nine-node isoparametric element. The principal of virtual work is used to obtain the weak form of the governing differential equations and the material and geometric stiffness matrices are derived through a linearization process. The validity and the accuracy of the method are illustrated by numerical examples.

The outline of this paper is as follows. In Section 2 the theory is explained. In Section 3 the problem is proposed for nine-node isoparametric element and the stiffness matrices are derived. In Section 4 several numerical examples are presented and the results are compared with literature. Finally, conclusions are drawn in Section 5.

2. Theory

Fig. 1 shows geometry of a three dimensional shell with a mid-surface \( M \). On the mid-surface the convective coordinate system \( \theta^1, \theta^2 \) is considered which has the base vectors \( \mathbf{a}_1, \mathbf{a}_2 \) and \( \mathbf{a}_3 \) which is orthogonal to \( \mathbf{a}_1 \) and \( \mathbf{a}_2 \). The position vector of any point with respect to \( O \) is [13]:

\[
\mathbf{R} = r(\theta^1, \theta^2) + \theta^3 \mathbf{a}_3. \tag{1}
\]

Fig. 2 shows the mid-surface of an arbitrary shell in equilibrium states before and after deformation \((t = 0, t, \text{ respectively})\). In this figure \( x, y \) and \( z \) represent reference Cartesian coordinate system and \( \theta^1, \theta^2 \) are the convective coordinate system.

As shown in Fig. 2, the base vectors of the convective coordinate system in initial configuration are denoted by \( \mathbf{a}_i \). Similarly, \( \mathbf{a}_i \) denotes the base vectors of the convective coordinate system at time “t”. It should be noted that the director vector is constrained to be perpendicular to the mid surface at each time; hence \( \mathbf{a}_3 = \mathbf{d} \). The position vector of a material point, which is a function of \( \theta^1 \) and \( \theta^2 \), becomes:

![Fig. 1. Geometry of a three dimensional shell.](image-url)
Therefore the base vectors can be written as:

\[
\begin{align*}
\mathbf{t}_{aa} &= \mathbf{t}_r, \\
\mathbf{a}_3 &= \frac{\mathbf{t}_{aa} \cdot \mathbf{a}_2}{|\mathbf{t}_{aa}|}.
\end{align*}
\]

Components of the first and the second fundamental tensors of the surface are:

\[
\begin{align*}
\mathbf{t}_{aa}^{ab} &= \frac{1}{2} (\mathbf{t}_r \cdot \mathbf{r}_x - \mathbf{x}_a \cdot \mathbf{x}_b), \\
\mathbf{t}_{ab}^{0} &= \mathbf{t}_{0}^{ab} - 0_{ab}.
\end{align*}
\]

In the above relation, \( \mathbf{t}_{ab}^{0} = \mathbf{t}_{0}^{ab} - 0_{ab} \) is the curvature of the surface where \( \mathbf{t}_{ab} \) is determinant of \( \mathbf{t}_{ab}^{0} \).

In the above relations, the lower left subscript in the strain components denotes the reference configuration. For convenience in numerical solution, let computational strain vector, \( \varepsilon_0 \); accordingly:

\[
\varepsilon_0 = \begin{bmatrix}
\varepsilon_{11}^0, \varepsilon_{22}^0, \varepsilon_{12}^0, \rho_{13}^0, \rho_{23}^0, \rho_{12}^0
\end{bmatrix}^T,
\]

where in the above relation, "\(^T\)" indicates transpose symbol.
In Cosserat theory, membrane and bending stresses are defined in terms of stress resultants in the direction of thickness [13]. Figs. 3 shows the effective Cauchy stresses at a material point of the deformed configuration and also the effective symmetric Piola stresses corresponding to the Cauchy stresses.

In Fig. 3, \( \mathbf{n}^m \) and \( \mathbf{m}^m \) are membrane stresses and bending moments per unit length in the deformed configuration or Cauchy stresses. The invariant forms for these stresses are:

\[
\mathbf{n}^m = \mathbf{n}^{\text{de}} \otimes \mathbf{a}_i, \\
\mathbf{m}^m = \mathbf{m}^{\text{de}} \otimes \mathbf{a}_i.
\]  

Similarly, \( \mathbf{n}^{\text{de}} \) and \( \mathbf{m}^{\text{de}} \) are membrane stresses and bending moments per unit length in the reference configuration or symmetric Piola stresses. The invariant forms for these stresses are:

\[
\mathbf{n}^{\text{de}} = \mathbf{n}^{\text{sym}} \otimes \mathbf{a}_i, \\
\mathbf{m}^{\text{de}} = \mathbf{m}^{\text{sym}} \otimes \mathbf{a}_i.
\]

These stresses are related to each other according to following relations:

\[
\begin{align*}
\mathbf{n}^m &= \mathbf{J} \mathbf{n}^{\text{de}}, \\
\mathbf{m}^m &= \mathbf{J} \mathbf{m}^{\text{de}}.
\end{align*}
\]

where \( \mathbf{J} = \det(\mathbf{F}) \) is the transformation Jacobian, namely, the ratio between the element surface after and before deformation. For a convective coordinate, the Jacobian term can be written as:

\[
\mathbf{J} = \det(\mathbf{F}) = \mathbf{i} \mathbf{a}_i \otimes \mathbf{0} a'.
\]

In this formulation, \( \mathbf{F} \) is the deformation gradient tensor.

For convenience in numerical solution, let the computational stress vector, \( \mathbf{\sigma} \) according to:

\[
\mathbf{\sigma} = (\mathbf{n}^{\text{sy}m} \mathbf{e}_x e_x + \mathbf{m}^{\text{sy}m} \mathbf{e}_y e_y) dS = \mathbf{t} \mathbf{R}_{\text{ext}}.
\]

Else it can be written as:

\[
\int_S (\mathbf{n}^{\text{de}} \delta e_x e_x + \mathbf{m}^{\text{de}} \delta e_y e_y) dS = \mathbf{t} \mathbf{R}_{\text{ext}}.
\]

where \( \mathbf{t} \mathbf{R}_{\text{ext}} \) is the virtual work of the external forces and can be written in terms of boundary tractions according to the following relation:

\[
\int_S (\mathbf{n} \cdot \mathbf{\delta U} + \mathbf{m} \cdot \mathbf{\delta d}) dS + \int_{\partial S} (\mathbf{n} \cdot \mathbf{\delta U} + \mathbf{m} \cdot \mathbf{\delta d}) d\partial S = \mathbf{t} \mathbf{R}_{\text{ext}}.
\]

In the above relation, \( \mathbf{n} \) and \( \mathbf{m} \) are distributed force and moment vectors at time \( \mathbf{t} \) per unit length of boundary, \( \partial S \), respectively.

Computational stress and strain vectors, \( \mathbf{\sigma} \) and \( \mathbf{e} \) are defined as:

\[
\mathbf{\sigma} = (\mathbf{n}^{11}, \mathbf{n}^{22}, \mathbf{n}^{12}, \mathbf{m}^{11}, \mathbf{m}^{22}, \mathbf{m}^{12})^T, \\
\mathbf{e} = (\mathbf{e}^{11}, \mathbf{e}^{22}, \mathbf{e}^{12}, \mathbf{e}^{11}, \mathbf{e}^{22}, \mathbf{e}^{12})^T.
\]

Therefore Eq. (12) can be written as:

\[
\int_S \mathbf{\sigma} \mathbf{\delta e} dS = \mathbf{t} \mathbf{R}_{\text{ext}}.
\]

For a linear elastic material the stress can be written as:

\[
\mathbf{\sigma} = \mathbf{C} \mathbf{e},
\]

where

\[
\mathbf{C} = \begin{bmatrix} C_{11} & 0 \\ 0 & C_{12} \end{bmatrix}
\]
Thus the first fundamental form of the surface in reference configuration can be achieved.

From Eq. (5) variation of the strain components are given by [14]:

\[
\begin{align*}
\delta \epsilon_{\alpha\beta} &= \frac{1}{2} \left( \delta \epsilon_{\alpha\gamma} \Gamma_{\gamma\beta}^{\rho} + \delta \epsilon_{\beta\gamma} \Gamma_{\gamma\alpha}^{\rho} + \delta \epsilon_{\alpha\beta} \right), \\
\delta \epsilon_{\alpha\beta} &= \frac{1}{2} \left( \delta \epsilon_{\alpha\gamma} \Gamma_{\gamma\beta}^{\rho} + \delta \epsilon_{\beta\gamma} \Gamma_{\gamma\alpha}^{\rho} + \delta \epsilon_{\alpha\beta} \right),
\end{align*}
\]

where \( a \) is determinant of \( a_{\alpha\beta} \) and:

\[
\delta \alpha = 2a_{22} \delta \mathbf{r}_1 + 2a_{12} \delta \mathbf{r}_2 + 2a_{11} \delta \mathbf{r}_1 \times \delta \mathbf{r}_2.
\]

### 3. Numerical solution

An arbitrary element with line supports in Cartesian coordinates may be mapped into a standard nine-node isoparametric element. For simplicity natural coordinates \( \xi, \eta \) are taken to be convective coordinate (see Fig. 4).

\[
\begin{align*}
0\mathbf{x} &= N_1^0 \mathbf{x}_1 + N_2^0 \mathbf{x}_2 + \ldots + N_9^0 \mathbf{x}_9, \\
0\mathbf{y} &= N_1^0 \mathbf{y}_1 + N_2^0 \mathbf{y}_2 + \ldots + N_9^0 \mathbf{y}_9,
\end{align*}
\]

where \( N_i \) is \( i \)th shape function of the nine-node isoparametric element.

So the base vectors in reference system are:

\[
\begin{align*}
0a_1 &= \begin{bmatrix}
N_{1,1}^0 \mathbf{x}_1 + N_{2,1}^0 \mathbf{x}_2 + \ldots + N_{9,1}^0 \mathbf{x}_9 \\
N_{1,2}^0 \mathbf{y}_1 + N_{2,2}^0 \mathbf{y}_2 + \ldots + N_{9,2}^0 \mathbf{y}_9
\end{bmatrix}, \\
and \quad 0a_2 &= \begin{bmatrix}
N_{1,2}^0 \mathbf{x}_1 + N_{2,2}^0 \mathbf{x}_2 + \ldots + N_{9,2}^0 \mathbf{x}_9 \\
N_{1,2}^0 \mathbf{y}_1 + N_{2,2}^0 \mathbf{y}_2 + \ldots + N_{9,2}^0 \mathbf{y}_9
\end{bmatrix}.
\end{align*}
\]

\[
\text{Thus the first fundamental form of the surface in reference configuration can be achieved.}
\]

Let \( U \) be the displacement vector, then the position of any point can be written as:

\[
\begin{align*}
\Gamma(\theta^1, \theta^2) &= \Gamma(\theta^1, \theta^2) + U(\theta^1, \theta^2), \\
\Gamma &= \begin{bmatrix}
u \\
w
\end{bmatrix}.
\end{align*}
\]

\[
\text{Fig. 4. Nine-node isoparametric element.}
\]
In-plane displacements, \( u \) and \( v \), are interpolated as follows:
\[
\begin{align*}
  u &= N_1 u_1 + N_2 u_2 + N_3 u_3 + \cdots + N_9 u_9, \\
  v &= N_1 v_1 + N_2 v_2 + N_3 v_3 + \cdots + N_9 v_9,
\end{align*}
\]
(27)
where \( N_i \) is \( i \)th shape function of the nine-node isoparametric element.

For out of plane displacements the Hermitian shape functions are employed for the four corners as follows:
\[
\begin{align*}
  w &= \sum_{i=1}^{4} \left( N_i^1 w_i + N_i^2 \frac{\partial w}{\partial \eta} + N_i^3 \frac{\partial w}{\partial \zeta} + N_i^4 \frac{\partial^2 w}{\partial \eta \partial \zeta} \right),
\end{align*}
\]
(28)
where \( N_i^1 - N_i^4 \) are Hermitian functions. For example
\[
\begin{align*}
  N_i^1 &= H01(\zeta)H01(\eta), \\
  N_i^2 &= H01(\zeta)H11(\eta) \quad H01(\zeta) = 1/(2 - 3\zeta + \zeta^2), \\
  N_i^3 &= H11(\zeta)H01(\eta) \quad H11(\zeta) = 1/(4 - 3\zeta^2 + \zeta^3), \\
  N_i^4 &= H11(\zeta)H11(\eta).
\end{align*}
\]
(29)

These shape functions satisfy consistency of displacements and to some extent rotations.

Let us assume:
\[
\mathbf{U} = \mathbf{N} \hat{\mathbf{U}}.
\]
(30)

In the above relation, \( \mathbf{N} \) is shape function matrix and can be written as follows:
\[
\mathbf{N} = \begin{bmatrix} \mathbf{Nu} & 0 & 0 \\ 0 & \mathbf{Nv} & 0 \\ 0 & 0 & \mathbf{Nw} \end{bmatrix},
\]
(31)
where
\[
\mathbf{Nu} = [N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_7 \quad N_9],
\]
(32)
\[
\mathbf{Nv} = [N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_7 \quad N_9],
\]
(33)
\[
\mathbf{Nw} = \left[ H_{11}^{01} H_{11}^{01} H_{11}^{01} H_{11}^{01} H_{11}^{01} H_{11}^{01} H_{11}^{01} \cdots H_{14}^{01} H_{14}^{01} H_{14}^{01} H_{14}^{01} \right].
\]
(34)

By this definition from Eq. (19) we have:
\[
\begin{align*}
  \delta \mathbf{e}_{xy} &= \left( \mathbf{r}_{1x} \mathbf{N}_{xy} + \mathbf{r}_{1y} \mathbf{N}_{xy} \right) \delta \hat{\mathbf{U}} = \mathbf{E}_{xy} \delta \hat{\mathbf{U}}, \\
  \delta \mathbf{e}_{xy} &= \left( \frac{1}{2 \sqrt{a}} \mathbf{Q}_{xy} - \frac{q_{xy}}{2 \sqrt{a}} \mathbf{A} \right) \delta \hat{\mathbf{U}} = \mathbf{K}_{xy} \delta \hat{\mathbf{U}}, \\
  \mathbf{Q}_{xy} &= \left( \mathbf{r}_{1x} \times \mathbf{r}_{2x} \right) \mathbf{N}_{xy} + \left( \mathbf{r}_{1y} \times \mathbf{r}_{2y} \right) \mathbf{N}_{xy}, \\
  q_{xy} &= \mathbf{r}_{1x} \times \mathbf{r}_{2x}.
\end{align*}
\]
(35)

The symbol “\( T \)” is transpose symbol.

Also from Eq. (3) we have:
\[
\begin{align*}
  \delta \mathbf{d} &= \delta \mathbf{a}_{3x} = \left( \frac{1}{\sqrt{a}} \mathbf{b}_{12} - \frac{1}{2 \sqrt{a}} \mathbf{A} \left( \mathbf{r}_{1x} \times \mathbf{r}_{2x} \right) \right) \delta \hat{\mathbf{U}} = \mathbf{D} \delta \hat{\mathbf{U}}.
\end{align*}
\]
(39)

Such that
\[
\mathbf{b}_{12} = \begin{bmatrix} 0 & \mathbf{t}z_1 N_1^2 - \mathbf{t}z_2 N_2^2 & \mathbf{t}y_1 N_1^2 - \mathbf{t}y_2 N_2^2 \\
\mathbf{t}z_2 N_1^2 - \mathbf{t}z_1 N_2^2 & 0 & \mathbf{t}x_1 N_1^2 - \mathbf{t}x_2 N_2^2 \\
\mathbf{t}y_2 N_1^2 - \mathbf{t}y_1 N_2^2 & \mathbf{t}x_2 N_1^2 - \mathbf{t}x_1 N_2^2 & 0 \end{bmatrix}
\]
(40)
and
\[
\mathbf{A} = 2a_1 a_2 r_{1x}^2 \cdot N_2 + 2a_1 a_2 r_{1y}^2 \cdot N_1 - 2a_1^2 r_{1x}^2 \cdot N_1 - 2a_1 a_2 r_{1y}^2 \cdot N_2.
\]
(41)

where “\( a \)” is determinant of \( a_{xy} \).

By replacing \( \mathbf{U} = \mathbf{N} \hat{\mathbf{U}} \), Eq. (13) can be written as:
\[
\int_{S_5} \left( \mathbf{E}_{xy} \cdot \mathbf{e}_{xy} + \mathbf{K}_{xy} \cdot \mathbf{e}_{xy} \right) dS = \int_{S} \left( \mathbf{B}^T \mathbf{N} + \mathbf{B}^T \mathbf{D} \right) dS + \int_{S} \left( \mathbf{B}^T \mathbf{N} + \mathbf{B}^T \mathbf{D} \right) dS.
\]
(42)
For simplicity, the curvilinear convective coordinate system \((\theta^1, \theta^2)\) can be mapped to a local Cartesian system \([11]\). Define a local Cartesian system \(\{x^1, x^2, x^3\}\) with base vectors \(\{n_1, n_2, n_3\}\) by means of the orthogonal transformation:

\[
\Lambda' = (E_1, E_2, E_3) = E_3 \times (E_3 \times \mathbf{n}) = \frac{1}{1 + E_3 \cdot \mathbf{n}} (E_3 \times \mathbf{n}) \otimes (E_3 \times \mathbf{n}),
\]

where \(\mathbf{n} = a^3 = \frac{r_2 \times r_1}{|r_2 \times r_1|}\) is the normal to the mid-surface and \(E_1, E_2\) and \(E_3\) are the base vectors of the reference Cartesian coordinate:

\[
E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
E_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},
E_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.
\]

Also, \([E_3 \times \mathbf{n}]\) is skew-symmetric tensor corresponding to \(E_3 \times \mathbf{n}\) vector.

Observe that \(\Lambda'\) maps \(E_1 \rightarrow \mathbf{n} = \Lambda'E_3\) without drilling and \(\mathbf{r}_x \cdot \mathbf{n} = 0\) and \(\mathbf{n}_a = \Lambda'E_a\) such that \(\mathbf{n}_a \cdot \mathbf{n}_b = \delta_{ab}\), where \(\delta_{ab}\) is the Kronecker delta function.

Also, it can be seen that at time “\(t\)”: \(\frac{\partial \mathbf{a}}{\partial \theta^2} = \mathbf{n}_a \mathbf{r}_x\) and \(\mathbf{r}_x = \frac{\partial \mathbf{a}}{\partial \theta^2} \mathbf{n}_a\), \(\quad a = 1, 2\).

Therefore in the local Cartesian system \(\frac{\partial \mathbf{a}}{\partial \theta^2} = \delta_{ab}\).

Eq. (42) at time \(t\) is non-linear in term of \(\mathbf{U}\) and should be linearized for the numerical analyses.

For this purpose, the Newton Raphson method is employed as follows:

Rename:

\[
\int_{\Omega} \left( \int_{T} \mathbf{E}_{s}\mathbf{B}_{s} \right) d\mathbf{S} = \mathbf{K}_{m} \mathbf{U}_{t} + \mathbf{F}_{s} - \mathbf{F}_{r}
\]

It is assumed that \(\mathbf{U}_{t}^{i}\) is known, where “\(i\)” is the iteration number; then:

\[
\mathbf{U}_{t}^{i+1} = \mathbf{U}_{t}^{i} + \Delta \mathbf{U}.
\]

By linearization of the function, \(\mathbf{F}(\mathbf{U})\); \(\Delta \mathbf{U}\) becomes:

\[
\Delta \mathbf{U} = (\mathbf{K}_{m} + \mathbf{K}_{c})^{-1}(\mathbf{F}_{s} - \mathbf{F}_{r}),
\]

where \(\mathbf{K}_{m}\) is material stiffness matrix and can be computed as below:

\[
\mathbf{K}_{m} = \int_{\Omega} \mathbf{B}^{T} \mathbf{C}^{m} \mathbf{B}^{m} d\mathbf{S}.
\]

In the above relation, \(\mathbf{B}\) is:

\[
\begin{bmatrix}
E_{11} \\
E_{22} \\
E_{12} \\
K_{11} \\
K_{12} \\
K_{22}
\end{bmatrix},
\]

where vectors \(E_{ij}\) and \(K_{ij}\) are determined through Eqs. (35) and (36), respectively.

Also \(\mathbf{K}_{c} = \int_{\partial \Omega} \sigma_{\theta^2} \frac{\partial \mathbf{a}}{\partial \theta^2} d\mathbf{S}\), the geometric stiffness matrix, can be computed as:

\[
\mathbf{K}_{c} = \int_{\Omega} \left( \int_{T} \mathbf{E}_{s}\mathbf{B}_{s} \right) d\mathbf{S}.
\]

\(\mathbf{E}_{s}\) and \(\mathbf{K}_{s}\) are given in Appendix A.

Also

\[
\mathbf{F}_{r} = \left( \int_{\Omega} \left( \mathbf{E}_{s}\mathbf{B}_{s} \right) \right) \mathbf{U}_{t},
\]

\[
\mathbf{F}_{s} = \int_{\Omega} \left( \mathbf{E}_{s}\mathbf{B}_{s} \right) d\mathbf{S} + \int_{\partial \Omega} \left( \mathbf{E}_{s}\mathbf{B}_{s} \right) d\mathbf{S}.
\]

Therefore the algorithm of solution can be summarized as below:

(1) Consider interpolation matrix, \(\mathbf{N}\), from Eq. (31).

(2) For \(i = 0\), let \(\mathbf{U}_{t}^{i} = 0\) and \(\Delta \mathbf{U} = 0\).
Compute $K_M$ and $K_G$ and $F_{ext}$ and $F_{res}$ from Eqs. (49), (51), (53) and (52).

(4) Compute $\Delta U$ from Eq. (48).

(5) Check for convergence, if $\text{norm}(\Delta \mathbf{U}) < \text{tolerance}$ exit else, let $\mathbf{U}_{i+1} = \mathbf{U}_i + \Delta \mathbf{U}$ and $i = i + 1$ then go to 3.

4. Numerical examples

In this section, the presented method is tested with some numerical examples. The main advantage of this method is that the convergence rate is very good in comparison with Cosserat theory, because in the present method, only the displacement field is interpolated and interpolation of director vector is not required. As discussed in Section 1, shear locking is eliminated as will be shown in the examples.

4.1. Scordellis–Lo barrel vault

The performance of the present method is evaluated on a standard test problem of a barrel vault roof shown in Fig. 5. The shell is loaded by own uniform vertical gravity load. It is supported by rigid diaphragms along the curved ends, which allow displacement in the axial direction and rotation about the tangent to the shell boundary, but is free along the straight edges. The following parameters are used: length $L = 50$ ft, radius $R = 25$ ft, thickness $h = 0.25$ ft and the span angle of the section is $\phi = 80^\circ$. The material properties are $E = 4.32 \times 10^7$ and $v = 0$ also the load is 90 lb/ft$^2$. This problem is extremely useful for determining the ability of the shell formulation to accurately solve complex states of membrane strain. Using symmetry, only one-quarter of panel needs to be modeled. There is a convergent numerical solution of magnitude $0.3024$ ft for the vertical deflection at point A, which is used to normalize the result. This value is reported by Scordellis [15]. The results are shown in the Fig. 5. It can be seen that with the $15 \times 15$ mesh, for one-quarter of the shell, the answer converges to 100%. Converging to exact answer indicates that omitting transverse shear strains does not cause significant error. As shown in literature [16], the percentage of shear energy is negligible. So by presented method in addition eliminating shear locking very good answers can be achieved.

4.2. Pinched cylinder with end diaphragms

This problem involves a thin cylindrical shell loaded by two centrally located and diametrically opposing concentrated forces as shown in Fig. 6. The ends of the cylinder are supported by rigid diaphragms. The length of the cylinder is $L = 600$, the radius is $R = 300$ and the thickness is $h = 3$. The material properties are $E = 3 \times 10^6$ and $v = 0.3$.

This problem is one of the most critical tests for both inextensional bending and complex membrane states of stress. Only one octant of the cylinder needs to be modeled because of symmetry conditions. There is a convergent numerical solution of $1.8248e-5$ for the radial displacement at the loaded points, which was used to normalize the results in Fig. 6. As shown in the figure, the rate of convergence is suitable.

4.3. A simply supported plate with concentrated load

In this example the deformation of a rectangular plate, under a central concentrated load is studied. Fig. 7 shows the deflection of central point of the plate. The following parameters are used: $h/a = 0.01$ and $v = 1/3$.

These results are derived for $14 \times 14$ mesh for one-quarter of the model. As shown in the figure, there is very good agreement with the results derived from Cosserat theory [17]. In Cosserat theory shear strains across the thickness are taken into account.
4.4. Buckling of a square plate

In this example, buckling of a square plate under single axes pressure is studied. Fig. 8 shows the deflection of central point of the plate. These results are derived for 15 x 15 mesh for one-quarter of the model.

4.5. A simply supported trapezoidal plate under uniform lateral load

In this example deformation of a trapezoidal plate under uniform lateral load is studied. Fig. 9 shows the deflection of central point of the plate.
As seen in the figure results are very close to Cosserat theory considered shear strains across the thickness.

The Cosserat theory cannot converge for very thin shells but by the current method, very thin shells can be highly deformed without shear locking. This point has been shown in Fig. 10. In this figure $a = 1 \text{ m}$, $b = 0.6$, $E = 2 \times 10^{11} \text{ N/m}^2$, $v = 0.3$, and $\frac{b^4}{a^4} \times 10^{-8} = 0.0024$.

Fig. 10 shows that the deformed shape of the plate can be achieved even for very thin shells.

5. Conclusion

A new non-linear method based on Cosserat theory, with constrained director, has been presented for large deformation of thin shells. The material and geometry stiffness matrices for finite-element solution have been derived. For nine-node isoparametric element, the theory has been solved. The advantage of this method is that the problem of shear locking has been eliminated also interpolation of the director vector is not required; hence speed of solution is very good. The method is computationally efficient and the numerical results exhibited a good agreement with the known values in the literature.
Appendix A

\[ K_G = \int_{S_0} \left( \frac{1}{2} m^{xy} E_{xy} + \frac{1}{2} m^{yy} K_{xy} \right) d^2 S, \tag{A.1} \]

\[ E_{xy} = \frac{1}{2} \left( N_y N_y^T + N_y^T N_y \right), \tag{A.2} \]

\[ K_{xy} = \left( \frac{3}{4} \frac{q_{xy}}{\sqrt{\langle q \rangle}^3} \right) A^T A - \left( \frac{1}{2} \frac{q_{xy}}{\sqrt{\langle q \rangle}^3} \right) A^T Q_{xy} - \left( \frac{1}{2} \frac{q_{xy}}{\sqrt{\langle q \rangle}^3} \right) Q_{xy}^T A + \frac{1}{\sqrt{\langle q \rangle}} Q_{xy}. \tag{A.3} \]

\[ A = 4N_y^T r_1^T r_1 N_1 + 4N_y^T r_1^T r_2 N_2 - 2N_y^T r_1^T r_1 N_2 - 2N_y^T r_1^T r_1 N_1 - 2N_y^T r_1^T r_1 N_2 - 2N_y^T r_1^T r_2 N_1. \tag{A.4} \]

\[ Q_{xy} = B_y N_y^T + B_{xy} N_1 + B_{xy} N_2. \tag{A.5} \]

\[ B_{xy} = \begin{bmatrix}
0 & i z_y N_{x y}^1 - i z_x N_{y x}^1 & i y_{y x} N_{x y}^1 - i y_{x y} N_{x x}^1 \\
i y_{x y} N_{x x}^1 - i y_{y x} N_{x y}^1 & 0 & i x_{x y} N_{x y}^1 - i x_{y x} N_{x x}^1 \\
0 & i z_x N_{y x}^1 - i z_y N_{x y}^1 & 0 \\
i y_{y x} N_{x x}^1 - i y_{x y} N_{x y}^1 & 0 & i x_{x y} N_{x y}^1 - i x_{y x} N_{x x}^1 \\
i x_{y x} N_{x x}^1 - i x_{x y} N_{x y}^1 & 0 & 0 \\
i y_{x y} N_{x x}^1 - i y_{y x} N_{x y}^1 & 0 & 0
\end{bmatrix}. \tag{A.6} \]

\[ B_{xy} = \begin{bmatrix}
0 & i z_y N_{x y}^2 - i z_x N_{y x}^2 & i y_{y x} N_{x y}^2 - i y_{x y} N_{x x}^2 \\
i y_{x y} N_{x x}^2 - i y_{y x} N_{x y}^2 & 0 & i x_{x y} N_{x y}^2 - i x_{y x} N_{x x}^2 \\
i x_{y x} N_{x x}^2 - i x_{x y} N_{x y}^2 & 0 & 0
\end{bmatrix}. \tag{A.7} \]

References