The science of thermodynamics deals with the amount of heat transfer as a system undergoes a process from one equilibrium state to another, and makes no reference to how long the process will take. But in engineering, we are often interested in the rate of heat transfer, which is the topic of the science of heat transfer.

We start this chapter with a review of the fundamental concepts of thermodynamics that form the framework for heat transfer. We first present the relation of heat to other forms of energy and review the first law of thermodynamics. We then present the three basic mechanisms of heat transfer, which are conduction, convection, and radiation, and discuss thermal conductivity. Conduction is the transfer of energy from the more energetic particles of a substance to the adjacent, less energetic ones as a result of interactions between the particles. Convection is the mode of heat transfer between a solid surface and the adjacent liquid or gas that is in motion, and it involves the combined effects of conduction and fluid motion. Radiation is the energy emitted by matter in the form of electromagnetic waves (or photons) as a result of the changes in the electronic configurations of the atoms or molecules. We close this chapter with a discussion of simultaneous heat transfer.
We all know from experience that a cold canned drink left in a room warms up and a warm canned drink left in a refrigerator cools down. This is accomplished by the transfer of energy from the warm medium to the cold one. The energy transfer is always from the higher temperature medium to the lower temperature one, and the energy transfer stops when the two mediums reach the same temperature.

You will recall from thermodynamics that energy exists in various forms. In this text we are primarily interested in heat, which is the form of energy that can be transferred from one system to another as a result of temperature difference. The science that deals with the determination of the rates of such energy transfers is heat transfer.

You may be wondering why we need to undertake a detailed study on heat transfer. After all, we can determine the amount of heat transfer for any system undergoing any process using a thermodynamic analysis alone. The reason is that thermodynamics is concerned with the amount of heat transfer as a system undergoes a process from one equilibrium state to another, and it gives no indication about how long the process will take. A thermodynamic analysis simply tells us how much heat must be transferred to realize a specified change of state to satisfy the conservation of energy principle.

In practice we are more concerned about the rate of heat transfer (heat transfer per unit time) than we are with the amount of it. For example, we can determine the amount of heat transferred from a thermos bottle as the hot coffee inside cools from 90°C to 80°C by a thermodynamic analysis alone. But a typical user or designer of a thermos is primarily interested in how long it will be before the hot coffee inside cools to 80°C, and a thermodynamic analysis cannot answer this question. Determining the rates of heat transfer to or from a system and thus the times of cooling or heating, as well as the variation of the temperature, is the subject of heat transfer (Fig. 1–1).

Thermodynamics deals with equilibrium states and changes from one equilibrium state to another. Heat transfer, on the other hand, deals with systems that lack thermal equilibrium, and thus it is a nonequilibrium phenomenon. Therefore, the study of heat transfer cannot be based on the principles of thermodynamics alone. However, the laws of thermodynamics lay the framework for the science of heat transfer. The first law requires that the rate of energy transfer into a system be equal to the rate of increase of the energy of that system. The second law requires that heat be transferred in the direction of decreasing temperature (Fig. 1–2). This is like a car parked on an inclined road that must go downhill in the direction of decreasing elevation when its brakes are released. It is also analogous to the electric current flowing in the direction of decreasing voltage or the fluid flowing in the direction of decreasing total pressure.

The basic requirement for heat transfer is the presence of a temperature difference. There can be no net heat transfer between two mediums that are at the same temperature. The temperature difference is the driving force for heat transfer, just as the voltage difference is the driving force for electric current flow and pressure difference is the driving force for fluid flow. The rate of heat transfer in a certain direction depends on the magnitude of the temperature gradient (the temperature difference per unit length or the rate of change of...
temperature) in that direction. The larger the temperature gradient, the higher the rate of heat transfer.

**Application Areas of Heat Transfer**

Heat transfer is commonly encountered in engineering systems and other aspects of life, and one does not need to go very far to see some application areas of heat transfer. In fact, one does not need to go anywhere. The human body is constantly rejecting heat to its surroundings, and human comfort is closely tied to the rate of this heat rejection. We try to control this heat transfer rate by adjusting our clothing to the environmental conditions.

Many ordinary household appliances are designed, in whole or in part, by using the principles of heat transfer. Some examples include the electric or gas range, the heating and air-conditioning system, the refrigerator and freezer, the water heater, the iron, and even the computer, the TV, and the VCR. Of course, energy-efficient homes are designed on the basis of minimizing heat loss in winter and heat gain in summer. Heat transfer plays a major role in the design of many other devices, such as car radiators, solar collectors, various components of power plants, and even spacecraft. The optimal insulation thickness in the walls and roofs of the houses, on hot water or steam pipes, or on water heaters is again determined on the basis of a heat transfer analysis with economic consideration (Fig. 1–3).

**Historical Background**

Heat has always been perceived to be something that produces in us a sensation of warmth, and one would think that the nature of heat is one of the first things understood by mankind. But it was only in the middle of the nineteenth century that the nature of heat was finally understood. The word "heat" was given a scientific meaning by using the term "temperature," which is a measure of the average kinetic energy of the particles in a substance. The discovery of the relationship between temperature and heat transfer was the beginning of the study of heat transfer. The discovery of the relationship between temperature and heat transfer was the beginning of the study of heat transfer. The discovery of the relationship between temperature and heat transfer was the beginning of the study of heat transfer.
century that we had a true physical understanding of the nature of heat, thanks to the development at that time of the kinetic theory, which treats molecules as tiny balls that are in motion and thus possess kinetic energy. Heat is then defined as the energy associated with the random motion of atoms and molecules. Although it was suggested in the eighteenth and early nineteenth centuries that heat is the manifestation of motion at the molecular level (called the live force), the prevailing view of heat until the middle of the nineteenth century was based on the caloric theory proposed by the French chemist Antoine Lavoisier (1743–1794) in 1789. The caloric theory asserts that heat is a fluid-like substance called the caloric that is a massless, colorless, odorless, and tasteless substance that can be poured from one body into another (Fig. 1–4). When caloric was added to a body, its temperature increased; and when caloric was removed from a body, its temperature decreased. When a body could not contain any more caloric, much the same way as when a glass of water could not dissolve any more salt or sugar, the body was said to be saturated with caloric. This interpretation gave rise to the terms saturated liquid and saturated vapor that are still in use today.

The caloric theory came under attack soon after its introduction. It maintained that heat is a substance that could not be created or destroyed. Yet it was known that heat can be generated indefinitely by rubbing one’s hands together or rubbing two pieces of wood together. In 1798, the American Benjamin Thompson (Count Rumford) (1753–1814) showed in his papers that heat can be generated continuously through friction. The validity of the caloric theory was also challenged by several others. But it was the careful experiments of the Englishman James P. Joule (1818–1889) published in 1843 that finally convinced the skeptics that heat was not a substance after all, and thus put the caloric theory to rest. Although the caloric theory was totally abandoned in the middle of the nineteenth century, it contributed greatly to the development of thermodynamics and heat transfer.

1–2 ENGINEERING HEAT TRANSFER

Heat transfer equipment such as heat exchangers, boilers, condensers, radiators, heaters, furnaces, refrigerators, and solar collectors are designed primarily on the basis of heat transfer analysis. The heat transfer problems encountered in practice can be considered in two groups: (1) rating and (2) sizing problems. The rating problems deal with the determination of the heat transfer rate for an existing system at a specified temperature difference. The sizing problems deal with the determination of the size of a system in order to transfer heat at a specified rate for a specified temperature difference.

A heat transfer process or equipment can be studied either experimentally (testing and taking measurements) or analytically (by analysis or calculations). The experimental approach has the advantage that we deal with the actual physical system, and the desired quantity is determined by measurement, within the limits of experimental error. However, this approach is expensive, time-consuming, and often impractical. Besides, the system we are analyzing may not even exist. For example, the size of a heating system of a building must usually be determined before the building is actually built on the basis of the dimensions and specifications given. The analytical approach (including numerical approach) has the advantage that it is fast and
inexpensive, but the results obtained are subject to the accuracy of the assumptions and idealizations made in the analysis. In heat transfer studies, often a good compromise is reached by reducing the choices to just a few by analysis, and then verifying the findings experimentally.

Modeling in Heat Transfer

The descriptions of most scientific problems involve expressions that relate the changes in some key variables to each other. Usually the smaller the increment chosen in the changing variables, the more general and accurate the description. In the limiting case of infinitesimal or differential changes in variables, we obtain *differential equations* that provide precise mathematical formulations for the physical principles and laws by representing the rates of changes as *derivatives*. Therefore, differential equations are used to investigate a wide variety of problems in sciences and engineering, including heat transfer. However, most heat transfer problems encountered in practice can be solved without resorting to differential equations and the complications associated with them.

The study of physical phenomena involves two important steps. In the first step, all the variables that affect the phenomena are identified, reasonable assumptions and approximations are made, and the interdependence of these variables is studied. The relevant physical laws and principles are invoked, and the problem is formulated mathematically. The equation itself is very instructive as it shows the degree of dependence of some variables on others, and the relative importance of various terms. In the second step, the problem is solved using an appropriate approach, and the results are interpreted.

Many processes that seem to occur in nature randomly and without any order are, in fact, being governed by some visible or not-so-visible physical laws. Whether we notice them or not, these laws are there, governing consistently and predictably what seem to be ordinary events. Most of these laws are well defined and well understood by scientists. This makes it possible to predict the course of an event before it actually occurs, or to study various aspects of an event mathematically without actually running expensive and time-consuming experiments. This is where the power of analysis lies. Very accurate results to meaningful practical problems can be obtained with relatively little effort by using a suitable and realistic mathematical model. The preparation of such models requires an adequate knowledge of the natural phenomena involved and the relevant laws, as well as a sound judgment. An unrealistic model will obviously give inaccurate and thus unacceptable results.

An analyst working on an engineering problem often finds himself or herself in a position to make a choice between a very accurate but complex model, and a simple but not-so-accurate model. The right choice depends on the situation at hand. The right choice is usually the simplest model that yields adequate results. For example, the process of baking potatoes or roasting a round chunk of beef in an oven can be studied analytically in a simple way by modeling the potato or the roast as a spherical solid ball that has the properties of water (Fig. 1–5). The model is quite simple, but the results obtained are sufficiently accurate for most practical purposes. As another example, when we analyze the heat losses from a building in order to select the right size for a heater, we determine the heat losses under anticipated worst conditions and select a furnace that will provide sufficient heat to make up for those losses.
Often we tend to choose a larger furnace in anticipation of some future expansion, or just to provide a factor of safety. A very simple analysis will be adequate in this case.

When selecting heat transfer equipment, it is important to consider the actual operating conditions. For example, when purchasing a heat exchanger that will handle hard water, we must consider that some calcium deposits will form on the heat transfer surfaces over time, causing fouling and thus a gradual decline in performance. The heat exchanger must be selected on the basis of operation under these adverse conditions instead of under new conditions.

Preparing very accurate but complex models is usually not so difficult. But such models are not much use to an analyst if they are very difficult and time-consuming to solve. At the minimum, the model should reflect the essential features of the physical problem it represents. There are many significant real-world problems that can be analyzed with a simple model. But it should always be kept in mind that the results obtained from an analysis are as accurate as the assumptions made in simplifying the problem. Therefore, the solution obtained should not be applied to situations for which the original assumptions do not hold.

A solution that is not quite consistent with the observed nature of the problem indicates that the mathematical model used is too crude. In that case, a more realistic model should be prepared by eliminating one or more of the questionable assumptions. This will result in a more complex problem that, of course, is more difficult to solve. Thus any solution to a problem should be interpreted within the context of its formulation.

1–3 HEAT AND OTHER FORMS OF ENERGY

Energy can exist in numerous forms such as thermal, mechanical, kinetic, potential, electrical, magnetic, chemical, and nuclear, and their sum constitutes the **total energy** \( E \) (or \( e \) on a unit mass basis) of a system. The forms of energy related to the molecular structure of a system and the degree of the molecular activity are referred to as the **microscopic energy**. The sum of all microscopic forms of energy is called the **internal energy** of a system, and is denoted by \( U \) (or \( u \) on a unit mass basis).

The international unit of energy is **joule** \( (J) \) or **kilojoule** \((1 \text{kJ} = 1000 \text{J})\). In the English system, the unit of energy is the **British thermal unit** \((\text{Btu})\), which is defined as the energy needed to raise the temperature of 1 lbm of water at 60°F by 1°F. The magnitudes of kJ and Btu are almost identical \((1 \text{ Btu} = 1.055056 \text{ kJ})\). Another well-known unit of energy is the **calorie** \((1 \text{ cal} = 4.1868 \text{ J})\), which is defined as the energy needed to raise the temperature of 1 gram of water at 14.5°C by 1°C.

Internal energy may be viewed as the sum of the kinetic and potential energies of the molecules. The portion of the internal energy of a system associated with the kinetic energy of the molecules is called **sensible energy** or **sensible heat**. The average velocity and the degree of activity of the molecules are proportional to the temperature. Thus, at higher temperatures the molecules will possess higher kinetic energy, and as a result, the system will have a higher internal energy.

The internal energy is also associated with the intermolecular forces between the molecules of a system. These are the forces that bind the molecules
to each other, and, as one would expect, they are strongest in solids and weakest in gases. If sufficient energy is added to the molecules of a solid or liquid, they will overcome these molecular forces and simply break away, turning the system to a gas. This is a phase change process and because of this added energy, a system in the gas phase is at a higher internal energy level than it is in the solid or the liquid phase. The internal energy associated with the phase of a system is called latent energy or latent heat.

The changes mentioned above can occur without a change in the chemical composition of a system. Most heat transfer problems fall into this category, and one does not need to pay any attention to the forces binding the atoms in a molecule together. The internal energy associated with the atomic bonds in a molecule is called chemical (or bond) energy, whereas the internal energy associated with the bonds within the nucleus of the atom itself is called nuclear energy. The chemical and nuclear energies are absorbed or released during chemical or nuclear reactions, respectively.

In the analysis of systems that involve fluid flow, we frequently encounter the combination of properties \( u \) and \( Pv \). For the sake of simplicity and convenience, this combination is defined as enthalpy \( h \). That is, \( h = u + Pv \) where the term \( Pv \) represents the flow energy of the fluid (also called the flow work), which is the energy needed to push a fluid and to maintain flow. In the energy analysis of flowing fluids, it is convenient to treat the flow energy as part of the energy of the fluid and to represent the microscopic energy of a fluid stream by enthalpy \( h \) (Fig. 1–6).

### Specific Heats of Gases, Liquids, and Solids

You may recall that an ideal gas is defined as a gas that obeys the relation

\[
Pv = RT \quad \text{or} \quad P = \rho RT \tag{1-1}
\]

where \( P \) is the absolute pressure, \( v \) is the specific volume, \( T \) is the absolute temperature, \( \rho \) is the density, and \( R \) is the gas constant. It has been experimentally observed that the ideal gas relation given above closely approximates the \( P\cdot v\cdot T \) behavior of real gases at low densities. At low pressures and high temperatures, the density of a gas decreases and the gas behaves like an ideal gas. In the range of practical interest, many familiar gases such as air, nitrogen, oxygen, hydrogen, helium, argon, neon, and krypton and even heavier gases such as carbon dioxide can be treated as ideal gases with negligible error (often less than one percent). Dense gases such as water vapor in steam power plants and refrigerant vapor in refrigerators, however, should not always be treated as ideal gases since they usually exist at a state near saturation.

You may also recall that specific heat is defined as the energy required to raise the temperature of a unit mass of a substance by one degree (Fig. 1–7). In general, this energy depends on how the process is executed. In thermodynamics, we are interested in two kinds of specific heats: specific heat at constant volume \( C_v \) and specific heat at constant pressure \( C_p \). The specific heat at constant volume \( C_v \) can be viewed as the energy required to raise the temperature of a unit mass of a substance by one degree as the volume is held constant. The energy required to do the same as the pressure is held constant is the specific heat at constant pressure \( C_p \). The specific heat at constant
pressure \( C_p \) is greater than \( C_v \) because at constant pressure the system is allowed to expand and the energy for this expansion work must also be supplied to the system. For ideal gases, these two specific heats are related to each other by 
\[
\frac{C_p}{R} = 1 + \frac{C_v}{R}.
\]
A common unit for specific heats is \( \text{kJ/kg} \cdot ^\circ \text{C} \) or \( \text{kJ/kg} \cdot \text{K} \). Notice that these two units are identical since 
\[
\Delta T(\circ \text{C}) = \Delta T(\text{K}),
\]
and \( 1 \circ \text{C} \) change in temperature is equivalent to a change of \( 1 \text{ K} \). Also, 
\[
1 \text{ kJ/kg} \cdot \circ \text{C} = 1 \text{ J/g} \cdot \circ \text{C} = 1 \text{ kJ/kg} \cdot \text{K} = 1 \text{ J/g} \cdot \text{K}.
\]

The specific heats of a substance, in general, depend on two independent properties such as temperature and pressure. For an ideal gas, however, they depend on temperature only (Fig. 1–8). At low pressures all real gases approach ideal gas behavior, and therefore their specific heats depend on temperature only.

The differential changes in the internal energy \( u \) and enthalpy \( h \) of an ideal gas can be expressed in terms of the specific heats as 
\[
du = C_v dT \quad \text{and} \quad dh = C_p dT \tag{1-2}
\]

The finite changes in the internal energy and enthalpy of an ideal gas during a process can be expressed approximately by using specific heat values at the average temperature as 
\[
\Delta u = C_v,\text{ave} \Delta T \quad \text{and} \quad \Delta h = C_p,\text{ave} \Delta T \tag{1-3}
\]
or 
\[
\Delta U = mC_v,\text{ave} \Delta T \quad \text{and} \quad \Delta H = mC_p,\text{ave} \Delta T \tag{1-4}
\]
where \( m \) is the mass of the system.

A substance whose specific volume (or density) does not change with temperature or pressure is called an incompressible substance. The specific volumes of solids and liquids essentially remain constant during a process, and thus they can be approximated as incompressible substances without sacrificing much in accuracy.

The constant-volume and constant-pressure specific heats are identical for incompressible substances (Fig. 1–9). Therefore, for solids and liquids the subscripts on \( C_v \) and \( C_p \) can be dropped and both specific heats can be represented by a single symbol, \( C \). That is, \( C_p \equiv C_v \equiv C \). This result could also be deduced from the physical definitions of constant-volume and constant-pressure specific heats. Specific heats of several common gases, liquids, and solids are given in the Appendix.

The specific heats of incompressible substances depend on temperature only. Therefore, the change in the internal energy of solids and liquids can be expressed as 
\[
\Delta U = mC_v,\text{ave} \Delta T \tag{1-5}
\]
where $C_{ave}$ is the average specific heat evaluated at the average temperature. Note that the internal energy change of the systems that remain in a single phase (liquid, solid, or gas) during the process can be determined very easily using average specific heats.

**Energy Transfer**

Energy can be transferred to or from a given mass by two mechanisms: *heat* $Q$ and *work* $W$. An energy interaction is heat transfer if its driving force is a temperature difference. Otherwise, it is work. A rising piston, a rotating shaft, and an electrical wire crossing the system boundaries are all associated with work interactions. Work done *per unit time* is called *power*, and is denoted by $W$. The unit of power is $W$ or hp (1 hp = 746 W). Car engines and hydraulic, steam, and gas turbines produce work; compressors, pumps, and mixers consume work. Notice that the energy of a system decreases as it does work, and increases as work is done on it.

In daily life, we frequently refer to the sensible and latent forms of internal energy as *heat*, and we talk about the heat content of bodies (Fig. 1–10). In thermodynamics, however, those forms of energy are usually referred to as *thermal energy* to prevent any confusion with *heat transfer*.

The term *heat* and the associated phrases such as *heat flow*, *heat addition*, *heat rejection*, *heat absorption*, *heat gain*, *heat loss*, *heat storage*, *heat generation*, *electrical heating*, *latent heat*, *body heat*, and *heat source* are in common use today, and the attempt to replace *heat* in these phrases by *thermal energy* had only limited success. These phrases are deeply rooted in our vocabulary and they are used by both the ordinary people and scientists without causing any misunderstanding. For example, the phrase *body heat* is understood to mean the *thermal energy content* of a body. Likewise, *heat flow* is understood to mean the *transfer of thermal energy*, not the flow of a fluid-like substance called *heat*, although the latter incorrect interpretation, based on the caloric theory, is the origin of this phrase. Also, the transfer of heat into a system is frequently referred to as *heat addition* and the transfer of heat out of a system as *heat rejection*.

Keeping in line with current practice, we will refer to the thermal energy as *heat* and the transfer of thermal energy as *heat transfer*. The amount of heat transferred during the process is denoted by $Q$. The amount of heat transferred per unit time is called *heat transfer rate*, and is denoted by $\dot{Q}$. The overdot stands for the time derivative, or “per unit time.” The heat transfer rate $\dot{Q}$ has the unit J/s, which is equivalent to W.

When the *rate* of heat transfer $\dot{Q}$ is available, then the total amount of heat transfer $Q$ during a time interval $\Delta t$ can be determined from

$$ Q = \int_{0}^{\Delta t} \dot{Q} \, dt \quad (J) \quad (1-6) $$

provided that the variation of $\dot{Q}$ with time is known. For the special case of $\dot{Q} = \text{constant}$, the equation above reduces to

$$ Q = \dot{Q} \Delta t \quad (J) \quad (1-7) $$

The sensible and latent forms of internal energy can be transferred as a result of a temperature difference, and they are referred to as *heat* or *thermal energy*. 

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**FIGURE 1–10**

The sensible and latent forms of internal energy can be transferred as a result of a temperature difference, and they are referred to as *heat* or *thermal energy*.
The rate of heat transfer per unit area normal to the direction of heat transfer is called **heat flux**, and the average heat flux is expressed as (Fig. 1–11)

\[ q = \frac{\dot{Q}}{A} \quad \text{(W/m}^2\text{)} \]  

(1-8)

where \( A \) is the heat transfer area. The unit of heat flux in English units is Btu/h \cdot ft\(^2\). Note that heat flux may vary with time as well as position on a surface.

**EXAMPLE 1–1  Heating of a Copper Ball**

A 10-cm diameter copper ball is to be heated from 100°C to an average temperature of 150°C in 30 minutes (Fig. 1–12). Taking the average density and specific heat of copper in this temperature range to be \( \rho = 8950 \text{ kg/m}^3 \) and \( C_p = 0.395 \text{ kJ/kg \cdot °C} \), respectively, determine (a) the total amount of heat transfer to the copper ball, (b) the average rate of heat transfer to the ball, and (c) the average heat flux.

**SOLUTION** The copper ball is to be heated from 100°C to 150°C. The total heat transfer, the average rate of heat transfer, and the average heat flux are to be determined.

**Assumptions** Constant properties can be used for copper at the average temperature.

**Properties** The average density and specific heat of copper are given to be \( \rho = 8950 \text{ kg/m}^3 \) and \( C_p = 0.395 \text{ kJ/kg \cdot °C} \).

**Analysis** (a) The amount of heat transferred to the copper ball is simply the change in its internal energy, and is determined from

\[
Q = \Delta U = mC_{ave}(T_2 - T_1)
\]

where

\[
m = \rho V = \frac{\pi}{6} bD^3 = \frac{\pi}{6}(8950 \text{ kg/m}^3)(0.1 \text{ m})^3 = 4.69 \text{ kg}
\]

Substituting,

\[
Q = (4.69 \text{ kg})(0.395 \text{ kJ/kg \cdot °C})(150 - 100)\text{°C} = 92.6 \text{ kJ}
\]

Therefore, 92.6 kJ of heat needs to be transferred to the copper ball to heat it from 100°C to 150°C.

(b) The rate of heat transfer normally changes during a process with time. However, we can determine the **average** rate of heat transfer by dividing the total amount of heat transfer by the time interval. Therefore,

\[
\dot{Q}_{ave} = \frac{Q}{\Delta t} = \frac{92.6 \text{ kJ}}{1800 \text{ s}} = 0.0514 \text{ kJ/s} = 51.4 \text{ W}
\]
1–4 THE FIRST LAW OF THERMODYNAMICS

The first law of thermodynamics, also known as the conservation of energy principle, states that energy can neither be created nor destroyed; it can only change forms. Therefore, every bit of energy must be accounted for during a process. The conservation of energy principle (or the energy balance) for any system undergoing any process may be expressed as follows: The net change (increase or decrease) in the total energy of the system during a process is equal to the difference between the total energy entering and the total energy leaving the system during that process. That is,

\[ E_{\text{in}} - E_{\text{out}} = \Delta E_{\text{system}} \]  

or, in the rate form, as

\[ E_{\text{in}} - E_{\text{out}} = \frac{dE_{\text{system}}}{dt} \]

Energy is a property, and the value of a property does not change unless the state of the system changes. Therefore, the energy change of a system is zero \((\Delta E_{\text{system}} = 0)\) if the state of the system does not change during the process, that is, the process is steady. The energy balance in this case reduces to (Fig. 1–13)

\[ E_{\text{in}} = E_{\text{out}} \]

In the absence of significant electric, magnetic, motion, gravity, and surface tension effects (i.e., for stationary simple compressible systems), the change

Discussion Note that heat flux may vary with location on a surface. The value calculated above is the average heat flux over the entire surface of the ball.
in the total energy of a system during a process is simply the change in its internal energy. That is, \[ \Delta E_{\text{system}} = \Delta U_{\text{system}}. \]

In heat transfer analysis, we are usually interested only in the forms of energy that can be transferred as a result of a temperature difference, that is, heat or thermal energy. In such cases it is convenient to write a heat balance and to treat the conversion of nuclear, chemical, and electrical energies into thermal energy as heat generation. The energy balance in that case can be expressed as

\[
Q_{\text{in}} - Q_{\text{out}} + E_{\text{gen}} = \Delta E_{\text{thermal, system}} \quad (J) \tag{1-13}
\]

**Energy Balance for Closed Systems (Fixed Mass)**

A closed system consists of a fixed mass. The total energy \( E \) for most systems encountered in practice consists of the internal energy \( U \). This is especially the case for stationary systems since they don’t involve any changes in their velocity or elevation during a process. The energy balance relation in that case reduces to

Stationary closed system: \[ E_{\text{in}} - E_{\text{out}} = \Delta U = mC_v\Delta T \quad (J) \tag{1-14} \]

where we expressed the internal energy change in terms of mass \( m \), the specific heat at constant volume \( C_v \), and the temperature change \( \Delta T \) of the system. When the system involves heat transfer only and no work interactions across its boundary, the energy balance relation further reduces to (Fig. 1–14)

Stationary closed system, no work: \[ Q = mC_v\Delta T \quad (J) \tag{1-15} \]

where \( Q \) is the net amount of heat transfer to or from the system. This is the form of the energy balance relation we will use most often when dealing with a fixed mass.

**Energy Balance for Steady-Flow Systems**

A large number of engineering devices such as water heaters and car radiators involve mass flow in and out of a system, and are modeled as control volumes. Most control volumes are analyzed under steady operating conditions. The term steady means no change with time at a specified location. The opposite of steady is unsteady or transient. Also, the term uniform implies no change with position throughout a surface or region at a specified time. These meanings are consistent with their everyday usage (steady girlfriend, uniform distribution, etc.). The total energy content of a control volume during a steady-flow process remains constant (\( E_{\text{CV}} = \) constant). That is, the change in the total energy of the control volume during such a process is zero (\( \Delta E_{\text{CV}} = 0 \)). Thus the amount of energy entering a control volume in all forms (heat, work, mass transfer) for a steady-flow process must be equal to the amount of energy leaving it.

The amount of mass flowing through a cross section of a flow device per unit time is called the mass flow rate, and is denoted by \( \dot{m} \). A fluid may flow in and out of a control volume through pipes or ducts. The mass flow rate of a fluid flowing in a pipe or duct is proportional to the cross-sectional area \( A_p \) of

\[
\dot{m} = \frac{\rho A_p v}{\rho g} \quad (1-16)
\]

\[ \rho = \text{density}, \quad A_p = \text{cross-sectional area} \]

\[ v = \text{velocity} \]
the pipe or duct, the density $\rho$, and the velocity $\bar{V}$ of the fluid. The mass flow rate through a differential area $dA_c$ can be expressed as $\delta \dot{m} = \rho \bar{V} c dA_c$, where $\bar{V}_c$ is the velocity component normal to $dA_c$. The mass flow rate through the entire cross-sectional area is obtained by integration over $A_c$.

The flow of a fluid through a pipe or duct can often be approximated to be one-dimensional. That is, the properties can be assumed to vary in one direction only (the direction of flow). As a result, all properties are assumed to be uniform at any cross section normal to the flow direction, and the properties are assumed to have bulk average values over the entire cross section. Under the one-dimensional flow approximation, the mass flow rate of a fluid flowing in a pipe or duct can be expressed as (Fig. 1–15)

$$\dot{m} = \rho \bar{V} A_c \quad \text{(kg/s)}$$

(1-16)

where $\rho$ is the fluid density, $\bar{V}$ is the average fluid velocity in the flow direction, and $A_c$ is the cross-sectional area of the pipe or duct.

The volume of a fluid flowing through a pipe or duct per unit time is called the volume flow rate $\dot{V}$, and is expressed as

$$\dot{V} = \bar{V} A_c \frac{m}{\rho} \quad \text{(m}^3/\text{s)}$$

(1-17)

Note that the mass flow rate of a fluid through a pipe or duct remains constant during steady flow. This is not the case for the volume flow rate, however, unless the density of the fluid remains constant.

For a steady-flow system with one inlet and one exit, the rate of mass flow into the control volume must be equal to the rate of mass flow out of it. That is, $\dot{m}_{\text{in}} = \dot{m}_{\text{out}} = \dot{m}$. When the changes in kinetic and potential energies are negligible, which is usually the case, and there is no work interaction, the energy balance for such a steady-flow system reduces to (Fig. 1–16)

$$\dot{Q} = \dot{m} \Delta h = \dot{m} C_p \Delta T \quad \text{(kJ/s)}$$

(1-18)

where $\dot{Q}$ is the rate of net heat transfer into or out of the control volume. This is the form of the energy balance relation that we will use most often for steady-flow systems.

**Surface Energy Balance**

As mentioned in the chapter opener, heat is transferred by the mechanisms of conduction, convection, and radiation, and heat often changes vehicles as it is transferred from one medium to another. For example, the heat conducted to the outer surface of the wall of a house in winter is convected away by the cold outdoor air while being radiated to the cold surroundings. In such cases, it may be necessary to keep track of the energy interactions at the surface, and this is done by applying the conservation of energy principle to the surface.

A surface contains no volume or mass, and thus no energy. Therefore, a surface can be viewed as a fictitious system whose energy content remains constant during a process (just like a steady-state or steady-flow system). Then the energy balance for a surface can be expressed as

**Surface energy balance:**

$$\dot{E}_{\text{in}} = \dot{E}_{\text{out}}$$

(1-19)
This relation is valid for both steady and transient conditions, and the surface energy balance does not involve heat generation since a surface does not have a volume. The energy balance for the outer surface of the wall in Fig. 1–17, for example, can be expressed as

\[ \dot{Q}_1 = \dot{Q}_2 + \dot{Q}_3 \]  

(1-20)

where \( \dot{Q}_1 \) is conduction through the wall to the surface, \( \dot{Q}_2 \) is convection from the surface to the outdoor air, and \( \dot{Q}_3 \) is net radiation from the surface to the surroundings.

When the directions of interactions are not known, all energy interactions can be assumed to be towards the surface, and the surface energy balance can be expressed as \( \Sigma \dot{E}_{\text{in}} = 0 \). Note that the interactions in opposite direction will end up having negative values, and balance this equation.

**EXAMPLE 1–2  Heating of Water in an Electric Teapot**

1.2 kg of liquid water initially at 15°C is to be heated to 95°C in a teapot equipped with a 1200-W electric heating element inside (Fig. 1–18). The teapot is 0.5 kg and has an average specific heat of 0.7 kJ/kg \cdot °C. Taking the specific heat of water to be 4.18 kJ/kg \cdot °C and disregarding any heat loss from the teapot, determine how long it will take for the water to be heated.

**SOLUTION** Liquid water is to be heated in an electric teapot. The heating time is to be determined.

**Assumptions** 1. Heat loss from the teapot is negligible. 2. Constant properties can be used for both the teapot and the water.

**Properties** The average specific heats are given to be 0.7 kJ/kg \cdot °C for the teapot and 4.18 kJ/kg \cdot °C for water.

**Analysis** We take the teapot and the water in it as the system, which is a closed system (fixed mass). The energy balance in this case can be expressed as

\[ E_{\text{in}} - E_{\text{out}} = \Delta E_{\text{system}} \]

\[ E_{\text{in}} = \Delta U_{\text{system}} = \Delta U_{\text{water}} + \Delta U_{\text{teapot}} \]

Then the amount of energy needed to raise the temperature of water and the teapot from 15°C to 95°C is

\[ E_{\text{in}} = (mC\Delta T)_{\text{water}} + (mC\Delta T)_{\text{teapot}} \]

\[ = (1.2 \text{ kg})(4.18 \text{ kJ/kg \cdot °C})(95 - 15)\text{°C} + (0.5 \text{ kg})(0.7 \text{ kJ/kg \cdot °C}) \]

\[ = 429.3 \text{ kJ} \]

The 1200-W electric heating unit will supply energy at a rate of 1.2 kW or 1.2 kJ per second. Therefore, the time needed for this heater to supply 429.3 kJ of heat is determined from

\[ \Delta t = \frac{\text{Total energy transferred}}{\text{Rate of energy transfer}} = \frac{E_{\text{in}}}{E_{\text{transfer}}} = \frac{429.3 \text{ kJ}}{1.2 \text{ kJ/s}} = 358 \text{ s} = 6.0 \text{ min} \]
Discussion  In reality, it will take more than 6 minutes to accomplish this heating process since some heat loss is inevitable during heating.

EXAMPLE 1–3  Heat Loss from Heating Ducts in a Basement

A 5-m-long section of an air heating system of a house passes through an unheated space in the basement (Fig. 1–19). The cross section of the rectangular duct of the heating system is 20 cm × 25 cm. Hot air enters the duct at 100 kPa and 60°C at an average velocity of 5 m/s. The temperature of the air in the duct drops to 54°C as a result of heat loss to the cool space in the basement. Determine the rate of heat loss from the air in the duct to the basement under steady conditions. Also, determine the cost of this heat loss per hour if the house is heated by a natural gas furnace that has an efficiency of 80 percent, and the cost of the natural gas in that area is $0.60/therm (1 therm = 100,000 Btu = 105,500 kJ).

SOLUTION  The temperature of the air in the heating duct of a house drops as a result of heat loss to the cool space in the basement. The rate of heat loss from the hot air and its cost are to be determined.

Assumptions  1 Steady operating conditions exist. 2 Air can be treated as an ideal gas with constant properties at room temperature.

Properties  The constant pressure specific heat of air at the average temperature of (54 + 60)/2 = 57°C is 1.007 kJ/kg · °C (Table A-15).

Analysis  We take the basement section of the heating system as our system, which is a steady-flow system. The rate of heat loss from the air in the duct can be determined from

\[ \dot{Q} = \dot{m} C_p \Delta T \]

where \( \dot{m} \) is the mass flow rate and \( \Delta T \) is the temperature drop. The density of air at the inlet conditions is

\[ \rho = \frac{P}{RT} = \frac{100 \text{ kPa}}{(0.287 \text{ kPa} \cdot \text{m}^3/\text{kg} \cdot \text{K})(60 + 273)\text{K}} = 1.046 \text{ kg/m}^3 \]

The cross-sectional area of the duct is

\[ A_c = (0.20 \text{ m})(0.25 \text{ m}) = 0.05 \text{ m}^2 \]

Then the mass flow rate of air through the duct and the rate of heat loss become

\[ \dot{m} = \rho \mathcal{V} A_c = (1.046 \text{ kg/m}^3)(5 \text{ m/s})(0.05 \text{ m}^2) = 0.2615 \text{ kg/s} \]

and

\[ \dot{Q}_{\text{loss}} = \dot{m} C_p(T_{\text{in}} - T_{\text{out}}) = (0.2615 \text{ kg/s})(1.007 \text{ kJ/kg} \cdot \degree \text{C})(60 - 54)\degree \text{C} = 1.580 \text{ kJ/s} \]
or 5688 kJ/h. The cost of this heat loss to the homeowner is

\[
\text{Cost of heat loss} = \frac{(\text{Rate of heat loss})(\text{Unit cost of energy input})}{\text{Furnace efficiency}}
\]

\[
= \frac{(5688 \text{ kJ/h})(0.60/\text{therm})}{0.80}
\]

\[
= \frac{1 \text{ therm}}{105,500 \text{ kJ}}
\]

\[
= $0.040/\text{h}
\]

**Discussion** The heat loss from the heating ducts in the basement is costing the homeowner 4 cents per hour. Assuming the heater operates 2000 hours during a heating season, the annual cost of this heat loss adds up to $80. Most of this money can be saved by insulating the heating ducts in the unheated areas.

---

**EXAMPLE 1–4 Electric Heating of a House at High Elevation**

Consider a house that has a floor space of 2000 ft\(^2\) and an average height of 9 ft at 5000 ft elevation where the standard atmospheric pressure is 12.2 psia (Fig. 1–20). Initially the house is at a uniform temperature of 50°F. Now the electric heater is turned on, and the heater runs until the air temperature in the house rises to an average value of 70°F. Determine the amount of energy transferred to the air assuming (a) the house is air-tight and thus no air escapes during the heating process and (b) some air escapes through the cracks as the heated air in the house expands at constant pressure. Also determine the cost of this heat for each case if the cost of electricity in that area is $0.075/kWh.

**SOLUTION** The air in the house is heated from 50°F to 70°F by an electric heater. The amount and cost of the energy transferred to the air are to be determined for constant-volume and constant-pressure cases.

**Assumptions** 1. Air can be treated as an ideal gas with constant properties at room temperature. 2. Heat loss from the house during heating is negligible. 3. The volume occupied by the furniture and other things is negligible.

**Properties** The specific heats of air at the average temperature of \((50 + 70)/2 = 60°F\) are \(C_p = 0.240 \text{ Btu/lbm} \cdot °\text{F}\) and \(C_v = C_p - R = 0.171 \text{ Btu/lbm} \cdot °\text{F}\) (Tables A-1E and A-15E).

**Analysis** The volume and the mass of the air in the house are

\[
V = (\text{Floor area})(\text{Height}) = (2000 \text{ ft}^2)(9 \text{ ft}) = 18,000 \text{ ft}^3
\]

\[
m = \frac{PV}{RT} = \frac{(12.2 \text{ psia})(18,000 \text{ ft}^3)}{(0.3704 \text{ psia} \cdot \text{ft}^3/\text{lbm} \cdot \text{R})(50 + 460)\text{R}} = 1162 \text{ lbm}
\]

(a) The amount of energy transferred to air at constant volume is simply the change in its internal energy, and is determined from

\[
E_{\text{in}} - E_{\text{out}} = \Delta E_{\text{system}}
\]

\[
E_{\text{in, constant volume}} = \Delta U_{\text{air}} = mC_v \Delta T
\]

\[
= (1162 \text{ lbm})(0.171 \text{ Btu/lbm} \cdot °\text{F})(70 - 50)°\text{F}
\]

\[
= 3974 \text{ Btu}
\]

At a unit cost of $0.075/kWh, the total cost of this energy is
In Section 1–1 we defined heat as the form of energy that can be transferred from one system to another as a result of temperature difference. A thermodynamic analysis is concerned with the amount of heat transfer as a system undergoes a process from one equilibrium state to another. The science that deals with the determination of the rates of such energy transfers is the heat transfer. The transfer of energy as heat is always from the higher-temperature medium to the lower-temperature one, and heat transfer stops when the two mediums reach the same temperature.

Heat can be transferred in three different modes: conduction, convection, and radiation. All modes of heat transfer require the existence of a temperature difference, and all modes are from the high-temperature medium to a lower-temperature one. Below we give a brief description of each mode. A detailed study of these modes is given in later chapters of this text.

1–5 • HEAT TRANSFER MECHANISMS

In Section 1–1 we defined heat as the form of energy that can be transferred from one system to another as a result of temperature difference. A thermodynamic analysis is concerned with the amount of heat transfer as a system undergoes a process from one equilibrium state to another. The science that deals with the determination of the rates of such energy transfers is the heat transfer. The transfer of energy as heat is always from the higher-temperature medium to the lower-temperature one, and heat transfer stops when the two mediums reach the same temperature.

Heat can be transferred in three different modes: conduction, convection, and radiation. All modes of heat transfer require the existence of a temperature difference, and all modes are from the high-temperature medium to a lower-temperature one. Below we give a brief description of each mode. A detailed study of these modes is given in later chapters of this text.

1–6 • CONDUCTION

Conduction is the transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as a result of interactions between the particles. Conduction can take place in solids, liquids, or gases. In gases and liquids, conduction is due to the collisions and diffusion of the
molecules during their random motion. In solids, it is due to the combination of vibrations of the molecules in a lattice and the energy transport by free electrons. A cold canned drink in a warm room, for example, eventually warms up to the room temperature as a result of heat transfer from the room to the drink through the aluminum can by conduction.

The rate of heat conduction through a medium depends on the geometry of the medium, its thickness, and the material of the medium, as well as the temperature difference across the medium. We know that wrapping a hot water tank with glass wool (an insulating material) reduces the rate of heat loss from the tank. The thicker the insulation, the smaller the heat loss. We also know that a hot water tank will lose heat at a higher rate when the temperature of the room housing the tank is lowered. Further, the larger the tank, the larger the surface area and thus the rate of heat loss.

Consider steady heat conduction through a large plane wall of thickness $\Delta x = L$ and area $A$, as shown in Fig. 1–21. The temperature difference across the wall is $\Delta T = T_2 - T_1$. Experiments have shown that the rate of heat transfer $\dot{Q}$ through the wall is doubled when the temperature difference $\Delta T$ across the wall or the area $A$ normal to the direction of heat transfer is doubled, but is halved when the wall thickness $L$ is doubled. Thus we conclude that the rate of heat conduction through a plane layer is proportional to the temperature difference across the layer and the heat transfer area, but is inversely proportional to the thickness of the layer. That is,

$$ \text{Rate of heat conduction} \propto \frac{(\text{Area})(\text{Temperature difference})}{\text{Thickness}} $$

or,

$$ \dot{Q}_{\text{cond}} = kA \frac{T_1 - T_2}{\Delta x} = -kA \frac{\Delta T}{\Delta x} \quad (W) \quad (1-21) $$

where the constant of proportionality $k$ is the thermal conductivity of the material, which is a measure of the ability of a material to conduct heat (Fig. 1–22). In the limiting case of $\Delta x \to 0$, the equation above reduces to the differential form

$$ \dot{Q}_{\text{cond}} = -kA \frac{dT}{dx} \quad (W) \quad (1-22) $$

which is called Fourier’s law of heat conduction after J. Fourier, who expressed it first in his heat transfer text in 1822. Here $dT/dx$ is the temperature gradient, which is the slope of the temperature curve on a $T-x$ diagram (the rate of change of $T$ with $x$), at location $x$. The relation above indicates that the rate of heat conduction in a direction is proportional to the temperature gradient in that direction. Heat is conducted in the direction of decreasing temperature, and the temperature gradient becomes negative when temperature decreases with increasing $x$. The negative sign in Eq. 1–22 ensures that heat transfer in the positive $x$ direction is a positive quantity.

The heat transfer area $A$ is always normal to the direction of heat transfer. For heat loss through a 5-m-long, 3-m-high, and 25-cm-thick wall, for example, the heat transfer area is $A = 15 \text{ m}^2$. Note that the thickness of the wall has no effect on $A$ (Fig. 1–23).
Thermal Conductivity

We have seen that different materials store heat differently, and we have defined the property specific heat \( C_p \) as a measure of a material’s ability to store thermal energy. For example, \( C_p = 4.18 \text{ kJ/kg} \cdot \text{°C} \) for water and \( C_p = 0.45 \text{ kJ/kg} \cdot \text{°C} \) for iron at room temperature, which indicates that water can store almost 10 times the energy that iron can per unit mass. Likewise, the thermal conductivity \( k \) is a measure of a material’s ability to conduct heat. For example, \( k = 0.608 \text{ W/m} \cdot \text{°C} \) for water and \( k = 80.2 \text{ W/m} \cdot \text{°C} \) for iron at room temperature, which indicates that iron conducts heat more than 100 times faster than water can. Thus we say that water is a poor heat conductor relative to iron, although water is an excellent medium to store thermal energy.

Equation 1–22 for the rate of conduction heat transfer under steady conditions can also be viewed as the defining equation for thermal conductivity. Thus the thermal conductivity of a material can be defined as the rate of...
heat transfer through a unit thickness of the material per unit area per unit temperature difference. The thermal conductivity of a material is a measure of the ability of the material to conduct heat. A high value for thermal conductivity indicates that the material is a good heat conductor, and a low value indicates that the material is a poor heat conductor or insulator. The thermal conductivities of some common materials at room temperature are given in Table 1–1. The thermal conductivity of pure copper at room temperature is \( k = 401 \text{ W/m} \cdot \text{°C} \), which indicates that a 1-m-thick copper wall will conduct heat at a rate of 401 W per m² area per °C temperature difference across the wall. Note that materials such as copper and silver that are good electric conductors are also good heat conductors, and have high values of thermal conductivity. Materials such as rubber, wood, and styrofoam are poor conductors of heat and have low conductivity values.

A layer of material of known thickness and area can be heated from one side by an electric resistance heater of known output. If the outer surfaces of the heater are well insulated, all the heat generated by the resistance heater will be transferred through the material whose conductivity is to be determined. Then measuring the two surface temperatures of the material when steady heat transfer is reached and substituting them into Eq. 1–22 together with other known quantities give the thermal conductivity (Fig. 1–25).

The thermal conductivities of materials vary over a wide range, as shown in Fig. 1–26. The thermal conductivities of gases such as air vary by a factor of 10⁴ from those of pure metals such as copper. Note that pure crystals and metals have the highest thermal conductivities, and gases and insulating materials the lowest.

Temperature is a measure of the kinetic energies of the particles such as the molecules or atoms of a substance. In a liquid or gas, the kinetic energy of the molecules is due to their random translational motion as well as their vibrational and rotational motions. When two molecules possessing different kinetic energies collide, part of the kinetic energy of the more energetic (higher-temperature) molecule is transferred to the less energetic (lower-temperature) molecule, much the same as when two elastic balls of the same mass at different velocities collide, part of the kinetic energy of the faster ball is transferred to the slower one. The higher the temperature, the faster the molecules move and the higher the number of such collisions, and the better the heat transfer.

The kinetic theory of gases predicts and the experiments confirm that the thermal conductivity of gases is proportional to the square root of the absolute temperature \( T \), and inversely proportional to the square root of the molar mass \( M \). Therefore, the thermal conductivity of a gas increases with increasing temperature and decreasing molar mass. So it is not surprising that the thermal conductivity of helium \( (M = 4) \) is much higher than those of air \( (M = 29) \) and argon \( (M = 40) \).

The thermal conductivities of gases at 1 atm pressure are listed in Table A-16. However, they can also be used at pressures other than 1 atm, since the thermal conductivity of gases is independent of pressure in a wide range of pressures encountered in practice.

The mechanism of heat conduction in a liquid is complicated by the fact that the molecules are more closely spaced, and they exert a stronger intermolecular force field. The thermal conductivities of liquids usually lie between those

### Table 1–1

<table>
<thead>
<tr>
<th>Material</th>
<th>( k ), W/m \cdot °C*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diamond</td>
<td>2300</td>
</tr>
<tr>
<td>Silver</td>
<td>429</td>
</tr>
<tr>
<td>Copper</td>
<td>401</td>
</tr>
<tr>
<td>Gold</td>
<td>317</td>
</tr>
<tr>
<td>Aluminum</td>
<td>237</td>
</tr>
<tr>
<td>Iron</td>
<td>80.2</td>
</tr>
<tr>
<td>Mercury (l)</td>
<td>8.54</td>
</tr>
<tr>
<td>Glass</td>
<td>0.78</td>
</tr>
<tr>
<td>Brick</td>
<td>0.72</td>
</tr>
<tr>
<td>Water (l)</td>
<td>0.613</td>
</tr>
<tr>
<td>Human skin</td>
<td>0.37</td>
</tr>
<tr>
<td>Wood (oak)</td>
<td>0.17</td>
</tr>
<tr>
<td>Helium (g)</td>
<td>0.152</td>
</tr>
<tr>
<td>Soft rubber</td>
<td>0.13</td>
</tr>
<tr>
<td>Glass fiber</td>
<td>0.043</td>
</tr>
<tr>
<td>Air (g)</td>
<td>0.026</td>
</tr>
<tr>
<td>Urethane, rigid foam</td>
<td>0.026</td>
</tr>
</tbody>
</table>

*Multiply by 0.5778 to convert to Btu/h · ft · °F.
of solids and gases. The thermal conductivity of a substance is normally highest in the solid phase and lowest in the gas phase. Unlike gases, the thermal conductivities of most liquids decrease with increasing temperature, with water being a notable exception. Like gases, the conductivity of liquids decreases with increasing molar mass. Liquid metals such as mercury and sodium have high thermal conductivities and are very suitable for use in applications where a high heat transfer rate to a liquid is desired, as in nuclear power plants.

In solids, heat conduction is due to two effects: the lattice vibrational waves induced by the vibrational motions of the molecules positioned at relatively fixed positions in a periodic manner called a lattice, and the energy transported via the free flow of electrons in the solid (Fig. 1–27). The thermal conductivity of a solid is obtained by adding the lattice and electronic components. The relatively high thermal conductivities of pure metals are primarily due to the electronic component. The lattice component of thermal conductivity strongly depends on the way the molecules are arranged. For example, diamond, which is a highly ordered crystalline solid, has the highest known thermal conductivity at room temperature.

Unlike metals, which are good electrical and heat conductors, crystalline solids such as diamond and semiconductors such as silicon are good heat conductors but poor electrical conductors. As a result, such materials find widespread use in the electronics industry. Despite their higher price, diamond heat sinks are used in the cooling of sensitive electronic components because of the
excellent thermal conductivity of diamond. Silicon oils and gaskets are commonly used in the packaging of electronic components because they provide both good thermal contact and good electrical insulation.

Pure metals have high thermal conductivities, and one would think that metal alloys should also have high conductivities. One would expect an alloy made of two metals of thermal conductivities \( k_1 \) and \( k_2 \) to have a conductivity \( k \) between \( k_1 \) and \( k_2 \). But this turns out not to be the case. The thermal conductivity of an alloy of two metals is usually much lower than that of either metal, as shown in Table 1–2. Even small amounts in a pure metal of “foreign” molecules that are good conductors themselves seriously disrupt the flow of heat in that metal. For example, the thermal conductivity of steel containing just 1 percent of chrome is 62 W/m \( \cdot \) °C, while the thermal conductivities of iron and chromium are 83 and 95 W/m \( \cdot \) °C, respectively.

The thermal conductivities of materials vary with temperature (Table 1–3). The variation of thermal conductivity over certain temperature ranges is negligible for some materials, but significant for others, as shown in Fig. 1–28. The thermal conductivities of certain solids exhibit dramatic increases at temperatures near absolute zero, when these solids become superconductors. For example, the conductivity of copper reaches a maximum value of about 20,000 W/m \( \cdot \) °C at 20 K, which is about 50 times the conductivity at room temperature. The thermal conductivities and other thermal properties of various materials are given in Tables A-3 to A-16.

![FIGURE 1–28](Image)

The variation of the thermal conductivity of various solids, liquids, and gases with temperature (from White, Ref. 10).

### TABLE 1–2

<table>
<thead>
<tr>
<th>Pure metal or alloy</th>
<th>( k ), W/m ( \cdot ) °C, at 300 K</th>
</tr>
</thead>
<tbody>
<tr>
<td>Copper</td>
<td>401</td>
</tr>
<tr>
<td>Nickel</td>
<td>91</td>
</tr>
<tr>
<td>Constantan (55% Cu, 45% Ni)</td>
<td>23</td>
</tr>
<tr>
<td>Copper</td>
<td>401</td>
</tr>
<tr>
<td>Aluminum</td>
<td>237</td>
</tr>
<tr>
<td>Commercial bronze</td>
<td>52</td>
</tr>
</tbody>
</table>

### TABLE 1–3

<table>
<thead>
<tr>
<th>( T ), K</th>
<th>Copper</th>
<th>Aluminum</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>482</td>
<td>302</td>
</tr>
<tr>
<td>200</td>
<td>413</td>
<td>237</td>
</tr>
<tr>
<td>300</td>
<td>401</td>
<td>237</td>
</tr>
<tr>
<td>400</td>
<td>393</td>
<td>240</td>
</tr>
<tr>
<td>500</td>
<td>379</td>
<td>231</td>
</tr>
<tr>
<td>600</td>
<td>366</td>
<td>218</td>
</tr>
</tbody>
</table>

The thermal conductivity of an alloy is usually much lower than the thermal conductivity of either metal of which it is composed.
The temperature dependence of thermal conductivity causes considerable complexity in conduction analysis. Therefore, it is common practice to evaluate the thermal conductivity \( k \) at the average temperature and treat it as a constant in calculations.

In heat transfer analysis, a material is normally assumed to be isotropic; that is, to have uniform properties in all directions. This assumption is realistic for most materials, except those that exhibit different structural characteristics in different directions, such as laminated composite materials and wood. The thermal conductivity of wood across the grain, for example, is different than that parallel to the grain.

**Thermal Diffusivity**

The product \( \rho C_p \), which is frequently encountered in heat transfer analysis, is called the heat capacity of a material. Both the specific heat \( C_p \) and the heat capacity \( \rho C_p \) represent the heat storage capability of a material. But \( C_p \) expresses it per unit mass whereas \( \rho C_p \) expresses it per unit volume, as can be noticed from their units J/kg \( \cdot \) °C and J/m\(^3\) \( \cdot \) °C, respectively.

Another material property that appears in the transient heat conduction analysis is the thermal diffusivity, which represents how fast heat diffuses through a material and is defined as

\[
\alpha = \frac{\text{Heat conducted}}{\text{Heat stored}} = \frac{k}{\rho C_p} \quad (\text{m}^2/\text{s})
\]

\( (1-23) \)

Note that the thermal conductivity \( k \) represents how well a material conducts heat, and the heat capacity \( \rho C_p \) represents how much energy a material stores per unit volume. Therefore, the thermal diffusivity of a material can be viewed as the ratio of the heat conducted through the material to the heat stored per unit volume. A material that has a high thermal conductivity or a low heat capacity will obviously have a large thermal diffusivity. The larger the thermal diffusivity, the faster the propagation of heat into the medium. A small value of thermal diffusivity means that heat is mostly absorbed by the material and a small amount of heat will be conducted further.

The thermal diffusivities of some common materials at 20°C are given in Table 1–4. Note that the thermal diffusivity ranges from \( \alpha = 0.14 \times 10^{-6} \text{ m}^2/\text{s} \) for water to \( 174 \times 10^{-6} \text{ m}^2/\text{s} \) for silver, which is a difference of more than a thousand times. Also note that the thermal diffusivities of beef and water are the same. This is not surprising, since meat as well as fresh vegetables and fruits are mostly water, and thus they possess the thermal properties of water.

**TABLE 1–4**

<table>
<thead>
<tr>
<th>Material</th>
<th>( \alpha, \text{ m}^2/\text{s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Silver</td>
<td>( 149 \times 10^{-6} )</td>
</tr>
<tr>
<td>Gold</td>
<td>( 127 \times 10^{-6} )</td>
</tr>
<tr>
<td>Copper</td>
<td>( 113 \times 10^{-6} )</td>
</tr>
<tr>
<td>Aluminum</td>
<td>( 97.5 \times 10^{-6} )</td>
</tr>
<tr>
<td>Iron</td>
<td>( 22.8 \times 10^{-6} )</td>
</tr>
<tr>
<td>Mercury (l)</td>
<td>( 4.7 \times 10^{-6} )</td>
</tr>
<tr>
<td>Marble</td>
<td>( 1.2 \times 10^{-6} )</td>
</tr>
<tr>
<td>Ice</td>
<td>( 1.2 \times 10^{-6} )</td>
</tr>
<tr>
<td>Concrete</td>
<td>( 0.75 \times 10^{-6} )</td>
</tr>
<tr>
<td>Brick</td>
<td>( 0.52 \times 10^{-6} )</td>
</tr>
<tr>
<td>Heavy soil (dry)</td>
<td>( 0.52 \times 10^{-6} )</td>
</tr>
<tr>
<td>Glass</td>
<td>( 0.34 \times 10^{-6} )</td>
</tr>
<tr>
<td>Glass wool</td>
<td>( 0.23 \times 10^{-6} )</td>
</tr>
<tr>
<td>Water (l)</td>
<td>( 0.14 \times 10^{-6} )</td>
</tr>
<tr>
<td>Beef</td>
<td>( 0.14 \times 10^{-6} )</td>
</tr>
<tr>
<td>Wood (oak)</td>
<td>( 0.13 \times 10^{-6} )</td>
</tr>
</tbody>
</table>

*Multiply by 10.76 to convert to ft\(^2\)/s.

**EXAMPLE 1–6 Measuring the Thermal Conductivity of a Material**

A common way of measuring the thermal conductivity of a material is to sandwich an electric thermafoil heater between two identical samples of the material, as shown in Fig. 1–29. The thickness of the resistance heater, including its cover, which is made of thin silicon rubber, is usually less than 0.5 mm. A circulating fluid such as tap water keeps the exposed ends of the samples at constant temperature. The lateral surfaces of the samples are well insulated to ensure that heat transfer through the samples is one-dimensional. Two thermocouples are embedded into each sample some distance \( L \) apart, and a
diff

differential thermometer reads the temperature drop $\Delta T$ across this distance along each sample. When steady operating conditions are reached, the total rate of heat transfer through both samples becomes equal to the electric power drawn by the heater, which is determined by multiplying the electric current by the voltage.

In a certain experiment, cylindrical samples of diameter 5 cm and length 10 cm are used. The two thermocouples in each sample are placed 3 cm apart. After initial transients, the electric heater is observed to draw 0.4 A at 110 V, and both differential thermometers read a temperature difference of 15°C. Determine the thermal conductivity of the sample.

SOLUTION The thermal conductivity of a material is to be determined by ensuring one-dimensional heat conduction, and by measuring temperatures when steady operating conditions are reached.

Assumptions 1 Steady operating conditions exist since the temperature readings do not change with time. 2 Heat losses through the lateral surfaces of the apparatus are negligible since those surfaces are well insulated, and thus the entire heat generated by the heater is conducted through the samples. 3 The apparatus possesses thermal symmetry.

Analysis The electrical power consumed by the resistance heater and converted to heat is

$$
\dot{W}_e = VI = (110 \text{ V})(0.4 \text{ A}) = 44 \text{ W}
$$

The rate of heat flow through each sample is

$$
\dot{Q} = \frac{1}{2} \dot{W}_e = \frac{1}{2} \times (44 \text{ W}) = 22 \text{ W}
$$

since only half of the heat generated will flow through each sample because of symmetry. Reading the same temperature difference across the same distance in each sample also confirms that the apparatus possesses thermal symmetry.

The heat transfer area is the area normal to the direction of heat flow, which is the cross-sectional area of the cylinder in this case:

$$
A = \frac{1}{4} \pi D^2 = \frac{1}{4} \pi (0.05 \text{ m})^2 = 0.00196 \text{ m}^2
$$

Noting that the temperature drops by 15°C within 3 cm in the direction of heat flow, the thermal conductivity of the sample is determined to be

$$
\dot{Q} = kA \frac{\Delta T}{L} \quad \rightarrow \quad k = \frac{\dot{Q}L}{A \Delta T} = \frac{(22 \text{ W})(0.03 \text{ m})}{(0.00196 \text{ m}^2)(15^\circ \text{C})} = 22.4 \text{ W/m} \cdot ^\circ \text{C}
$$

Discussion Perhaps you are wondering if we really need to use two samples in the apparatus, since the measurements on the second sample do not give any additional information. It seems like we can replace the second sample by insulation. Indeed, we do not need the second sample; however, it enables us to verify the temperature measurements on the first sample and provides thermal symmetry, which reduces experimental error.

EXAMPLE 1–7 Conversion between SI and English Units

An engineer who is working on the heat transfer analysis of a brick building in English units needs the thermal conductivity of brick. But the only value he can
Convection is the mode of energy transfer between a solid surface and the adjacent liquid or gas that is in motion, and it involves the combined effects of conduction and fluid motion. The faster the fluid motion, the greater the convection heat transfer. In the absence of any bulk fluid motion, heat transfer between a solid surface and the adjacent fluid is by pure conduction. The presence of bulk motion of the fluid enhances the heat transfer between the solid surface and the fluid, but it also complicates the determination of heat transfer rates.

**SOLUTION** The situation this engineer is facing is not unique, and most engineers often find themselves in a similar position. A person must be very careful during unit conversion not to fall into some common pitfalls and to avoid some costly mistakes. Although unit conversion is a simple process, it requires utmost care and careful reasoning.

The conversion factors for W and m are straightforward and are given in conversion tables to be

\[
1 \text{ W} = 3.41214 \text{ Btu/h} \\
1 \text{ m} = 3.2808 \text{ ft}
\]

But the conversion of °C into °F is not so simple, and it can be a source of error if one is not careful. Perhaps the first thought that comes to mind is to replace °C by \((°F - 32)/1.8\) since \(T(°C) = \left(T(°F) - 32\right)/1.8\). But this will be wrong since the °C in the unit W/m · °C represents per °C change in temperature. Noting that 1°C change in temperature corresponds to 1.8°F, the proper conversion factor to be used is

\[1°C = 1.8°F\]

Substituting, we get

\[
1 \text{ W/m · °C} = \frac{3.41214 \text{ Btu/h}}{(3.2808 \text{ ft})(1.8°F)} = 0.5778 \text{ Btu/h · ft · °F}
\]

which is the desired conversion factor. Therefore, the thermal conductivity of the brick in English units is

\[
k_{\text{brick}} = 0.72 \text{ W/m · °C} \\
= 0.72 \times (0.5778 \text{ Btu/h · ft · °F}) \\
= 0.42 \text{ Btu/h · ft · °F}
\]

**Discussion** Note that the thermal conductivity value of a material in English units is about half that in SI units (Fig. 1–30). Also note that we rounded the result to two significant digits (the same number in the original value) since expressing the result in more significant digits (such as 0.4160 instead of 0.42) would falsely imply a more accurate value than the original one.
Consider the cooling of a hot block by blowing cool air over its top surface (Fig. 1–31). Energy is first transferred to the air layer adjacent to the block by conduction. This energy is then carried away from the surface by convection, that is, by the combined effects of conduction within the air that is due to random motion of air molecules and the bulk or macroscopic motion of the air that removes the heated air near the surface and replaces it by the cooler air.

Convection is called **forced convection** if the fluid is forced to flow over the surface by external means such as a fan, pump, or the wind. In contrast, convection is called **natural** (or **free**) convection if the fluid motion is caused by buoyancy forces that are induced by density differences due to the variation of temperature in the fluid (Fig. 1–32). For example, in the absence of a fan, heat transfer from the surface of the hot block in Fig. 1–31 will be by natural convection since any motion in the air in this case will be due to the rise of the warmer (and thus lighter) air near the surface and the fall of the cooler (and thus heavier) air to fill its place. Heat transfer between the block and the surrounding air will be by conduction if the temperature difference between the air and the block is not large enough to overcome the resistance of air to movement and thus to initiate natural convection currents.

Heat transfer processes that involve **change of phase** of a fluid are also considered to be convection because of the fluid motion induced during the process, such as the rise of the vapor bubbles during boiling or the fall of the liquid droplets during condensation.

Despite the complexity of convection, the rate of **convection heat transfer** is observed to be proportional to the temperature difference, and is conveniently expressed by **Newton’s law of cooling** as

\[
\dot{Q}_{\text{conv}} = hA_s(T_s - T_e) \quad (\text{W})
\]

where \( h \) is the convection heat transfer coefficient in W/m\(^2\)·°C or Btu/h·ft\(^2\)·°F, \( A_s \) is the surface area through which convection heat transfer takes place, \( T_s \) is the surface temperature, and \( T_e \) is the temperature of the fluid sufficiently far from the surface. Note that at the surface, the fluid temperature equals the surface temperature of the solid.

The convection heat transfer coefficient \( h \) is not a property of the fluid. It is an experimentally determined parameter whose value depends on all the variables influencing convection such as the surface geometry, the nature of fluid motion, the properties of the fluid, and the bulk fluid velocity. Typical values of \( h \) are given in Table 1–5.

Some people do not consider convection to be a fundamental mechanism of heat transfer since it is essentially heat conduction in the presence of fluid motion. But we still need to give this combined phenomenon a name, unless we are willing to keep referring to it as “conduction with fluid motion.” Thus, it is practical to recognize convection as a separate heat transfer mechanism despite the valid arguments to the contrary.

**TABLE 1–5**

Typical values of convection heat transfer coefficient

<table>
<thead>
<tr>
<th>Type of convection</th>
<th>( h ), W/m(^2)·°C*</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free convection of gases</td>
<td>2–25</td>
</tr>
<tr>
<td>Free convection of liquids</td>
<td>10–1000</td>
</tr>
<tr>
<td>Forced convection of gases</td>
<td>25–250</td>
</tr>
<tr>
<td>Forced convection of liquids</td>
<td>50–20,000</td>
</tr>
<tr>
<td>Boiling and condensation</td>
<td>2500–100,000</td>
</tr>
</tbody>
</table>

*Multiply by 0.176 to convert to Btu/h·ft\(^2\)·°F.

**EXAMPLE 1–8  Measuring Convection Heat Transfer Coefficient**

A 2-m-long, 0.3-cm-diameter electrical wire extends across a room at 15°C, as shown in Fig. 1–33. Heat is generated in the wire as a result of resistance heating, and the surface temperature of the wire is measured to be 152°C in steady...
Radiation is the energy emitted by matter in the form of electromagnetic waves (or photons) as a result of the changes in the electronic configurations of the atoms or molecules. Unlike conduction and convection, the transfer of energy by radiation does not require the presence of an intervening medium. In fact, energy transfer by radiation is fastest (at the speed of light) and it suffers no attenuation in a vacuum. This is how the energy of the sun reaches the earth.

In heat transfer studies we are interested in thermal radiation, which is the form of radiation emitted by bodies because of their temperature. It differs from other forms of electromagnetic radiation such as x-rays, gamma rays, microwaves, radio waves, and television waves that are not related to temperature. All bodies at a temperature above absolute zero emit thermal radiation.

Radiation is a volumetric phenomenon, and all solids, liquids, and gases emit, absorb, or transmit radiation to varying degrees. However, radiation is
usually considered to be a surface phenomenon for solids that are opaque to thermal radiation such as metals, wood, and rocks since the radiation emitted by the interior regions of such material can never reach the surface, and the radiation incident on such bodies is usually absorbed within a few microns from the surface.

The maximum rate of radiation that can be emitted from a surface at an absolute temperature \( T_s \) (in K or R) is given by the Stefan–Boltzmann law as

\[
\dot{Q}_{\text{emit, max}} = \sigma A T_s^4 \quad \text{(W)}
\]

where \( \sigma = 5.67 \times 10^{-8} \, \text{W/m}^2 \cdot \text{K}^4 \) or 0.1714 \( \times 10^{-8} \) Btu/h \cdot ft\(^2\) \cdot R\(^4\) is the Stefan–Boltzmann constant. The idealized surface that emits radiation at this maximum rate is called a blackbody, and the radiation emitted by a blackbody is called blackbody radiation (Fig. 1–34). The radiation emitted by all real surfaces is less than the radiation emitted by a blackbody at the same temperature, and is expressed as

\[
\dot{Q}_{\text{emit}} = \varepsilon \sigma A T_s^4 \quad \text{(W)}
\]

where \( \varepsilon \) is the emissivity of the surface. The property emissivity, whose value is in the range \( 0 \leq \varepsilon \leq 1 \), is a measure of how closely a surface approximates a blackbody for which \( \varepsilon = 1 \). The emissivities of some surfaces are given in Table 1–6.

Another important radiation property of a surface is its absorptivity \( \alpha \), which is the fraction of the radiation energy incident on a surface that is absorbed by the surface. Like emissivity, its value is in the range \( 0 \leq \alpha \leq 1 \). A blackbody absorbs the entire radiation incident on it. That is, a blackbody is a perfect absorber (\( \alpha = 1 \)) as it is a perfect emitter.

In general, both \( \varepsilon \) and \( \alpha \) of a surface depend on the temperature and the wavelength of the radiation. Kirchhoff’s law of radiation states that the emissivity and the absorptivity of a surface at a given temperature and wavelength are equal. In many practical applications, the surface temperature and the temperature of the source of incident radiation are of the same order of magnitude, and the average absorptivity of a surface is taken to be equal to its average emissivity. The rate at which a surface absorbs radiation is determined from (Fig. 1–35)

\[
\dot{Q}_{\text{abs}} = \alpha \dot{Q}_{\text{incident}} \quad \text{(W)}
\]

where \( \dot{Q}_{\text{incident}} \) is the rate at which radiation is incident on the surface and \( \alpha \) is the absorptivity of the surface. For opaque (nontransparent) surfaces, the portion of incident radiation not absorbed by the surface is reflected back.

The difference between the rates of radiation emitted by the surface and the radiation absorbed is the net radiation heat transfer. If the rate of radiation absorption is greater than the rate of radiation emission, the surface is said to be gaining energy by radiation. Otherwise, the surface is said to be losing energy by radiation. In general, the determination of the net rate of heat transfer by radiation between two surfaces is a complicated matter since it depends on the properties of the surfaces, their orientation relative to each other, and the interaction of the medium between the surfaces with radiation.
When a surface of emissivity ε and surface area \( A_s \) at an absolute temperature \( T_s \) is completely enclosed by a much larger (or black) surface at absolute temperature \( T_{surr} \) separated by a gas (such as air) that does not intervene with radiation, the net rate of radiation heat transfer between these two surfaces is given by (Fig. 1–36)

\[
Q_{\text{rad}} = εσA_s (T_s^4 - T_{surr}^4) \quad (W)
\]

In this special case, the emissivity and the surface area of the surrounding surface do not have any effect on the net radiation heat transfer.

Radiation heat transfer to or from a surface surrounded by a gas such as air occurs parallel to conduction (or convection, if there is bulk gas motion) between the surface and the gas. Thus the total heat transfer is determined by adding the contributions of both heat transfer mechanisms. For simplicity and convenience, this is often done by defining a combined heat transfer coefficient \( h_{\text{combined}} \) that includes the effects of both convection and radiation. Then the total heat transfer rate to or from a surface by convection and radiation is expressed as

\[
Q_{\text{total}} = h_{\text{combined}}A_s (T_s - T_{surr}) \quad (W)
\]

Note that the combined heat transfer coefficient is essentially a convection heat transfer coefficient modified to include the effects of radiation.

Radiation is usually significant relative to conduction or natural convection, but negligible relative to forced convection. Thus radiation in forced convection applications is usually disregarded, especially when the surfaces involved have low emissivities and low to moderate temperatures.

**EXAMPLE 1–9 Radiation Effect on Thermal Comfort**

It is a common experience to feel “chilly” in winter and “warm” in summer in our homes even when the thermostat setting is kept the same. This is due to the so-called “radiation effect” resulting from radiation heat exchange between our bodies and the surrounding surfaces of the walls and the ceiling.

Consider a person standing in a room maintained at 22°C at all times. The inner surfaces of the walls, floors, and the ceiling of the house are observed to be at an average temperature of 10°C in winter and 25°C in summer. Determine the rate of radiation heat transfer between this person and the surrounding surfaces if the exposed surface area and the average outer surface temperature of the person are 1.4 m² and 30°C, respectively (Fig. 1–37).

**SOLUTION** The rates of radiation heat transfer between a person and the surrounding surfaces at specified temperatures are to be determined in summer and winter.

**Assumptions** 1 Steady operating conditions exist. 2 Heat transfer by convection is not considered. 3 The person is completely surrounded by the interior surfaces of the room. 4 The surrounding surfaces are at a uniform temperature.

**Properties** The emissivity of a person is \( ε = 0.95 \) (Table 1–6).

**Analysis** The net rates of radiation heat transfer from the body to the surrounding walls, ceiling, and floor in winter and summer are

**FIGURE 1–36** Radiation heat transfer between a surface and the surfaces surrounding it.

**FIGURE 1–37** Schematic for Example 1–9.
We mentioned that there are three mechanisms of heat transfer, but not all three can exist simultaneously in a medium. For example, heat transfer is only by conduction in opaque solids, but by conduction and radiation in semitransparent solids. Thus, a solid may involve conduction and radiation but not convection. However, a solid may involve heat transfer by convection and/or radiation on its surfaces exposed to a fluid or other surfaces. For example, the outer surfaces of a cold piece of rock will warm up in a warmer environment as a result of heat gain by convection (from the air) and radiation (from the sun or the warmer surrounding surfaces). But the inner parts of the rock will warm up as this heat is transferred to the inner region of the rock by conduction.

Heat transfer is by conduction and possibly by radiation in a still fluid (no bulk fluid motion) and by convection and radiation in a flowing fluid. In the absence of radiation, heat transfer through a fluid is either by conduction or convection, depending on the presence of any bulk fluid motion. Convection can be viewed as combined conduction and fluid motion, and conduction in a fluid can be viewed as a special case of convection in the absence of any fluid motion (Fig. 1–38).

Thus, when we deal with heat transfer through a fluid, we have either conduction or convection, but not both. Also, gases are practically transparent to radiation, except that some gases are known to absorb radiation strongly at certain wavelengths. Ozone, for example, strongly absorbs ultraviolet radiation. But in most cases, a gas between two solid surfaces does not interfere with radiation and acts effectively as a vacuum. Liquids, on the other hand, are usually strong absorbers of radiation.

Finally, heat transfer through a vacuum is by radiation only since conduction or convection requires the presence of a material medium.

\[
\hat{Q}_{\text{rad, winter}} = \varepsilon_\sigma A_s \left(T_4^{4} - T_{\text{surr, winter}}^{4}\right)
= (0.95)(5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(1.4 \text{ m}^2)
\times [(30 + 273)^4 - (10 + 273)^4] \text{ K}^4
= 152 \text{ W}
\]

and

\[
\hat{Q}_{\text{rad, summer}} = \varepsilon_\sigma A_s \left(T_4^{4} - T_{\text{surr, summer}}^{4}\right)
= (0.95)(5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(1.4 \text{ m}^2)
\times [(30 + 273)^4 - (25 + 273)^4] \text{ K}^4
= 40.9 \text{ W}
\]

Discussion Note that we must use absolute temperatures in radiation calculations. Also, note that the rate of heat loss from the person by radiation is almost four times as large in winter than it is in summer, which explains the “chill” we feel in winter even if the thermostat setting is kept the same.
EXAMPLE 1–10 Heat Loss from a Person

Consider a person standing in a breezy room at 20°C. Determine the total rate of heat transfer from this person if the exposed surface area and the average outer surface temperature of the person are 1.6 m² and 29°C, respectively, and the convection heat transfer coefficient is 6 W/m²·°C (Fig. 1–39).

SOLUTION The total rate of heat transfer from a person by both convection and radiation to the surrounding air and surfaces at specified temperatures is to be determined.

Assumptions
1. Steady operating conditions exist.
2. The person is completely surrounded by the interior surfaces of the room.
3. The surrounding surfaces are at the same temperature as the air in the room.
4. Heat conduction to the floor through the feet is negligible.

Properties
The emissivity of a person is \( \varepsilon = 0.95 \) (Table 1–6).

Analysis
The heat transfer between the person and the air in the room will be by convection (instead of conduction) since it is conceivable that the air in the vicinity of the skin or clothing will warm up and rise as a result of heat transfer from the body, initiating natural convection currents. It appears that the experimentally determined value for the rate of convection heat transfer in this case is 6 W per unit surface area (m²) per unit temperature difference (in K or °C) between the person and the air away from the person. Thus, the rate of convection heat transfer from the person to the air in the room is

\[
\dot{Q}_{\text{conv}} = hA_s(T_s - T_a) \\
= (6 \text{ W/m}^2 \cdot \text{°C})(1.6 \text{ m}^2)(29 - 20)\text{°C} \\
= 86.4 \text{ W}
\]

The person will also lose heat by radiation to the surrounding wall surfaces. We take the temperature of the surfaces of the walls, ceiling, and floor to be equal to the air temperature in this case for simplicity, but we recognize that this does not need to be the case. These surfaces may be at a higher or lower temperature than the average temperature of the room air, depending on the outdoor conditions and the structure of the walls. Considering that air does not intervene with radiation and the person is completely enclosed by the surrounding surfaces, the net rate of radiation heat transfer from the person to the surrounding walls, ceiling, and floor is

\[
\dot{Q}_{\text{rad}} = \varepsilon\sigma A_s (T_s^4 - T_{\text{sur}}^4) \\
= (0.95)(5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(1.6 \text{ m}^2) \\
\times [(29 + 273)^4 - (20 + 273)^4] \text{ K}^4 \\
= 81.7 \text{ W}
\]

Note that we must use absolute temperatures in radiation calculations. Also note that we used the emissivity value for the skin and clothing at room temperature since the emissivity is not expected to change significantly at a slightly higher temperature.

Then the rate of total heat transfer from the body is determined by adding these two quantities:

\[
\dot{Q}_{\text{total}} = \dot{Q}_{\text{conv}} + \dot{Q}_{\text{rad}} = (86.4 + 81.7) \text{ W} = 168.1 \text{ W}
\]
**Discussion** The heat transfer would be much higher if the person were not dressed since the exposed surface temperature would be higher. Thus, an important function of the clothes is to serve as a barrier against heat transfer.

In these calculations, heat transfer through the feet to the floor by conduction, which is usually very small, is neglected. Heat transfer from the skin by perspiration, which is the dominant mode of heat transfer in hot environments, is not considered here.

**EXAMPLE 1–11 Heat Transfer between Two Isothermal Plates**

Consider steady heat transfer between two large parallel plates at constant temperatures of $T_1 = 300$ K and $T_2 = 200$ K that are $L = 1$ cm apart, as shown in Fig. 1–40. Assuming the surfaces to be black (emissivity $\varepsilon = 1$), determine the rate of heat transfer between the plates per unit surface area assuming the gap between the plates is (a) filled with atmospheric air, (b) evacuated, (c) filled with urethane insulation, and (d) filled with superinsulation that has an apparent thermal conductivity of 0.00002 W/m · °C.

**SOLUTION** The total rate of heat transfer between two large parallel plates at specified temperatures is to be determined for four different cases.

**Assumptions** 1 Steady operating conditions exist. 2 There are no natural convection currents in the air between the plates. 3 The surfaces are black and thus $\varepsilon = 1$.

**Properties** The thermal conductivity at the average temperature of 250 K is $k = 0.0219$ W/m · °C for air (Table A-11), 0.026 W/m · °C for urethane insulation (Table A-6), and 0.00002 W/m · °C for the superinsulation.

**Analysis** (a) The rates of conduction and radiation heat transfer between the plates through the air layer are

$$Q_{\text{cond}} = kA \frac{T_1 - T_2}{L} = (0.0219 \text{ W/m} \cdot \text{°C})(1 \text{ m}^2) \frac{(300 - 200) \text{°C}}{0.01 \text{ m}} = 219 \text{ W}$$

and

$$Q_{\text{rad}} = \varepsilon\sigma A(T_1^4 - T_2^4) = (1)(5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(1 \text{ m}^2)(300 \text{ K})^4 - (200 \text{ K})^4 = 368 \text{ W}$$

Therefore,

$$Q_{\text{total}} = Q_{\text{cond}} + Q_{\text{rad}} = 219 + 368 = 587 \text{ W}$$

The heat transfer rate in reality will be higher because of the natural convection currents that are likely to occur in the air space between the plates.

(b) When the air space between the plates is evacuated, there will be no conduction or convection, and the only heat transfer between the plates will be by radiation. Therefore,

$$Q_{\text{total}} = Q_{\text{rad}} = 368 \text{ W}$$

(c) An opaque solid material placed between two plates blocks direct radiation heat transfer between the plates. Also, the thermal conductivity of an insulating material accounts for the radiation heat transfer that may be occurring through
the voids in the insulating material. The rate of heat transfer through the urethane insulation is

\[
\dot{Q}_{\text{total}} = \dot{Q}_{\text{cond}} = kA \frac{T_1 - T_2}{L} (0.026 \text{ W/m} \cdot \text{°C})(1 \text{ m}^2)\frac{(300 - 200)\text{°C}}{0.01 \text{ m}} = 260 \text{ W}
\]

Note that heat transfer through the urethane material is less than the heat transfer through the air determined in (a), although the thermal conductivity of the insulation is higher than that of air. This is because the insulation blocks the radiation whereas air transmits it.

(d) The layers of the superinsulation prevent any direct radiation heat transfer between the plates. However, radiation heat transfer between the sheets of superinsulation does occur, and the apparent thermal conductivity of the superinsulation accounts for this effect. Therefore,

\[
\dot{Q}_{\text{total}} = kA \frac{T_1 - T_2}{L} = (0.00002 \text{ W/m} \cdot \text{°C})(1 \text{ m}^2)\frac{(300 - 200)\text{°C}}{0.01 \text{ m}} = 0.2 \text{ W}
\]

which is \( \frac{1}{1000} \) of the heat transfer through the vacuum. The results of this example are summarized in Fig. 1–41 to put them into perspective.

Discussion This example demonstrates the effectiveness of superinsulations, which are discussed in the next chapter, and explains why they are the insulation of choice in critical applications despite their high cost.

**EXAMPLE 1–12 Heat Transfer in Conventional and Microwave Ovens**

The fast and efficient cooking of microwave ovens made them one of the essential appliances in modern kitchens (Fig. 1–42). Discuss the heat transfer mechanisms associated with the cooking of a chicken in microwave and conventional ovens, and explain why cooking in a microwave oven is more efficient.

**SOLUTION** Food is cooked in a microwave oven by absorbing the electromagnetic radiation energy generated by the microwave tube, called the magnetron.
The radiation emitted by the magnetron is not thermal radiation, since its emission is not due to the temperature of the magnetron; rather, it is due to the conversion of electrical energy into electromagnetic radiation at a specified wavelength. The wavelength of the microwave radiation is such that it is reflected by metal surfaces; transmitted by the cookware made of glass, ceramic, or plastic; and absorbed and converted to internal energy by food (especially the water, sugar, and fat) molecules.

In a microwave oven, the radiation that strikes the chicken is absorbed by the skin of the chicken and the outer parts. As a result, the temperature of the chicken at and near the skin rises. Heat is then conducted toward the inner parts of the chicken from its outer parts. Of course, some of the heat absorbed by the outer surface of the chicken is lost to the air in the oven by convection.

In a conventional oven, the air in the oven is first heated to the desired temperature by the electric or gas heating element. This preheating may take several minutes. The heat is then transferred from the air to the skin of the chicken by natural convection in most ovens or by forced convection in the newer convection ovens that utilize a fan. The air motion in convection ovens increases the convection heat transfer coefficient and thus decreases the cooking time. Heat is then conducted toward the inner parts of the chicken from its outer parts as in microwave ovens.

Microwave ovens replace the slow convection heat transfer process in conventional ovens by the instantaneous radiation heat transfer. As a result, microwave ovens transfer energy to the food at full capacity the moment they are turned on, and thus they cook faster while consuming less energy.

**EXAMPLE 1–13 Heating of a Plate by Solar Energy**

A thin metal plate is insulated on the back and exposed to solar radiation at the front surface (Fig. 1–43). The exposed surface of the plate has an absorptivity of 0.6 for solar radiation. If solar radiation is incident on the plate at a rate of 700 W/m² and the surrounding air temperature is 25°C, determine the surface temperature of the plate when the heat loss by convection and radiation equals the solar energy absorbed by the plate. Assume the combined convection and radiation heat transfer coefficient to be 50 W/m²°C.

**SOLUTION** The back side of the thin metal plate is insulated and the front side is exposed to solar radiation. The surface temperature of the plate is to be determined when it stabilizes.

**Assumptions** 1 Steady operating conditions exist. 2 Heat transfer through the insulated side of the plate is negligible. 3 The heat transfer coefficient remains constant.

**Properties** The solar absorptivity of the plate is given to be \( \alpha = 0.6 \).

**Analysis** The absorptivity of the plate is 0.6, and thus 60 percent of the solar radiation incident on the plate will be absorbed continuously. As a result, the temperature of the plate will rise, and the temperature difference between the plate and the surroundings will increase. This increasing temperature difference will cause the rate of heat loss from the plate to the surroundings to increase. At some point, the rate of heat loss from the plate will equal the rate of solar
1–10 PROBLEM-SOLVING TECHNIQUE

The first step in learning any science is to grasp the fundamentals, and to gain a sound knowledge of it. The next step is to master the fundamentals by putting this knowledge to test. This is done by solving significant real-world problems. Solving such problems, especially complicated ones, requires a systematic approach. By using a step-by-step approach, an engineer can reduce the solution of a complicated problem into the solution of a series of simple problems (Fig. 1-44). When solving a problem, we recommend that you use the following steps zealously as applicable. This will help you avoid some of the common pitfalls associated with problem solving.

Step 1: Problem Statement
In your own words, briefly state the problem, the key information given, and the quantities to be found. This is to make sure that you understand the problem and the objectives before you attempt to solve the problem.

Step 2: Schematic
Draw a realistic sketch of the physical system involved, and list the relevant information on the figure. The sketch does not have to be something elaborate, but it should resemble the actual system and show the key features. Indicate any energy and mass interactions with the surroundings. Listing the given information on the sketch helps one to see the entire problem at once. Also, check for properties that remain constant during a process (such as temperature during an isothermal process), and indicate them on the sketch.

Step 3: Assumptions
State any appropriate assumptions made to simplify the problem to make it possible to obtain a solution. Justify the questionable assumptions. Assume reasonable values for missing quantities that are necessary. For example, in the absence of specific data for atmospheric pressure, it can be taken to be

\[
\hat{E}_{\text{gained}} = \hat{E}_{\text{lost}} \quad \text{or} \quad \alpha A_s q_{\text{incident, solar}} = h_{\text{combined}} A_s (T_s - T_a)
\]

Solving for \( T_s \) and substituting, the plate surface temperature is determined to be

\[
T_s = T_a + \frac{q_{\text{incident, solar}}}{h_{\text{combined}}} = 25^\circ C + \frac{0.6 \times (700 \text{ W/m}^2)}{50 \text{ W/m}^2 \cdot ^\circ C} = 33.4^\circ C
\]

Discussion
Note that the heat losses will prevent the plate temperature from rising above 33.4°C. Also, the combined heat transfer coefficient accounts for the effects of both convection and radiation, and thus it is very convenient to use in heat transfer calculations when its value is known with reasonable accuracy.
1 atm. However, it should be noted in the analysis that the atmospheric pressure decreases with increasing elevation. For example, it drops to 0.83 atm in Denver (elevation 1610 m) (Fig. 1–45).

**Step 4: Physical Laws**
Apply all the relevant basic physical laws and principles (such as the conservation of energy), and reduce them to their simplest form by utilizing the assumptions made. However, the region to which a physical law is applied must be clearly identified first. For example, the heating or cooling of a canned drink is usually analyzed by applying the conservation of energy principle to the entire can.

**Step 5: Properties**
Determine the unknown properties at known states necessary to solve the problem from property relations or tables. List the properties separately, and indicate their source, if applicable.

**Step 6: Calculations**
Substitute the known quantities into the simplified relations and perform the calculations to determine the unknowns. Pay particular attention to the units and unit cancellations, and remember that a dimensional quantity without a unit is meaningless. Also, don’t give a false implication of high accuracy by copying all the digits from the screen of the calculator—round the results to an appropriate number of significant digits.

**Step 7: Reasoning, Verification, and Discussion**
Check to make sure that the results obtained are reasonable and intuitive, and verify the validity of the questionable assumptions. Repeat the calculations that resulted in unreasonable values. For example, insulating a water heater that uses $80 worth of natural gas a year cannot result in savings of $200 a year (Fig. 1–46).

Also, point out the significance of the results, and discuss their implications. State the conclusions that can be drawn from the results, and any recommendations that can be made from them. Emphasize the limitations under which the results are applicable, and caution against any possible misunderstandings and using the results in situations where the underlying assumptions do not apply. For example, if you determined that wrapping a water heater with a $20 insulation jacket will reduce the energy cost by $30 a year, indicate that the insulation will pay for itself from the energy it saves in less than a year. However, also indicate that the analysis does not consider labor costs, and that this will be the case if you install the insulation yourself.

Keep in mind that you present the solutions to your instructors, and any engineering analysis presented to others is a form of communication. Therefore neatness, organization, completeness, and visual appearance are of utmost importance for maximum effectiveness. Besides, neatness also serves as a great checking tool since it is very easy to spot errors and inconsistencies in a neat work. Carelessness and skipping steps to save time often ends up costing more time and unnecessary anxiety.
The approach just described is used in the solved example problems without explicitly stating each step, as well as in the Solutions Manual of this text. For some problems, some of the steps may not be applicable or necessary. However, we cannot overemphasize the importance of a logical and orderly approach to problem solving. Most difficulties encountered while solving a problem are not due to a lack of knowledge; rather, they are due to a lack of coordination. You are strongly encouraged to follow these steps in problem solving until you develop your own approach that works best for you.

A Remark on Significant Digits
In engineering calculations, the information given is not known to more than a certain number of significant digits, usually three digits. Consequently, the results obtained cannot possibly be accurate to more significant digits. Reporting results in more significant digits implies greater accuracy than exists, and it should be avoided.

For example, consider a 3.75-L container filled with gasoline whose density is 0.845 kg/L, and try to determine its mass. Probably the first thought that comes to your mind is to multiply the volume and density to obtain 3.16875 kg for the mass, which falsely implies that the mass determined is accurate to six significant digits. In reality, however, the mass cannot be more accurate than three significant digits since both the volume and the density are accurate to three significant digits only. Therefore, the result should be rounded to three significant digits, and the mass should be reported to be 3.17 kg instead of what appears in the screen of the calculator. The result 3.16875 kg would be correct only if the volume and density were given to be 3.75000 L and 0.845000 kg/L, respectively. The value 3.75 L implies that we are fairly confident that the volume is accurate within ±0.01 L, and it cannot be 3.74 or 3.76 L. However, the volume can be 3.746, 3.750, 3.753, etc., since they all round to 3.75 L (Fig. 1–47). It is more appropriate to retain all the digits during intermediate calculations, and to do the rounding in the final step since this is what a computer will normally do.

When solving problems, we will assume the given information to be accurate to at least three significant digits. Therefore, if the length of a pipe is given to be 40 m, we will assume it to be 40.0 m in order to justify using three significant digits in the final results. You should also keep in mind that all experimentally determined values are subject to measurement errors, and such errors will reflect in the results obtained. For example, if the density of a substance has an uncertainty of 2 percent, then the mass determined using this density value will also have an uncertainty of 2 percent.

You should also be aware that we sometimes knowingly introduce small errors in order to avoid the trouble of searching for more accurate data. For example, when dealing with liquid water, we just use the value of 1000 kg/m³ for density, which is the density value of pure water at 0°C. Using this value at 75°C will result in an error of 2.5 percent since the density at this temperature is 975 kg/m³. The minerals and impurities in the water will introduce additional error. This being the case, you should have no reservation in rounding the final results to a reasonable number of significant digits. Besides, having a few percent uncertainty in the results of engineering analysis is usually the norm, not the exception.
Perhaps you are wondering why we are about to undertake a painstaking study of the fundamentals of heat transfer. After all, almost all such problems we are likely to encounter in practice can be solved using one of several sophisticated software packages readily available in the market today. These software packages not only give the desired numerical results, but also supply the outputs in colorful graphical form for impressive presentations. It is unthinkable to practice engineering today without using some of these packages. This tremendous computing power available to us at the touch of a button is both a blessing and a curse. It certainly enables engineers to solve problems easily and quickly, but it also opens the door for abuses and misinformation. In the hands of poorly educated people, these software packages are as dangerous as sophisticated powerful weapons in the hands of poorly trained soldiers.

Thinking that a person who can use the engineering software packages without proper training on fundamentals can practice engineering is like thinking that a person who can use a wrench can work as a car mechanic. If it were true that the engineering students do not need all these fundamental courses they are taking because practically everything can be done by computers quickly and easily, then it would also be true that the employers would no longer need high-salaried engineers since any person who knows how to use a word-processing program can also learn how to use those software packages. However, the statistics show that the need for engineers is on the rise, not on the decline, despite the availability of these powerful packages.

We should always remember that all the computing power and the engineering software packages available today are just tools, and tools have meaning only in the hands of masters. Having the best word-processing program does not make a person a good writer; it simply makes a good writer a better and more efficient writer. Hand calculators did not eliminate the need to teach our children how to add or subtract, and the sophisticated medical software packages did not take the place of medical school training. Neither will engineering software packages replace the traditional engineering education. They will simply cause a shift in emphasis in the courses from mathematics to physics. That is, more time will be spent in the classroom discussing the physical aspects of the problems in greater detail, and less time on the mechanics of solution procedures.

All these marvelous and powerful tools available today put an extra burden on today’s engineers. They must still have a thorough understanding of the fundamentals, develop a “feel” of the physical phenomena, be able to put the data into proper perspective, and make sound engineering judgments, just like their predecessors. However, they must do it much better, and much faster, using more realistic models because of the powerful tools available today. The engineers in the past had to rely on hand calculations, slide rules, and later hand calculators and computers. Today they rely on software packages. The easy access to such power and the possibility of a simple misunderstanding or misinterpretation causing great damage make it more important today than ever to have a solid training in the fundamentals of engineering. In this text we make an extra effort to put the emphasis on developing an intuitive and physical understanding of natural phenomena instead of on the mathematical details of solution procedures.
Engineering Equation Solver (EES)
EES is a program that solves systems of linear or nonlinear algebraic or differential equations numerically. It has a large library of built-in thermodynamic property functions as well as mathematical functions, and allows the user to supply additional property data. Unlike some software packages, EES does not solve thermodynamic problems; it only solves the equations supplied by the user. Therefore, the user must understand the problem and formulate it by applying any relevant physical laws and relations. EES saves the user considerable time and effort by simply solving the resulting mathematical equations. This makes it possible to attempt significant engineering problems not suitable for hand calculations, and to conduct parametric studies quickly and conveniently. EES is a very powerful yet intuitive program that is very easy to use, as shown in the examples below. The use and capabilities of EES are explained in Appendix 3.

Heat Transfer Tools (HTT)
One software package specifically designed to help bridge the gap between the textbook fundamentals and these powerful software packages is Heat Transfer Tools, which may be ordered “bundled” with this text. The software included in that package was developed for instructional use only and thus is applicable only to fundamental problems in heat transfer. While it does not have the power and functionality of the professional, commercial packages, HTT uses research-grade numerical algorithms behind the scenes and modern graphical user interfaces. Each module is custom designed and applicable to a single, fundamental topic in heat transfer to ensure that almost all time at the computer is spent learning heat transfer. Nomenclature and all inputs and outputs are consistent with those used in this and most other textbooks in the field. In addition, with the capability of testing parameters so readily available, one can quickly gain a physical feel for the effects of all the non-dimensional numbers that arise in heat transfer.

EXAMPLE 1–14 Solving a System of Equations with EES

The difference of two numbers is 4, and the sum of the squares of these two numbers is equal to the sum of the numbers plus 20. Determine these two numbers.

SOLUTION Relations are given for the difference and the sum of the squares of two numbers. They are to be determined.

Analysis We start the EES program by double-clicking on its icon, open a new file, and type the following on the blank screen that appears:

\[
\begin{align*}
x-y &= 4 \\
x^2+y^2 &= x+y+20
\end{align*}
\]

which is an exact mathematical expression of the problem statement with \(x\) and \(y\) denoting the unknown numbers. The solution to this system of two
nonlinear equations with two unknowns is obtained by a single click on the "calculator" symbol on the taskbar. It gives

\[ x = 5 \quad \text{and} \quad y = 1 \]

**Discussion**  Note that all we did is formulate the problem as we would on paper; EES took care of all the mathematical details of solution. Also note that equations can be linear or nonlinear, and they can be entered in any order with unknowns on either side. Friendly equation solvers such as EES allow the user to concentrate on the physics of the problem without worrying about the mathematical complexities associated with the solution of the resulting system of equations.

Throughout the text, problems that are unsuitable for hand calculations and are intended to be solved using EES are indicated by a computer icon.

*TOPIC OF SPECIAL INTEREST*

**Thermal Comfort**

Unlike animals such as a fox or a bear that are born with built-in furs, human beings come into this world with little protection against the harsh environmental conditions (Fig. 1–49). Therefore, we can claim that the search for thermal comfort dates back to the beginning of human history. It is believed that early human beings lived in caves that provided shelter as well as protection from extreme thermal conditions. Probably the first form of heating system used was *open fire*, followed by fire in dwellings through the use of a *chimney* to vent out the combustion gases. The concept of *central heating* dates back to the times of the Romans, who heated homes by utilizing double-floor construction techniques and passing the fire’s fumes through the opening between the two floor layers. The Romans were also the first to use *transparent windows* made of mica or glass to keep the wind and rain out while letting the light in. Wood and coal were the primary energy sources for heating, and oil and candles were used for lighting. The ruins of south-facing houses indicate that the value of *solar heating* was recognized early in the history.

The term *air-conditioning* is usually used in a restricted sense to imply cooling, but in its broad sense it means *to condition* the air to the desired level by heating, cooling, humidifying, dehumidifying, cleaning, and deodorizing. The purpose of the air-conditioning system of a building is to provide *complete thermal comfort* for its occupants. Therefore, we need to understand the thermal aspects of the *human body* in order to design an effective air-conditioning system.

The building blocks of living organisms are *cells*, which resemble miniature factories performing various functions necessary for the survival of organisms. The human body contains about 100 trillion cells with an average diameter of 0.01 mm. In a typical cell, thousands of chemical reactions

*This section can be skipped without a loss in continuity.*
occur every second during which some molecules are broken down and energy is released and some new molecules are formed. The high level of chemical activity in the cells that maintain the human body temperature at a temperature of 37.0°C (98.6°F) while performing the necessary bodily functions is called the metabolism. In simple terms, metabolism refers to the burning of foods such as carbohydrates, fat, and protein. The metabolizable energy content of foods is usually expressed by nutritionists in terms of the capitalized Calorie. One Calorie is equivalent to 1 Cal = 1 kcal = 4.1868 kJ.

The rate of metabolism at the resting state is called the basal metabolic rate, which is the rate of metabolism required to keep a body performing the necessary bodily functions such as breathing and blood circulation at zero external activity level. The metabolic rate can also be interpreted as the energy consumption rate for a body. For an average man (30 years old, 70 kg, 1.73 m high, 1.8 m² surface area), the basal metabolic rate is 84 W. That is, the body is converting chemical energy of the food (or of the body fat if the person had not eaten) into heat at a rate of 84 J/s, which is then dissipated to the surroundings. The metabolic rate increases with the level of activity, and it may exceed 10 times the basal metabolic rate when someone is doing strenuous exercise. That is, two people doing heavy exercising in a room may be supplying more energy to the room than a 1-kW resistance heater (Fig. 1–50). An average man generates heat at a rate of 108 W while reading, writing, typing, or listening to a lecture in a classroom in a seated position. The maximum metabolic rate of an average man is 1250 W at age 20 and 730 at age 70. The corresponding rates for women are about 30 percent lower. Maximum metabolic rates of trained athletes can exceed 2000 W.

Metabolic rates during various activities are given in Table 1–7 per unit body surface area. The surface area of a nude body was given by D. DuBois in 1916 as

$$A_s = 0.202m^{0.425} h^{0.725} \text{ (m}^2\text{)}$$

where \(m\) is the mass of the body in kg and \(h\) is the height in m. Clothing increases the exposed surface area of a person by up to about 50 percent. The metabolic rates given in the table are sufficiently accurate for most purposes, but there is considerable uncertainty at high activity levels. More accurate values can be determined by measuring the rate of respiratory oxygen consumption, which ranges from about 0.25 L/min for an average resting man to more than 2 L/min during extremely heavy work. The entire energy released during metabolism can be assumed to be released as heat (in sensible or latent forms) since the external mechanical work done by the muscles is very small. Besides, the work done during most activities such as walking or riding an exercise bicycle is eventually converted to heat through friction.

The comfort of the human body depends primarily on three environmental factors: the temperature, relative humidity, and air motion. The temperature of the environment is the single most important index of comfort. Extensive research is done on human subjects to determine the “thermal comfort zone” and to identify the conditions under which the body feels


TABLE 1–7
Metabolic rates during various activities (from ASHRAE Handbook of Fundamentals, Ref. 1, Chap. 8, Table 4).

<table>
<thead>
<tr>
<th>Activity</th>
<th>Metabolic rate* W/m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resting:</td>
<td></td>
</tr>
<tr>
<td>Sleeping</td>
<td>40</td>
</tr>
<tr>
<td>Reclining</td>
<td>45</td>
</tr>
<tr>
<td>Seated, quiet</td>
<td>60</td>
</tr>
<tr>
<td>Standing, relaxed</td>
<td>70</td>
</tr>
<tr>
<td>Walking (on the level):</td>
<td></td>
</tr>
<tr>
<td>2 mph (0.89 m/s)</td>
<td>115</td>
</tr>
<tr>
<td>3 mph (1.34 m/s)</td>
<td>150</td>
</tr>
<tr>
<td>4 mph (1.79 m/s)</td>
<td>220</td>
</tr>
<tr>
<td>Office Activities:</td>
<td></td>
</tr>
<tr>
<td>Reading, seated</td>
<td>55</td>
</tr>
<tr>
<td>Writing</td>
<td>60</td>
</tr>
<tr>
<td>Typing</td>
<td>65</td>
</tr>
<tr>
<td>Filing, seated</td>
<td>70</td>
</tr>
<tr>
<td>Filing, standing</td>
<td>80</td>
</tr>
<tr>
<td>Walking about</td>
<td>100</td>
</tr>
<tr>
<td>Lifting/packing</td>
<td>120</td>
</tr>
<tr>
<td>Driving/Flying:</td>
<td></td>
</tr>
<tr>
<td>Car</td>
<td>60–115</td>
</tr>
<tr>
<td>Aircraft, routine</td>
<td>70</td>
</tr>
<tr>
<td>Heavy vehicle</td>
<td>185</td>
</tr>
<tr>
<td>Miscellaneous Occupational Activities:</td>
<td></td>
</tr>
<tr>
<td>Cooking</td>
<td>95–115</td>
</tr>
<tr>
<td>Cleaning house</td>
<td>115–140</td>
</tr>
<tr>
<td>Machine work: Light</td>
<td>115–140</td>
</tr>
<tr>
<td>Heavy</td>
<td>235</td>
</tr>
<tr>
<td>Handling 50-kg bags</td>
<td>235</td>
</tr>
<tr>
<td>Pick and shovel work</td>
<td>235–280</td>
</tr>
<tr>
<td>Miscellaneous Leisure Activities:</td>
<td></td>
</tr>
<tr>
<td>Dancing, social</td>
<td>140–255</td>
</tr>
<tr>
<td>Calisthenics/exercise</td>
<td>175–235</td>
</tr>
<tr>
<td>Tennis, singles</td>
<td>210–270</td>
</tr>
<tr>
<td>Basketball</td>
<td>290–440</td>
</tr>
<tr>
<td>Wrestling, competitive</td>
<td>410–505</td>
</tr>
</tbody>
</table>

*Multiply by 1.8 m² to obtain metabolic rates for an average man. Multiply by 0.3171 to convert to Btu/h · ft².

comfortable in an environment. It has been observed that most normally clothed people resting or doing light work feel comfortable in the operative temperature (roughly, the average temperature of air and surrounding surfaces) range of 23°C to 27°C or 73°F to 80°F (Fig. 1–51). For unclothed people, this range is 29°C to 31°C. Relative humidity also has a considerable effect on comfort since it is a measure of air’s ability to absorb moisture and thus it affects the amount of heat a body can dissipate by evaporation. High relative humidity slows down heat rejection by evaporation, especially at high temperatures, and low relative humidity speeds it up. The desirable level of relative humidity is the broad range of 30 to 70 percent, with 50 percent being the most desirable level. Most people at these conditions feel neither hot nor cold, and the body does not need to activate any of the defense mechanisms to maintain the normal body temperature (Fig. 1–52).

Another factor that has a major effect on thermal comfort is excessive air motion or draft, which causes undesired local cooling of the human body. Draft is identified by many as a most annoying factor in work places, automobiles, and airplanes. Experiencing discomfort by draft is most common among people wearing indoor clothing and doing light sedentary work, and least common among people with high activity levels. The air velocity should be kept below 9 m/min (30 ft/min) in winter and 15 m/min (50 ft/min) in summer to minimize discomfort by draft, especially when the air is cool. A low level of air motion is desirable as it removes the warm, moist air that builds around the body and replaces it with fresh air. Therefore, air motion should be strong enough to remove heat and moisture from the vicinity of the body, but gentle enough to be unnoticed. High speed air motion causes discomfort outdoors as well. For example, an environment at 10°C (50°F) with 48 km/h winds feels as cold as an environment at −7°C (20°F) with 3 km/h winds because of the chilling effect of the air motion (the wind-chill factor).

A comfort system should provide uniform conditions throughout the living space to avoid discomfort caused by nonuniformities such as drafts, asymmetric thermal radiation, hot or cold floors, and vertical temperature stratification. Asymmetric thermal radiation is caused by the cold surfaces of large windows, uninsulated walls, or cold products and the warm surfaces of gas or electric radiant heating panels on the walls or ceiling, solar-heated masonry walls or ceilings, and warm machinery. Asymmetric radiation causes discomfort by exposing different sides of the body to surfaces at different temperatures and thus to different heat loss or gain by radiation. A person whose left side is exposed to a cold window, for example, will feel like heat is being drained from that side of his or her body (Fig. 1–53). For thermal comfort, the radiant temperature asymmetry should not exceed 5°C in the vertical direction and 10°C in the horizontal direction. The unpleasant effect of radiation asymmetry can be minimized by properly sizing and installing heating panels, using double-pane windows, and providing generous insulation at the walls and the roof.

Direct contact with cold or hot floor surfaces also causes localized discomfort in the feet. The temperature of the floor depends on the way it is constructed (being directly on the ground or on top of a heated room, being made of wood or concrete, the use of insulation, etc.) as well as the floor.
covering used such as pads, carpets, rugs, and linoleum. A floor temperature of 23 to 25°C is found to be comfortable to most people. The floor asymmetry loses its significance for people with footwear. An effective and economical way of raising the floor temperature is to use radiant heating panels instead of turning the thermostat up. Another nonuniform condition that causes discomfort is temperature stratification in a room that exposes the head and the feet to different temperatures. For thermal comfort, the temperature difference between the head and foot levels should not exceed 3°C. This effect can be minimized by using destratification fans.

It should be noted that no thermal environment will please everyone. No matter what we do, some people will express some discomfort. The thermal comfort zone is based on a 90 percent acceptance rate. That is, an environment is deemed comfortable if only 10 percent of the people are dissatisfied with it. Metabolism decreases somewhat with age, but it has no effect on the comfort zone. Research indicates that there is no appreciable difference between the environments preferred by old and young people. Experiments also show that men and women prefer almost the same environment. The metabolism rate of women is somewhat lower, but this is compensated by their slightly lower skin temperature and evaporative loss. Also, there is no significant variation in the comfort zone from one part of the world to another and from winter to summer. Therefore, the same thermal comfort conditions can be used throughout the world in any season. Also, people cannot acclimatize themselves to prefer different comfort conditions.

In a cold environment, the rate of heat loss from the body may exceed the rate of metabolic heat generation. Average specific heat of the human body is 3.49 kJ/kg · °C, and thus each 1°C drop in body temperature corresponds to a deficit of 244 kJ in body heat content for an average 70-kg man. A drop of 0.5°C in mean body temperature causes noticeable but acceptable discomfort. A drop of 2.6°C causes extreme discomfort. A sleeping person will wake up when his or her mean body temperature drops by 1.3°C (which normally shows up as a 0.5°C drop in the deep body and 3°C in the skin area). The drop of deep body temperature below 35°C may damage the body temperature regulation mechanism, while a drop below 28°C may be fatal. Sedentary people reported to feel comfortable at a mean skin temperature of 33.3°C, uncomfortably cold at 31°C, shivering cold at 30°C, and extremely cold at 29°C. People doing heavy work reported to feel comfortable at much lower temperatures, which shows that the activity level affects human performance and comfort. The extremities of the body such as hands and feet are most easily affected by cold weather, and their temperature is a better indication of comfort and performance. A hand-skin temperature of 20°C is perceived to be uncomfortably cold, 15°C to be extremely cold, and 5°C to be painfully cold. Useful work can be performed by hands without difficulty as long as the skin temperature of fingers remains above 16°C (ASHRAE Handbook of Fundamentals, Ref. 1, Chapter 8).

The first line of defense of the body against excessive heat loss in a cold environment is to reduce the skin temperature and thus the rate of heat loss from the skin by constricting the veins and decreasing the blood flow to the skin. This measure decreases the temperature of the tissues subjacent to the skin, but maintains the inner body temperature. The next preventive
measure is increasing the rate of metabolic heat generation in the body by shivering, unless the person does it voluntarily by increasing his or her level of activity or puts on additional clothing. Shivering begins slowly in small muscle groups and may double the rate of metabolic heat production of the body at its initial stages. In the extreme case of total body shivering, the rate of heat production may reach six times the resting levels (Fig. 1–54). If this measure also proves inadequate, the deep body temperature starts falling. Body parts furthest away from the core such as the hands and feet are at greatest danger for tissue damage.

In hot environments, the rate of heat loss from the body may drop below the metabolic heat generation rate. This time the body activates the opposite mechanisms. First the body increases the blood flow and thus heat transport to the skin, causing the temperature of the skin and the subjacent tissues to rise and approach the deep body temperature. Under extreme heat conditions, the heart rate may reach 180 beats per minute in order to maintain adequate blood supply to the brain and the skin. At higher heart rates, the volumetric efficiency of the heart drops because of the very short time between the beats to fill the heart with blood, and the blood supply to the skin and more importantly to the brain drops. This causes the person to faint as a result of heat exhaustion. Dehydration makes the problem worse. A similar thing happens when a person working very hard for a long time stops suddenly. The blood that has flooded the skin has difficulty returning to the heart in this case since the relaxed muscles no longer force the blood back to the heart, and thus there is less blood available for pumping to the brain.

The next line of defense is releasing water from sweat glands and resorting to evaporative cooling, unless the person removes some clothing and reduces the activity level (Fig. 1–55). The body can maintain its core temperature at 37 °C in this evaporative cooling mode indefinitely, even in environments at higher temperatures (as high as 200 °C during military endurance tests), if the person drinks plenty of liquids to replenish his or her water reserves and the ambient air is sufficiently dry to allow the sweat to evaporate instead of rolling down the skin. If this measure proves inadequate, the body will have to start absorbing the metabolic heat and the deep body temperature will rise. A person can tolerate a temperature rise of 1.4 °C without major discomfort but may collapse when the temperature rise reaches 2.8 °C. People feel sluggish and their efficiency drops considerably when the core body temperature rises above 39 °C. A core temperature above 41 °C may damage hypothalamic proteins, resulting in cessation
of sweating, increased heat production by shivering, and a heat stroke with irreversible and life-threatening damage. Death can occur above 43°C.

A surface temperature of 46°C causes pain on the skin. Therefore, direct contact with a metal block at this temperature or above is painful. However, a person can stay in a room at 100°C for up to 30 min without any damage or pain on the skin because of the convective resistance at the skin surface and evaporative cooling. We can even put our hands into an oven at 200°C for a short time without getting burned.

Another factor that affects thermal comfort, health, and productivity is ventilation. Fresh outdoor air can be provided to a building naturally by doing nothing, or forcefully by a mechanical ventilation system. In the first case, which is the norm in residential buildings, the necessary ventilation is provided by infiltration through cracks and leaks in the living space and by the opening of the windows and doors. The additional ventilation needed in the bathrooms and kitchens is provided by air vents with dampers or exhaust fans. With this kind of uncontrolled ventilation, however, the fresh air supply will be either too high, wasting energy, or too low, causing poor indoor air quality. But the current practice is not likely to change for residential buildings since there is not a public outcry for energy waste or air quality, and thus it is difficult to justify the cost and complexity of mechanical ventilation systems.

Mechanical ventilation systems are part of any heating and air conditioning system in commercial buildings, providing the necessary amount of fresh outdoor air and distributing it uniformly throughout the building. This is not surprising since many rooms in large commercial buildings have no windows and thus rely on mechanical ventilation. Even the rooms with windows are in the same situation since the windows are tightly sealed and cannot be opened in most buildings. It is not a good idea to oversize the ventilation system just to be on the “safe side” since exhausting the heated or cooled indoor air wastes energy. On the other hand, reducing the ventilation rates below the required minimum to conserve energy should also be avoided so that the indoor air quality can be maintained at the required levels. The minimum fresh air ventilation requirements are listed in Table 1–8. The values are based on controlling the CO₂ and other contaminants with an adequate margin of safety, which requires each person be supplied with at least 7.5 L/s (15 ft³/min) of fresh air.

Another function of the mechanical ventilation system is to clean the air by filtering it as it enters the building. Various types of filters are available for this purpose, depending on the cleanliness requirements and the allowable pressure drop.

**TABLE 1–8**

<table>
<thead>
<tr>
<th>Application</th>
<th>Requirement (per person)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Classrooms, libraries, supermarkets</td>
<td>8 L/s (15 ft³/min)</td>
</tr>
<tr>
<td>Dining rooms, conference rooms, offices</td>
<td>10 L/s (20 ft³/min)</td>
</tr>
<tr>
<td>Hospital rooms</td>
<td>13 L/s (25 ft³/min)</td>
</tr>
<tr>
<td>Hotel rooms (per room)</td>
<td>15 L/s (30 ft³/min)</td>
</tr>
<tr>
<td>Smoking lounges</td>
<td>30 L/s (60 ft³/min)</td>
</tr>
<tr>
<td>Retail stores (per m²)</td>
<td>1.0–1.5 L/s (0.2–0.3 ft³/min)</td>
</tr>
<tr>
<td>Residential buildings</td>
<td>0.35 air change per hour, but not less than 7.5 L/s (or 15 ft³/min) per person</td>
</tr>
</tbody>
</table>
In this chapter, the basics of heat transfer are introduced and discussed. The science of thermodynamics deals with the amount of heat transfer as a system undergoes a process from one equilibrium state to another, whereas the science of heat transfer deals with the rate of heat transfer, which is the main quantity of interest in the design and evaluation of heat transfer equipment. The sum of all forms of energy of a system is called total energy, and it includes the internal, kinetic, and potential energies. The internal energy represents the molecular energy of a system, and it consists of sensible, latent, chemical, and nuclear forms. The sensible and latent forms of internal energy can be transferred from one medium to another as a result of a temperature difference, and are referred to as heat or thermal energy. Thus, heat transfer is the exchange of the sensible and latent forms of internal energy between two mediums as a result of a temperature difference. The amount of heat transferred per unit time is called heat transfer rate and is denoted by \( Q \). The rate of heat transfer per unit area is called heat flux, \( q \).

A system of fixed mass is called a closed system and a system that involves mass transfer across its boundaries is called an open system or control volume. The first law of thermodynamics or the energy balance for any system undergoing any process can be expressed as

\[
E_{\text{in}} - E_{\text{out}} = \Delta E_{\text{system}}
\]

When a stationary closed system involves heat transfer only and no work interactions across its boundary, the energy balance relation reduces to

\[
Q = mC_p\Delta T
\]

where \( Q \) is the amount of net heat transfer to or from the system. When heat is transferred at a constant rate of \( Q \), the amount of heat transfer during a time interval \( \Delta t \) can be determined from \( Q = \dot{Q} \Delta t \).

Under steady conditions and in the absence of any work interactions, the conservation of energy relation for a control volume with one inlet and one exit with negligible changes in kinetic and potential energies can be expressed as

\[
\dot{Q} = mC_p\Delta T
\]

where \( m = \rho V A_s \) is the mass flow rate and \( \dot{Q} \) is the rate of net heat transfer into or out of the control volume.

Heat can be transferred in three different modes: conduction, convection, and radiation. Conduction is the transfer of energy from the more energetic particles of a substance to the adjacent less energetic ones as a result of interactions between the particles, and is expressed by Fourier’s law of heat conduction as

\[
\dot{Q}_{\text{cond}} = -kA\frac{dT}{dx}
\]

where \( k \) is the thermal conductivity of the material, \( A \) is the area normal to the direction of heat transfer, and \( dT/dx \) is the temperature gradient. The magnitude of the rate of heat conduction across a plane layer of thickness \( L \) is given by

\[
\dot{Q}_{\text{cond}} = kA\frac{\Delta T}{L}
\]

where \( \Delta T \) is the temperature difference across the layer.

Convection is the mode of heat transfer between a solid surface and the adjacent liquid or gas that is in motion, and involves the combined effects of conduction and fluid motion. The rate of convection heat transfer is expressed by Newton’s law of cooling as

\[
\dot{Q}_{\text{convection}} = hA_s(T_s - T_{\text{sur}})
\]

where \( h \) is the convection heat transfer coefficient in W/m² · °C or Btu/h · ft² · °F, \( A_s \) is the surface area through which convection heat transfer takes place, \( T_s \) is the surface temperature, and \( T_{\text{sur}} \) is the temperature of the fluid sufficiently far from the surface.

Radiation is the energy emitted by matter in the form of electromagnetic waves (or photons) as a result of the changes in the electronic configurations of the atoms or molecules. The maximum rate of radiation that can be emitted from a surface at an absolute temperature \( T_s \) is given by the Stefan–Boltzmann law as \( \dot{Q}_{\text{emitted}} = \sigma A_sT_s^4 \) where \( \sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4 \) or 0.1714 \text{ Btu/h · ft}^2 · \text{°R} \) is the Stefan–Boltzmann constant.

When a surface of emissivity \( \varepsilon \) and surface area \( A_s \) at an absolute temperature \( T_s \) is completely enclosed by a much larger (or black) surface at absolute temperature \( T_{\text{sur}} \) separated by a gas (such as air) that does not intervene with radiation, the net rate of radiation heat transfer between these two surfaces is given by

\[
\dot{Q}_{\text{rad}} = \varepsilon\sigma A_s(T_s^4 - T_{\text{sur}}^4)
\]

In this case, the emissivity and the surface area of the surrounding surface do not have any effect on the net radiation heat transfer.

The rate at which a surface absorbs radiation is determined from \( \dot{Q}_{\text{absorbed}} = \alpha\dot{Q}_{\text{incident}} \) where \( \dot{Q}_{\text{incident}} \) is the rate at which radiation is incident on the surface and \( \alpha \) is the absorptivity of the surface.
REFERENCES AND SUGGESTED READING


PROBLEMS*

Thermodynamics and Heat Transfer

1–1C What are the mechanisms of energy transfer to a closed system? How is heat transfer distinguished from the other forms of energy transfer?

1–10C How are heat, internal energy, and thermal energy related to each other?

1–11C An ideal gas is heated from 50°C to 80°C (a) at constant volume and (b) at constant pressure. For which case do you think the energy required will be greater? Why?

1–12 A cylindrical resistor element on a circuit board dissipates 0.6 W of power. The resistor is 1.5 cm long, and has a diameter of 0.4 cm. Assuming heat to be transferred uniformly from all surfaces, determine (a) the amount of heat this resistor dissipates during a 24-hour period, (b) the heat flux, and (c) the fraction of heat dissipated from the top and bottom surfaces.

1–13E A logic chip used in a computer dissipates 3 W of power in an environment at 120°F, and has a heat transfer surface area of 0.08 in². Assuming the heat transfer from the surface to be uniform, determine (a) the amount of heat this chip dissipates during an eight-hour work day, in kWh, and (b) the heat flux on the surface of the chip, in W/in².

1–14 Consider a 150-W incandescent lamp. The filament of the lamp is 5 cm long and has a diameter of 0.5 mm. The diameter of the glass bulb of the lamp is 8 cm. Determine the heat flux, in W/m², (a) on the surface of the filament and (b) on the surface of the glass bulb, and (c) calculate how much it will cost per year to keep that lamp on for eight hours a day every day if the unit cost of electricity is $0.08/kWh.

Answers: (a) 1.91 × 10⁴ W/m², (b) 7500 W/m², (c) $35.04/yr

1–15 A 1200-W iron is left on the ironing board with its base exposed to the air. About 90 percent of the heat generated in the iron is dissipated through its base whose surface area is 150 cm², and the remaining 10 percent through other surfaces. Assuming the heat transfer from the surface to be uniform,
determine (a) the amount of heat the iron dissipates during a 2-hour period, in kWh, (b) the heat flux on the surface of the iron base, in W/m², and (c) the total cost of the electrical energy consumed during this 2-hour period. Take the unit cost of electricity to be $0.07/kWh.

1–16 A 15-cm × 20-cm circuit board houses on its surface 120 closely spaced logic chips, each dissipating 0.12 W. If the heat transfer from the back surface of the board is negligible, determine (a) the amount of heat this circuit board dissipates during a 10-hour period, in kWh, and (b) the heat flux on the surface of the circuit board, in W/m².

1–17 A 15-cm-diameter aluminum ball is to be heated from 80°C to an average temperature of 200°C. Taking the average density and specific heat of aluminum in this temperature range to be ρ = 2700 kg/m³ and C_p = 0.90 kJ/kg · °C, respectively, determine the amount of energy that needs to be transferred to the aluminum ball. Answer: 515 kJ

1–18 The average specific heat of the human body is 3.6 kJ/kg · °C. If the body temperature of a 70-kg man rises from 37°C to 39°C during strenuous exercise, determine the increase in the thermal energy content of the body as a result of this rise in body temperature.

1–19 Infiltration of cold air into a warm house during winter through the cracks around doors, windows, and other openings is a major source of energy loss since the cold air that enters needs to be heated to the room temperature. The infiltration is often expressed in terms of ACH (air changes per hour). An ACH of 2 indicates that the entire air in the house is replaced twice every hour by the cold air outside.

Consider an electrically heated house that has a floor space of 200 m² and an average height of 3 m at 1000 m elevation, where the standard atmospheric pressure is 89.6 kPa. The house is maintained at a temperature of 22°C, and the infiltration losses are estimated to amount to 0.7 ACH. Assuming the pressure and the temperature in the house remain constant, determine the amount of energy loss from the house due to infiltration for a day during which the average outdoor temperature is 5°C. Also, determine the cost of this energy loss for that day if the unit cost of electricity in that area is $0.082/kWh.

Answers: 53.8 kWh/day, $4.41/day

1–20 Consider a house with a floor space of 200 m² and an average height of 3 m at sea level, where the standard atmospheric pressure is 101.3 kPa. Initially the house is at a uniform temperature of 10°C. Now the electric heater is turned on, and the heater runs until the air temperature in the house rises to an average value of 22°C. Determine how much heat is absorbed by the air assuming some air escapes through the cracks as the heated air in the house expands at constant pressure. Also, determine the cost of this heat if the unit cost of electricity in that area is $0.075/kWh.

1–21E Consider a 60-gallon water heater that is initially filled with water at 45°F. Determine how much energy needs to be transferred to the water to raise its temperature to 140°F. Take the density and specific heat of water to be 62 lbm/ft³ and 1.0 Btu/lbm · °F, respectively.

The First Law of Thermodynamics

1–22C On a hot summer day, a student turns his fan on when he leaves his room in the morning. When he returns in the evening, will his room be warmer or cooler than the neighboring rooms? Why? Assume all the doors and windows are kept closed.

1–23C Consider two identical rooms, one with a refrigerator in it and the other without one. If all the doors and windows are closed, will the room that contains the refrigerator be cooler or warmer than the other room? Why?

1–24C Define mass and volume flow rates. How are they related to each other?
1–25 Two 800-kg cars moving at a velocity of 90 km/h have a head-on collision on a road. Both cars come to a complete rest after the crash. Assuming all the kinetic energy of cars is converted to thermal energy, determine the average temperature rise of the remains of the cars immediately after the crash. Take the average specific heat of the cars to be 0.45 kJ/kg °C.

1–26 A classroom that normally contains 40 people is to be air-conditioned using window air-conditioning units of 5-kW cooling capacity. A person at rest may be assumed to dissipate heat at a rate of 360 kJ/h. There are 10 lightbulbs in the room, each with a rating of 100 W. The rate of heat transfer to the classroom through the walls and the windows is estimated to be 15,000 kJ/h. If the room air is to be maintained at a constant temperature of 21°C, determine the number of window air-conditioning units required. **Answer:** two units

1–27E A rigid tank contains 20 lbm of air at 50 psia and 80°F. The air is now heated until its pressure is doubled. Determine (a) the volume of the tank and (b) the amount of heat transfer. **Answers:** (a) 80 ft³, (b) 2035 Btu

1–28 A 1-m³ rigid tank contains hydrogen at 250 kPa and 420 K. The gas is now cooled until its temperature drops to 300 K. Determine (a) the final pressure in the tank and (b) the amount of heat transfer from the tank.

1–29 A 4-m × 5-m × 6-m room is to be heated by a baseboard resistance heater. It is desired that the resistance heater be able to raise the air temperature in the room from 7°C to 25°C within 15 minutes. Assuming no heat losses from the room and an atmospheric pressure of 100 kPa, determine the required power rating of the resistance heater. Assume constant specific heats at room temperature. **Answer:** 3.01 kW

1–30 A 4-m × 5-m × 7-m room is heated by the radiator of a steam heating system. The steam radiator transfers heat at a rate of 10,000 kJ/h and a 100-W fan is used to distribute the warm air in the room. The heat losses from the room are estimated to be at a rate of about 5000 kJ/h. If the initial temperature of the room air is 10°C, determine how long it will take for the air temperature to rise to 20°C. Assume constant specific heats at room temperature.

1–31 A student living in a 4-m × 6-m × 6-m dormitory room turns his 150-W fan on before she leaves her room on a summer day hoping that the room will be cooler when she comes back in the evening. Assuming all the doors and windows are tightly closed and disregarding any heat transfer through the walls and the windows, determine the temperature in the room when she comes back 10 hours later. Use specific heat values at room temperature and assume the room to be at 100 kPa and 15°C in the morning when she leaves. **Answer:** 58.1°C

1–32E A 10-ft³ tank contains oxygen initially at 14.7 psia and 80°F. A paddle wheel within the tank is rotated until the pressure inside rises to 20 psia. During the process 20 Btu of heat is lost to the surroundings. Neglecting the energy stored in the paddle wheel, determine the work done by the paddle wheel.

1–33 A room is heated by a baseboard resistance heater. When the heat losses from the room on a winter day amount to 7000 kJ/h, it is observed that the air temperature in the room remains constant even though the heater operates continuously. Determine the power rating of the heater, in kW.

1–34 A 50-kg mass of copper at 70°C is dropped into an insulated tank containing 80 kg of water at 25°C. Determine the final equilibrium temperature in the tank.

1–35 A 20-kg mass of iron at 100°C is brought into contact with 20 kg of aluminum at 200°C in an insulated enclosure. Determine the final equilibrium temperature of the combined system. **Answer:** 168°C

1–36 An unknown mass of iron at 90°C is dropped into an insulated tank that contains 80 L of water at 20°C. At the same
time, a paddle wheel driven by a 200-W motor is activated to stir the water. Thermal equilibrium is established after 25 minutes with a final temperature of 27°C. Determine the mass of the iron. Neglect the energy stored in the paddle wheel, and take the density of water to be 1000 kg/m³.

Answer: 72.1 kg

1–37E A 90-lbm mass of copper at 160°F and a 50-lbm mass of iron at 200°F are dropped into a tank containing 180 lbm of water at 70°F. If 600 Btu of heat is lost to the surroundings during the process, determine the final equilibrium temperature.

Heating System Passes through an Unheated Area

A hair dryer is basically a duct in which a few layers of electric resistors are placed. A small fan pulls the air in and forces it to flow over the resistors where it is heated. Air enters a 1200-W hair dryer at 100 kPa and 22°C, and leaves at 47°C. The cross-sectional area of the hair dryer at the exit is 60 cm². Neglecting the power consumed by the fan and the heat losses through the walls of the hair dryer, determine (a) the volume flow rate of air at the inlet and (b) the velocity of the air at the exit.

Answers: (a) 0.0404 m³/s, (b) 7.30 m/s

Heat Transfer Mechanisms

1–44C Define thermal conductivity and explain its significance in heat transfer.

1–45C What are the mechanisms of heat transfer? How are they distinguished from each other?

1–46C What is the physical mechanism of heat conduction in a solid, a liquid, and a gas?

1–47C Consider heat transfer through a windowless wall of a house in a winter day. Discuss the parameters that affect the rate of heat conduction through the wall.

1–48C Write down the expressions for the physical laws that govern each mode of heat transfer, and identify the variables involved in each relation.

1–49C How does heat conduction differ from convection?

1–50C Does any of the energy of the sun reach the earth by conduction or convection?

1–51C How does forced convection differ from natural convection?

1–52C Define emissivity and absorptivity. What is Kirchhoff’s law of radiation?

1–53C What is a blackbody? How do real bodies differ from blackbodies?

1–54C Judging from its unit W/m · °C, can we define thermal conductivity of a material as the rate of heat transfer through the material per unit thickness per unit temperature difference? Explain.

1–55C Consider heat loss through the two walls of a house on a winter night. The walls are identical, except that one of them has a tightly fit glass window. Through which wall will the house lose more heat? Explain.

1–56C Which is a better heat conductor, diamond or silver?
1–57C Consider two walls of a house that are identical except that one is made of 10-cm-thick wood, while the other is made of 25-cm-thick brick. Through which wall will the house lose more heat in winter?

1–58C How do the thermal conductivity of gases and liquids vary with temperature?

1–59C Why is the thermal conductivity of superinsulation orders of magnitude lower than the thermal conductivity of ordinary insulation?

1–60C Why do we characterize the heat conduction ability of insulators in terms of their apparent thermal conductivity instead of the ordinary thermal conductivity?

1–61C Consider an alloy of two metals whose thermal conductivities are \( k_1 \) and \( k_2 \). Will the thermal conductivity of the alloy be less than \( k_1 \), greater than \( k_2 \), or between \( k_1 \) and \( k_2 \)?

1–62 The inner and outer surfaces of a 5-m \( \times \) 6-m brick wall of thickness 30 cm and thermal conductivity 0.69 W/m \( \cdot \) °C are maintained at temperatures of 20°C and 5°C, respectively. Determine the rate of heat transfer through the wall, in W.

Answer: 1035 W

1–63 The inner and outer surfaces of a 0.5-cm-thick 2-m \( \times \) 2-m window glass in winter are 10°C and 3°C, respectively. If the thermal conductivity of the glass is 0.78 W/m \( \cdot \) °C, determine the amount of heat loss, in kJ, through the glass over a period of 5 hours. What would your answer be if the glass were 1 cm thick?

Answers: 78,624 kJ, 39,312 kJ

1–64 Reconsider Problem 1–63. Using EES (or other) software, plot the amount of heat loss through the glass as a function of the window glass thickness in the range of 0.1 cm to 1.0 cm. Discuss the results.

1–65 An aluminum pan whose thermal conductivity is 237 W/m \( \cdot \) °C has a flat bottom with diameter 20 cm and thickness 0.4 cm. Heat is transferred steadily to boiling water in the pan through its bottom at a rate of 800 W. If the inner surface of the bottom of the pan is at 105°C, determine the temperature of the outer surface of the bottom of the pan.

1–66E The north wall of an electrically heated home is 20 ft long, 10 ft high, and 1 ft thick, and is made of brick whose thermal conductivity is \( k = 0.42 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F} \). On a certain winter night, the temperatures of the inner and the outer surfaces of the wall are measured to be at about 62°F and 25°F, respectively, for a period of 8 hours. Determine \( a \) the rate of heat loss through the wall that night and \( b \) the cost of that heat loss to the homeowner if the cost of electricity is $0.07/kWh.

1–67 In a certain experiment, cylindrical samples of diameter 4 cm and length 7 cm are used (see Fig. 1–29). The two thermocouples in each sample are placed 3 cm apart. After initial transients, the electric heater is observed to draw 0.6 A at 110 V, and both differential thermometers read a temperature difference of 10°C. Determine the thermal conductivity of the sample.

Answer: 78.8 W/m \( \cdot \) °C

1–68 One way of measuring the thermal conductivity of a material is to sandwich an electric thermofoil heater between two identical rectangular samples of the material and to heavily insulate the four outer edges, as shown in the figure. Thermocouples attached to the inner and outer surfaces of the samples record the temperatures.

During an experiment, two 0.5-cm-thick samples 10 cm \( \times \) 10 cm in size are used. When steady operation is reached, the heater is observed to draw 35 W of electric power, and the temperature of each sample is observed to drop from 82°C at the inner surface to 74°C at the outer surface. Determine the thermal conductivity of the material at the average temperature.

1–69 Repeat Problem 1–68 for an electric power consumption of 28 W.
1–70 A heat flux meter attached to the inner surface of a 3-cm-thick refrigerator door indicates a heat flux of 25 W/m² through the door. Also, the temperatures of the inner and the outer surfaces of the door are measured to be 7°C and 15°C, respectively. Determine the average thermal conductivity of the refrigerator door.  
**Answer:** 0.0938 W/m·°C

1–71 Consider a person standing in a room maintained at 20°C at all times. The inner surfaces of the walls, floors, and ceiling of the house are observed to be at an average temperature of 12°C in winter and 23°C in summer. Determine the rates of radiation heat transfer between this person and the surrounding surfaces in both summer and winter if the exposed surface area, emissivity, and the average outer surface temperature of the person are 1.6 m², 0.95, and 32°C, respectively.

1–72 Reconsider Problem 1–71. Using EES (or other) software, plot the rate of radiation heat transfer in winter as a function of the temperature of the inner surface of the room in the range of 8°C to 18°C. Discuss the results.

1–73 For heat transfer purposes, a standing man can be modeled as a 30-cm-diameter, 170-cm-long vertical cylinder with both the top and bottom surfaces insulated and with the side surface at an average temperature of 34°C. For a convection heat transfer coefficient of 15 W/m²·°C, determine the rate of heat loss from this man by convection in an environment at 20°C.  
**Answer:** 336 W

1–74 Hot air at 80°C is blown over a 2-m × 4-m flat surface at 30°C. If the average convection heat transfer coefficient is 55 W/m²·°C, determine the rate of heat transfer from the air to the plate, in kW.  
**Answer:** 22 kW

1–75 Reconsider Problem 1–74. Using EES (or other) software, plot the rate of heat transfer as a function of the heat transfer coefficient in the range of 20 W/m²·°C to 100 W/m²·°C. Discuss the results.

1–76 The heat generated in the circuitry on the surface of a silicon chip (k = 130 W/m·°C) is conducted to the ceramic substrate to which it is attached. The chip is 6 mm × 6 mm in size and 0.5 mm thick and dissipates 3 W of power. Disregarding any heat transfer through the 0.5-mm-high side surfaces, determine the temperature difference between the front and back surfaces of the chip in steady operation.

1–77 A 50-cm-long, 800-W electric resistance heating element with diameter 0.5 cm and surface temperature 120°C is immersed in 60 kg of water initially at 20°C. Determine how long it will take for this heater to raise the water temperature to 80°C. Also, determine the convection heat transfer coefficients at the beginning and at the end of the heating process.

1–78 A 5-cm-external-diameter, 10-m-long hot water pipe at 80°C is losing heat to the surrounding air at 5°C by natural convection with a heat transfer coefficient of 25 W/m²·°C. Determine the rate of heat loss from the pipe by natural convection, in W.  
**Answer:** 2945 W

1–79 A hollow spherical iron container with outer diameter 20 cm and thickness 0.4 cm is filled with iced water at 0°C. If the outer surface temperature is 5°C, determine the approximate rate of heat loss from the sphere, in kW, and the rate at which ice melts in the container. The heat from fusion of water is 333.7 kJ/kg.

1–80 Reconsider Problem 1–79. Using EES (or other) software, plot the rate at which ice melts as a function of the container thickness in the range of 0.2 cm to 2.0 cm. Discuss the results.

1–81 Using EES (or other) software, plot the rate at which ice melts as a function of the container thickness in the range of 0.2 cm to 2.0 cm. Discuss the results.

1–82 Two surfaces of a 2-cm-thick plate are maintained at 0°C and 80°C, respectively. If it is determined that heat is transferred through the plate at a rate of 500 W/m², determine its thermal conductivity.

1–83 Four power transistors, each dissipating 15 W, are mounted on a thin vertical aluminum plate 22 cm × 22 cm in size. The heat generated by the transistors is to be dissipated by both surfaces of the plate to the surrounding air at 25°C, which is blown over the plate by a fan. The entire plate can be assumed to be nearly isothermal, and the exposed surface area of the transistor can be taken to be equal to its base area. If the average convection heat transfer coefficient is 25 W/m²·°C, determine the temperature of the aluminum plate. Disregard any radiation effects.
1–84 An ice chest whose outer dimensions are 30 cm \times 40 \text{cm} \times 40 \text{cm} is made of 3-cm-thick Styrofoam (k = 0.033 \text{ W/m} \cdot \text{°C}). Initially, the chest is filled with 40 kg of ice at 0°C, and the inner surface temperature of the ice chest can be taken to be 0°C at all times. The heat of fusion of ice at 0°C is 333.7 \text{ kJ/kg}, and the surrounding ambient air is at 30°C. Disregarding any heat transfer from the 40-cm \times 40-cm base of the ice chest, determine how long it will take for the ice in the chest to melt completely if the outer surfaces of the ice chest are at 8°C.

Answer: 32.7 days

1–85 A transistor with a height of 0.4 cm and a diameter of 0.6 cm is mounted on a circuit board. The transistor is cooled by air flowing over it with an average heat transfer coefficient of 30 \text{ W/m}^2 \cdot \text{°C}. If the air temperature is 55°C and the transistor case temperature is not to exceed 70°C, determine the amount of power this transistor can dissipate safely. Disregard any heat transfer from the transistor base.

1–86 Reconsider Problem 1–85. Using EES (or other) software, plot the amount of power the transistor can dissipate safely as a function of the maximum case temperature in the range of 60°C to 90°C. Discuss the results.

1–87E A 200-ft-long section of a steam pipe whose outer diameter is 4 inches passes through an open space at 50°F. The average temperature of the outer surface of the pipe is measured to be 280°F, and the average heat transfer coefficient on that surface is determined to be 6 \text{ Btu/h} \cdot \text{ft}^2 \cdot \text{°F}. Determine (a) the rate of heat loss from the steam pipe and (b) the annual cost of this energy loss if steam is generated in a natural gas furnace having an efficiency of 86 percent, and the price of natural gas is $0.58/therm (1 therm = 100,000 \text{ Btu}).

Answers: (a) 289,000 \text{ Btu/h}, (b) $17,074/yr

1–88 The boiling temperature of nitrogen at atmospheric pressure at sea level (1 atm) is −196°C. Therefore, nitrogen is commonly used in low temperature scientific studies since the temperature of liquid nitrogen in a tank open to the atmosphere will remain constant at −196°C until the liquid nitrogen in the tank is depleted. Any heat transfer to the tank will result in the evaporation of some liquid nitrogen, which has a heat of vaporization of 198 \text{ kJ/kg} and a density of 810 \text{ kg/m}^3 at 1 atm.

Consider a 4-m-diameter spherical tank initially filled with liquid nitrogen at 1 atm and −196°C. The tank is exposed to 20°C ambient air with a heat transfer coefficient of 25 \text{ W/m}^2 \cdot \text{°C}. The temperature of the thin-shelled spherical tank is observed to be almost the same as the temperature of the nitrogen inside. Disregarding any radiation heat exchange, determine the rate of evaporation of the liquid nitrogen in the tank as a result of the heat transfer from the ambient air.

1–89 Repeat Problem 1–88 for liquid oxygen, which has a boiling temperature of −183°C, a heat of vaporization of 213 \text{ kJ/kg}, and a density of 1140 \text{ kg/m}^3 at 1 atm pressure.

1–90 Reconsider Problem 1–88. Using EES (or other) software, plot the rate of evaporation of liquid nitrogen as a function of the ambient air temperature in the range of 0°C to 35°C. Discuss the results.

1–91 Consider a person whose exposed surface area is 1.7 \text{ m}^2, emissivity is 0.7, and surface temperature is 32°C.
HEAT TRANSFER

Determine the rate of heat loss from that person by radiation in a large room having walls at a temperature of (a) 300 K and (b) 280 K.  

**Answers:** (a) 37.4 W, (b) 169.2 W

1–92 A 0.3-cm-thick, 12-cm-high, and 18-cm-long circuit board houses 80 closely spaced logic chips on one side, each dissipating 0.06 W. The board is impregnated with copper fillings and has an effective thermal conductivity of 16 W/m · °C. All the heat generated in the chips is conducted across the circuit board and is dissipated from the back side of the board to the ambient air. Determine the temperature difference between the two sides of the circuit board.  

**Answer:** 0.042°C

1–93 Consider a sealed 20-cm-high electronic box whose base dimensions are 40 cm × 40 cm placed in a vacuum chamber. The emissivity of the outer surface of the box is 0.95. If the electronic components in the box dissipate a total of 100 W of power and the outer surface temperature of the box is not to exceed 55°C, determine the temperature at which the surrounding surfaces must be kept if this box is to be cooled by radiation alone. Assume the heat transfer from the bottom surface of the box to the stand to be negligible.

![FIGURE P1–93](image)

1–94 Using the conversion factors between W and Btu/h, m and ft, and K and °C, express the Stefan–Boltzmann constant \( \sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4 \) in the English unit Btu/h · ft\(^2\) · °F.

1–95 An engineer who is working on the heat transfer analysis of a house in English units needs the convection heat transfer coefficient on the outer surface of the house. But the only value he can find from his handbooks is 20 W/m\(^2\) · °C, which is in SI units. The engineer does not have a direct conversion factor between the two unit systems for the convection heat transfer coefficient. Using the conversion factors between W and Btu/h, m and ft, and °C and °F, express the given convection heat transfer coefficient in Btu/h · ft\(^2\) · °F.

**Answer:** 3.52 Btu/h · ft\(^2\) · °F

**Simultaneous Heat Transfer Mechanisms**

1–96C Can all three modes of heat transfer occur simultaneously (in parallel) in a medium?

1–97C Can a medium involve (a) conduction and convection, (b) conduction and radiation, or (c) convection and radiation simultaneously? Give examples for the “yes” answers.

1–98C The deep human body temperature of a healthy person remains constant at 37°C while the temperature and the humidity of the environment change with time. Discuss the heat transfer mechanisms between the human body and the environment both in summer and winter, and explain how a person can keep cooler in summer and warmer in winter.

1–99C We often turn the fan on in summer to help us cool. Explain how a fan makes us feel cooler in the summer. Also explain why some people use ceiling fans also in winter.

1–100 Consider a person standing in a room at 23°C. Determine the total rate of heat transfer from this person if the exposed surface area and the skin temperature of the person are 1.7 m\(^2\) and 32°C, respectively, and the convection heat transfer coefficient is 5 W/m\(^2\) · °C. Take the emissivity of the skin and the clothes to be 0.9, and assume the temperature of the inner surfaces of the room to be the same as the air temperature.

**Answer:** 161 W

1–101 Consider steady heat transfer between two large parallel plates at constant temperatures of \( T_1 = 290 \text{ K} \) and \( T_2 = 150 \text{ K} \) that are \( L = 2 \text{ cm} \) apart. Assuming the surfaces to be black (emissivity \( \varepsilon = 1 \)), determine the rate of heat transfer between the plates per unit surface area assuming the gap between the plates is (a) filled with atmospheric air, (b) evacuated, (c) filled with fiberglass insulation, and (d) filled with superinsulation having an apparent thermal conductivity of 0.00015 W/m · °C.

1–102 A 1.4-m-long, 0.2-cm-diameter electrical wire extends across a room that is maintained at 20°C. Heat is generated in the wire as a result of resistance heating, and the surface temperature of the wire is measured to be 240°C in steady operation. Also, the voltage drop and electric current through the wire are measured to be 110 V and 3 A, respectively. Disregarding any heat transfer by radiation, determine the convection heat transfer coefficient for heat transfer between the outer surface of the wire and the air in the room.

**Answer:** 170.5 W/m\(^2\) · °C

![FIGURE P1–102](image)

1–103 Reconsider Problem 1–102. Using EES (or other) software, plot the convection heat transfer coefficient as a function of the wire surface temperature in the range of 100°C to 300°C. Discuss the results.

1–104E A 2-in-diameter spherical ball whose surface is maintained at a temperature of 170°F is suspended in the middle of a room at 70°F. If the convection heat transfer coefficient is 12 Btu/h · ft\(^2\) · °F and the emissivity of the surface is 0.8, determine the total rate of heat transfer from the ball.
A 1000-W iron is left on the iron board with its base exposed to the air at 20°C. The convection heat transfer coefficient between the base surface and the surrounding air is 35 W/m²·°C. If the base has an emissivity of 0.6 and a surface area of 0.02 m², determine the temperature of the base of the iron. Answer: 674°C

The outer surface of a spacecraft in space has an emissivity of 0.8 and a solar absorptivity of 0.3. If solar radiation is incident on the spacecraft at a rate of 950 W/m², determine the rate of heat loss from the collector by convection and radiation during a calm day when the ambient air temperature is 70°F and the effective sky temperature for radiation exchange is 50°F. Take the convection heat transfer coefficient on the exposed surface to be 2.5 Btu/h·ft²·°F.

The outer surface of a spacecraft when the radiation emitted equals the solar energy absorbed.

A 3-m-internal-diameter spherical tank made of 1-cm-thick stainless steel is used to store iced water at 0°C. If the base has an emissivity of 0.6 and a surface area of 0.02 m², determine the temperature of the base exposed to the air at 20°C. The tank

Iron
1000 W
20°C

The roof of a house consists of a 15-cm-thick concrete slab (k = 2 W/m·°C) that is 15 m wide and 20 m long. The emissivity of the outer surface of the roof is 0.9, and the convection heat transfer coefficient on that surface is estimated to be 15 W/m²·°C. The inner surface of the roof is maintained at 15°C. On a clear winter night, the ambient air is reported to be at 10°C while the night sky temperature for radiation heat transfer is 255 K. Considering both radiation and convection heat transfer, determine the outer surface temperature and the rate of heat transfer through the roof.

If the house is heated by a furnace burning natural gas with an efficiency of 85 percent, and the unit cost of natural gas is $0.60/therm (1 therm = 105,500 kJ of energy content), determine the money lost through the roof that night during a 14-hour period.

Consider a flat plate solar collector placed horizontally on the flat roof of a house. The collector is 5 ft wide and 15 ft long, and the average temperature of the exposed surface of the collector is 100°F. The emissivity of the exposed surface of the collector is 0.9. Determine the rate of heat loss from the collector by convection and radiation during a calm day when the ambient air temperature is 70°F and the effective sky temperature for radiation exchange is 50°F. Take the convection heat transfer coefficient on the exposed surface to be 2.5 Btu/h·ft²·°F.

What is the value of the engineering software packages in (a) engineering education and (b) engineering practice?

Determine a positive real root of the following equation using EES:

\[ 2x^3 - 10x^{0.5} - 3x = -3 \]

Solve the following system of two equations with two unknowns using EES:

\[ x^3 - y^2 = 7.75 \]
\[ 3xy + y = 3.5 \]

Solve the following system of three equations with three unknowns using EES:

\[ 2x - y + z = 5 \]
\[ 3x^2 + 2y = z + 2 \]
\[ xy + 2z = 8 \]

Solve the following system of three equations with three unknowns using EES:

\[ xy - z = 1 \]
\[ x - 3y^{0.5} + xz = -2 \]
\[ x + y - z = 2 \]

What is metabolism? What is the range of metabolic rate for an average man? Why are we interested in metabolic rate?
rate of the occupants of a building when we deal with heating and air conditioning?

- **1–116C** Why is the metabolic rate of women, in general, lower than that of men? What is the effect of clothing on the environmental temperature that feels comfortable?

- **1–117C** What is asymmetric thermal radiation? How does it cause thermal discomfort in the occupants of a room?

- **1–118C** How do (a) draft and (b) cold floor surfaces cause discomfort for a room’s occupants?

- **1–119C** What is stratification? Is it likely to occur at places with low or high ceilings? How does it cause thermal discomfort for a room’s occupants? How can stratification be prevented?

- **1–120C** Why is it necessary to ventilate buildings? What is the effect of ventilation on energy consumption for heating in winter and for cooling in summer? Is it a good idea to keep the bathroom fans on all the time? Explain.

### Review Problems

#### 1–121
2.5 kg of liquid water initially at 18°C is to be heated to 96°C in a teapot equipped with a 1200-W electric heating element inside. The teapot is 0.8 kg and has an average specific heat of 0.6 kJ/kg · °C. Taking the specific heat of water to be 4.18 kJ/kg · °C and disregarding any heat loss from the teapot, determine how long it will take for the water to be heated.

#### 1–122
A 4-m-long section of an air heating system of a house passes through an unheated space in the attic. The inner diameter of the circular duct of the heating system is 20 cm. Hot air enters the duct at 100 kPa and 65°C at an average velocity of 3 m/s. The temperature of the air in the duct drops to 60°C as a result of heat loss to the cool space in the attic. Determine the rate of heat loss from the air in the duct to the attic under steady conditions. Also, determine the cost of this heat loss per hour if the house is heated by a natural gas furnace having an efficiency of 82 percent, and the cost of the natural gas in that area is $0.58/therm (1 therm = 105,500 kJ).

*Answers:* 0.488 kJ/s, $0.012/h

#### 1–123
Reconsider Problem 1–122. Using EES (or other) software, plot the cost of the heat loss per hour as a function of the average air velocity in the range of 1 m/s to 10 m/s. Discuss the results.

#### 1–124
Water flows through a shower head steadily at a rate of 10 L/min. An electric resistance heater placed in the water pipe heats the water from 16°C to 43°C. Taking the density of water to be 1 kg/L, determine the electric power input to the heater, in kW.

In an effort to conserve energy, it is proposed to pass the drained warm water at a temperature of 39°C through a heat exchanger to preheat the incoming cold water. If the heat exchanger has an effectiveness of 0.50 (that is, it recovers only half of the energy that can possibly be transferred from the drained water to incoming cold water), determine the electric power input required in this case. If the price of the electric energy is 8.5 ¢/kWh, determine how much money is saved during a 10-minute shower as a result of installing this heat exchanger.

*Answers:* 18.8 kW, 10.8 kW, $0.0113

#### 1–125
It is proposed to have a water heater that consists of an insulated pipe of 5 cm diameter and an electrical resistor inside. Cold water at 15°C enters the heating section steadily at a rate of 18 L/min. If water is to be heated to 50°C, determine (a) the power rating of the resistance heater and (b) the average velocity of the water in the pipe.

#### 1–126
A passive solar house that is losing heat to the outdoors at an average rate of 50,000 kJ/h is maintained at 22°C at all times during a winter night for 10 hours. The house is to be heated by 50 glass containers each containing 20 L of water heated to 80°C during the day by absorbing solar energy. A thermostat-controlled 15-kW back-up electric resistance heater turns on whenever necessary to keep the house at 22°C. (a) How long did the electric heating system run that night? (b) How long would the electric heater have run that night if the house incorporated no solar heating?

*Answers:* (a) 4.77 h, (b) 9.26 h

#### 1–127
It is well known that wind makes the cold air feel much colder as a result of the windchill effect that is due to the increase in the convection heat transfer coefficient with increasing air velocity. The windchill effect is usually expressed in terms of the windchill factor, which is the difference between the actual air temperature and the equivalent calm-air...
temperature. For example, a windchill factor of 20 °C for an actual air temperature of 5 °C means that the windy air at 5 °C feels as cold as the still air at −15 °C. In other words, a person will lose as much heat to air at 5 °C with a windchill factor of 20 °C as he or she would in calm air at −15 °C.

For heat transfer purposes, a standing man can be modeled as a 30-cm-diameter, 170-cm-long vertical cylinder with both the top and bottom surfaces insulated and with the side surface at an average temperature of 34 °C. For a convection heat transfer coefficient of 15 W/m²·°C, determine the rate of heat loss from this man by convection in still air at 20 °C. What would your answer be if the convection heat transfer coefficient is increased to 50 W/m²·°C as a result of winds? What is the windchill factor in this case? Answers: 336 W, 1120 W, 32.7 °C

A thin metal plate is insulated on the back and exposed to solar radiation on the front surface. The exposed surface of the plate has an absorptivity of 0.7 for solar radiation. If solar radiation is incident on the plate at a rate of 700 W/m² and the surrounding air temperature is 10 °C, determine the surface temperature of the plate when the heat loss by convection equals the solar energy absorbed by the plate. Take the convection heat transfer coefficient to be 30 W/m²·°C, and disregard any heat loss by radiation.

1–129 A 4-m × 5-m × 6-m room is to be heated by one ton (1000 kg) of liquid water contained in a tank placed in the room. The room is losing heat to the outside at an average rate of 10,000 kJ/h. The room is initially at 20 °C and 100 kPa, and is maintained at an average temperature of 20 °C at all times. If the hot water is to meet the heating requirements of this room for a 24-hour period, determine the minimum temperature of the water when it is first brought into the room. Assume constant specific heats for both air and water at room temperature. Answer: 77.4 °C

1–130 Consider a 3-m × 3-m × 3-m cubical furnace whose top and side surfaces closely approximate black surfaces at a temperature of 1200 K. The base surface has an emissivity of ε = 0.7, and is maintained at 800 K. Determine the net rate of radiation heat transfer from the top and side surfaces. Answer: 594,400 W

1–131 Consider a refrigerator whose dimensions are 1.8 m × 1.2 m × 0.8 m and whose walls are 3 cm thick. The refrigerator consumes 600 W of power when operating and has a COP of 2.5. It is observed that the motor of the refrigerator remains on for 5 minutes and then is off for 15 minutes periodically. If the average temperatures at the inner and outer surfaces of the refrigerator are 6 °C and 17 °C, respectively, determine the average thermal conductivity of the refrigerator walls. Also, determine the annual cost of operating this refrigerator if the unit cost of electricity is $0.08/kWh.

1–132 A 0.2-L glass of water at 20 °C is to be cooled with ice to 5 °C. Determine how much ice needs to be added to the water, in grams, if the ice is at 0 °C. Also, determine how much water would be needed if the cooling is to be done with cold water at 0 °C. The melting temperature and the heat of fusion of ice at atmospheric pressure are 0 °C and 333.7 kJ/kg, respectively, and the density of water is 1 kg/L.
Heat Transfer

Figure P1–132

Ice, 0°C

Water
0.2 L
20°C

1–133 Reconsider Problem 1–132. Using EES (or other) software, plot the amount of ice that needs to be added to the water as a function of the ice temperature in the range of −24°C to 0°C. Discuss the results.

1–134E In order to cool 1 short ton (2000 lbm) of water at 70°F in a tank, a person pours 160 lbm of ice at 25°F into the water. Determine the final equilibrium temperature in the tank. The melting temperature and the heat of fusion of ice at atmospheric pressure are 32°F and 143.5 Btu/lbm, respectively.

Answer: 56.3°F

1–135 Engine valves ($C_p = 440 \text{ J/kg} \cdot ^\circ\text{C}$ and $\rho = 7840 \text{ kg/m}^3$) are to be heated from 40°C to 800°C in 5 minutes in the heat treatment section of a valve manufacturing facility. The valves have a cylindrical stem with a diameter of 8 mm and a length of 10 cm. The valve head and the stem may be assumed to be of equal surface area, with a total mass of 0.0788 kg. For a single valve, determine (a) the amount of heat transfer, (b) the average rate of heat transfer, and (c) the average heat flux, (d) the number of valves that can be heat treated per day if the heating section can hold 25 valves, and it is used 10 hours per day.

1–136 The hot water needs of a household are met by an electric 60-L hot water tank equipped with a 1.6-kW heating element. The tank is initially filled with hot water at 80°C, and the cold water temperature is 20°C. Someone takes a shower by mixing constant flow rates of hot and cold waters. After a showering period of 8 minutes, the average water temperature in the tank is measured to be 60°C. The heater is kept on during the shower and hot water is replaced by cold water. If the cold water is mixed with the hot water stream at a rate of 0.06 kg/s, determine the flow rate of hot water and the average temperature of mixed water used during the shower.

1–137 Consider a flat plate solar collector placed at the roof of a house. The temperatures at the inner and outer surfaces of glass cover are measured to be 28°C and 25°C, respectively. The glass cover has a surface area of 2.2 m² and a thickness of 0.6 cm and a thermal conductivity of 0.7 W/m · °C. Heat is lost from the outer surface of the cover by convection and radiation with a convection heat transfer coefficient of 10 W/m² · °C and an ambient temperature of 15°C. Determine the fraction of heat lost from the glass cover by radiation.

1–138 The rate of heat loss through a unit surface area of a window per unit temperature difference between the indoors and the outdoors is called the $U$-factor. The value of the $U$-factor ranges from about 1.25 W/m² · °C (or 0.22 Btu/h · ft² · °F) for low-e coated, argon-filled, quadruple-pane windows to 6.25 W/m² · °C (or 1.1 Btu/h · ft² · °F) for a single-pane window with aluminum frames. Determine the range for the rate of heat loss through a 1.2-m × 1.8-m window of a house that is maintained at 20°C when the outdoor air temperature is −8°C.

Figure P1–138

Indoors
20°C

Outdoors
−8°C

1–139 Reconsider Problem 1–138. Using EES (or other) software, plot the rate of heat loss through the window as a function of the $U$-factor. Discuss the results.

Design and Essay Problems

1–140 Write an essay on how microwave ovens work, and explain how they cook much faster than conventional ovens. Discuss whether conventional electric or microwave ovens consume more electricity for the same task.

1–141 Using information from the utility bills for the coldest month last year, estimate the average rate of heat loss from your house for that month. In your analysis, consider the contribution of the internal heat sources such as people, lights, and appliances. Identify the primary sources of heat loss from your house and propose ways of improving the energy efficiency of your house.

1–142 Design a 1200-W electric hair dryer such that the air temperature and velocity in the dryer will not exceed 50°C and 3 ms, respectively.

1–143 Design an electric hot water heater for a family of four in your area. The maximum water temperature in the tank...
and the power consumption are not to exceed 60°C and 4 kW, respectively. There are two showers in the house, and the flow rate of water through each of the shower heads is about 10 L/min. Each family member takes a 5-minute shower every morning. Explain why a hot water tank is necessary, and determine the proper size of the tank for this family.

Conduct this experiment to determine the heat transfer coefficient between an incandescent lightbulb and the surrounding air using a 60-W lightbulb. You will need an indoor–outdoor thermometer, which can be purchased for about $10 in a hardware store, and a metal glue. You will also need a piece of string and a ruler to calculate the surface area of the lightbulb. First, measure the air temperature in the room, and then glue the tip of the thermocouple wire of the thermometer to the glass of the lightbulb. Turn the light on and wait until the temperature reading stabilizes. The temperature reading will give the surface temperature of the lightbulb. Assuming 10 percent of the rated power of the bulb is converted to light, calculate the heat transfer coefficient from Newton’s law of cooling.
Heat transfer has *direction* as well as *magnitude*. The rate of heat conduction in a specified direction is proportional to the *temperature gradient*, which is the change in temperature per unit length in that direction. Heat conduction in a medium, in general, is three-dimensional and time dependent. That is, \( T = T(x, y, z, t) \) and the temperature in a medium varies with position as well as time. Heat conduction in a medium is said to be *steady* when the temperature does not vary with time, and *unsteady* or *transient* when it does. Heat conduction in a medium is said to be *one-dimensional* when conduction is significant in one dimension only and negligible in the other two dimensions, *two-dimensional* when conduction in the third dimension is negligible, and *three-dimensional* when conduction in all dimensions is significant.

We start this chapter with a description of steady, unsteady, and multidimensional heat conduction. Then we derive the differential equation that governs heat conduction in a large plane wall, a long cylinder, and a sphere, and generalize the results to three-dimensional cases in rectangular, cylindrical, and spherical coordinates. Following a discussion of the boundary conditions, we present the formulation of heat conduction problems and their solutions. Finally, we consider heat conduction problems with variable thermal conductivity.

This chapter deals with the theoretical and mathematical aspects of heat conduction, and it can be covered selectively, if desired, without causing a significant loss in continuity. The more practical aspects of heat conduction are covered in the following two chapters.
2–1 INTRODUCTION

In Chapter 1 heat conduction was defined as the transfer of thermal energy from the more energetic particles of a medium to the adjacent less energetic ones. It was stated that conduction can take place in liquids and gases as well as solids provided that there is no bulk motion involved.

Although heat transfer and temperature are closely related, they are of a different nature. Unlike temperature, heat transfer has direction as well as magnitude, and thus it is a vector quantity (Fig. 2–1). Therefore, we must specify both direction and magnitude in order to describe heat transfer completely at a point. For example, saying that the temperature on the inner surface of a wall is 18°C describes the temperature at that location fully. But saying that the heat flux on that surface is 50 W/m² immediately prompts the question “in what direction?” We can answer this question by saying that heat conduction is toward the inside (indicating heat gain) or toward the outside (indicating heat loss).

To avoid such questions, we can work with a coordinate system and indicate direction with plus or minus signs. The generally accepted convention is that heat transfer in the positive direction of a coordinate axis is positive and in the opposite direction it is negative. Therefore, a positive quantity indicates heat transfer in the positive direction and a negative quantity indicates heat transfer in the negative direction (Fig. 2–2).

The driving force for any form of heat transfer is the temperature difference, and the larger the temperature difference, the larger the rate of heat transfer. Some heat transfer problems in engineering require the determination of the temperature distribution (the variation of temperature) throughout the medium in order to calculate some quantities of interest such as the local heat transfer rate, thermal expansion, and thermal stress at some critical locations at specified times. The specification of the temperature at a point in a medium first requires the specification of the location of that point. This can be done by choosing a suitable coordinate system such as the rectangular, cylindrical, or spherical coordinates, depending on the geometry involved, and a convenient reference point (the origin).

The location of a point is specified as \((x, y, z)\) in rectangular coordinates, as \((r, \phi, z)\) in cylindrical coordinates, and as \((r, \phi, \theta)\) in spherical coordinates, where the distances \(x, y, z\), and \(r\) and the angles \(\phi\) and \(\theta\) are as shown in Figure 2–3. Then the temperature at a point \((x, y, z)\) at time \(t\) in rectangular coordinates is expressed as \(T(x, y, z, t)\). The best coordinate system for a given geometry is the one that describes the surfaces of the geometry best. For example, a parallelepiped is best described in rectangular coordinates since each surface can be described by a constant value of the \(x\)-, \(y\)-, or \(z\)-coordinates. A cylinder is best suited for cylindrical coordinates since its lateral surface can be described by a constant value of the radius. Similarly, the entire outer surface of a spherical body can best be described by a constant value of the radius in spherical coordinates. For an arbitrarily shaped body, we normally use rectangular coordinates since it is easier to deal with distances than with angles.

The notation just described is also used to identify the variables involved in a heat transfer problem. For example, the notation \(T(x, y, z, t)\) implies that the temperature varies with the space variables \(x, y, z\) as well as time. The
notation $T(x)$, on the other hand, indicates that the temperature varies in the $x$-direction only and there is no variation with the other two space coordinates or time.

**Steady versus Transient Heat Transfer**

Heat transfer problems are often classified as being **steady** (also called steady-state) or **transient** (also called unsteady). The term **steady** implies no change with time at any point within the medium, while **transient** implies variation with time or time dependence. Therefore, the temperature or heat flux remains unchanged with time during steady heat transfer through a medium at any location, although both quantities may vary from one location to another (Fig. 2–4). For example, heat transfer through the walls of a house will be steady when the conditions inside the house and the outdoors remain constant for several hours. But even in this case, the temperatures on the inner and outer surfaces of the wall will be different unless the temperatures inside and outside the house are the same. The cooling of an apple in a refrigerator, on the other hand, is a transient heat transfer process since the temperature at any fixed point within the apple will change with time during cooling. During transient heat transfer, the temperature normally varies with time as well as position. In the special case of variation with time but not with position, the temperature of the medium changes uniformly with time. Such heat transfer systems are called **lumped systems**. A small metal object such as a thermocouple junction or a thin copper wire, for example, can be analyzed as a lumped system during a heating or cooling process.

Most heat transfer problems encountered in practice are transient in nature, but they are usually analyzed under some presumed steady conditions since steady processes are easier to analyze, and they provide the answers to our questions. For example, heat transfer through the walls and ceiling of a typical house is never steady since the outdoor conditions such as the temperature, the speed and direction of the wind, the location of the sun, and so on, change constantly. The conditions in a typical house are not so steady either. Therefore, it is almost impossible to perform a heat transfer analysis of a house accurately. But then, do we really need an in-depth heat transfer analysis? If the
purpose of a heat transfer analysis of a house is to determine the proper size of a heater, which is usually the case, we need to know the maximum rate of heat loss from the house, which is determined by considering the heat loss from the house under worst conditions for an extended period of time, that is, during steady operation under worst conditions. Therefore, we can get the answer to our question by doing a heat transfer analysis under steady conditions. If the heater is large enough to keep the house warm under the presumed worst conditions, it is large enough for all conditions. The approach described above is a common practice in engineering.

**Multidimensional Heat Transfer**

Heat transfer problems are also classified as being one-dimensional, two-dimensional, or three-dimensional, depending on the relative magnitudes of heat transfer rates in different directions and the level of accuracy desired. In the most general case, heat transfer through a medium is three-dimensional. That is, the temperature varies along all three primary directions within the medium during the heat transfer process. The temperature distribution throughout the medium at a specified time as well as the heat transfer rate at any location in this general case can be described by a set of three coordinates such as the \( x, y, \) and \( z \) in the rectangular (or Cartesian) coordinate system; the \( r, \phi, \) and \( z \) in the cylindrical coordinate system; and the \( r, \phi, \) and \( \theta \) in the spherical (or polar) coordinate system. The temperature distribution in this case is expressed as \( T(x, y, z, t) \), \( T(r, \phi, z, t) \), and \( T(r, \phi, \theta, t) \) in the respective coordinate systems.

The temperature in a medium, in some cases, varies mainly in two primary directions, and the variation of temperature in the third direction (and thus heat transfer in that direction) is negligible. A heat transfer problem in that case is said to be two-dimensional. For example, the steady temperature distribution in a long bar of rectangular cross section can be expressed as \( T(x, y) \) if the temperature variation in the \( z \)-direction (along the bar) is negligible and there is no change with time (Fig. 2–5).

A heat transfer problem is said to be one-dimensional if the temperature in the medium varies in one direction only and thus heat is transferred in one direction, and the variation of temperature and thus heat transfer in other directions are negligible or zero. For example, heat transfer through the glass of a window can be considered to be one-dimensional since heat transfer through the glass will occur predominantly in one direction (the direction normal to the surface of the glass) and heat transfer in other directions (from one side edge to the other and from the top edge to the bottom) is negligible (Fig. 2–6). Likewise, heat transfer through a hot water pipe can be considered to be one-dimensional since heat transfer through the pipe occurs predominantly in the radial direction from the hot water to the ambient, and heat transfer along the pipe and along the circumference of a cross section (\( z \)- and \( \phi \)-directions) is typically negligible. Heat transfer to an egg dropped into boiling water is also nearly one-dimensional because of symmetry. Heat will be transferred to the egg in this case in the radial direction, that is, along straight lines passing through the midpoint of the egg.

We also mentioned in Chapter 1 that the rate of heat conduction through a medium in a specified direction (say, in the \( x \)-direction) is proportional to the temperature difference across the medium and the area normal to the direction...
of heat transfer, but is inversely proportional to the distance in that direction. This was expressed in the differential form by Fourier’s law of heat conduction for one-dimensional heat conduction as

\[ \dot{Q}_{\text{cond}} = -kA \frac{dT}{dx} \quad \text{(W)} \tag{2-1} \]

where \( k \) is the thermal conductivity of the material, which is a measure of the ability of a material to conduct heat, and \( \frac{dT}{dx} \) is the temperature gradient, which is the slope of the temperature curve on a \( T-x \) diagram (Fig. 2–7). The thermal conductivity of a material, in general, varies with temperature. But sufficiently accurate results can be obtained by using a constant value for thermal conductivity at the average temperature.

Heat is conducted in the direction of decreasing temperature, and thus the temperature gradient is negative when heat is conducted in the positive \( x \)-direction. The negative sign in Eq. 2–1 ensures that heat transfer in the positive \( x \)-direction is a positive quantity.

To obtain a general relation for Fourier’s law of heat conduction, consider a medium in which the temperature distribution is three-dimensional. Figure 2–8 shows an isothermal surface in that medium. The heat flux vector at a point \( P \) on this surface must be perpendicular to the surface, and it must point in the direction of decreasing temperature. If \( n \) is the normal of the isothermal surface at point \( P \), the rate of heat conduction at that point can be expressed by Fourier’s law as

\[ \dot{Q}_x = -kA \frac{dT}{dn} \quad \text{(W)} \tag{2-2} \]

In rectangular coordinates, the heat conduction vector can be expressed in terms of its components as

\[ \vec{\dot{Q}} = \dot{Q}_x \hat{i} + \dot{Q}_y \hat{j} + \dot{Q}_z \hat{k} \tag{2-3} \]

where \( \hat{i}, \hat{j}, \) and \( \hat{k} \) are the unit vectors, and \( \dot{Q}_x, \dot{Q}_y, \) and \( \dot{Q}_z \) are the magnitudes of the heat transfer rates in the \( x-, y-, \) and \( z- \)directions, which again can be determined from Fourier’s law as

\[ \dot{Q}_x = -kA_x \frac{dT}{dx} \quad \dot{Q}_y = -kA_y \frac{dT}{dy} \quad \text{and} \quad \dot{Q}_z = -kA_z \frac{dT}{dz} \tag{2-4} \]

Here \( A_x, A_y, \) and \( A_z \) are heat conduction areas normal to the \( x-, y-, \) and \( z- \)directions, respectively (Fig. 2–8).

Most engineering materials are isotropic in nature, and thus they have the same properties in all directions. For such materials we do not need to be concerned about the variation of properties with direction. But in anisotropic materials such as the fibrous or composite materials, the properties may change with direction. For example, some of the properties of wood along the grain are different than those in the direction normal to the grain. In such cases the thermal conductivity may need to be expressed as a tensor quantity to account for the variation with direction. The treatment of such advanced topics is beyond the scope of this text, and we will assume the thermal conductivity of a material to be independent of direction.
Heat Generation

A medium through which heat is conducted may involve the conversion of electrical, nuclear, or chemical energy into heat (or thermal) energy. In heat conduction analysis, such conversion processes are characterized as heat generation.

For example, the temperature of a resistance wire rises rapidly when electric current passes through it as a result of the electrical energy being converted to heat at a rate of $I^2R$, where $I$ is the current and $R$ is the electrical resistance of the wire (Fig. 2–9). The safe and effective removal of this heat away from the sites of heat generation (the electronic circuits) is the subject of electronics cooling, which is one of the modern application areas of heat transfer.

Likewise, a large amount of heat is generated in the fuel elements of nuclear reactors as a result of nuclear fission that serves as the heat source for the nuclear power plants. The natural disintegration of radioactive elements in nuclear waste or other radioactive material also results in the generation of heat throughout the body. The heat generated in the sun as a result of the fusion of hydrogen into helium makes the sun a large nuclear reactor that supplies heat to the earth.

Another source of heat generation in a medium is exothermic chemical reactions that may occur throughout the medium. The chemical reaction in this case serves as a heat source for the medium. In the case of endothermic reactions, however, heat is absorbed instead of being released during reaction, and thus the chemical reaction serves as a heat sink. The heat generation term becomes a negative quantity in this case.

Often it is also convenient to model the absorption of radiation such as solar energy or gamma rays as heat generation when these rays penetrate deep into the body while being absorbed gradually. For example, the absorption of solar energy in large bodies of water can be treated as heat generation throughout the water at a rate equal to the rate of absorption, which varies with depth (Fig. 2–10). But the absorption of solar energy by an opaque body occurs within a few microns of the surface, and the solar energy that penetrates into the medium in this case can be treated as specified heat flux on the surface.

Note that heat generation is a volumetric phenomenon. That is, it occurs throughout the body of a medium. Therefore, the rate of heat generation in a medium is usually specified per unit volume and is denoted by $g$, whose unit is W/m³ or Btu/h·ft³.

The rate of heat generation in a medium may vary with time as well as position within the medium. When the variation of heat generation with position is known, the total rate of heat generation in a medium of volume $V$ can be determined from

$$\dot{G} = \int_V g \, dV \quad \text{(W)} \tag{2-5}$$

In the special case of uniform heat generation, as in the case of electric resistance heating throughout a homogeneous material, the relation in Eq. 2–5 reduces to $\dot{G} = \dot{g}V$, where $\dot{g}$ is the constant rate of heat generation per unit volume.
EXAMPLE 2–1  Heat Gain by a Refrigerator

In order to size the compressor of a new refrigerator, it is desired to determine the rate of heat transfer from the kitchen air into the refrigerated space through the walls, door, and the top and bottom section of the refrigerator (Fig. 2–11). In your analysis, would you treat this as a transient or steady-state heat transfer problem? Also, would you consider the heat transfer to be one-dimensional or multidimensional? Explain.

SOLUTION  The heat transfer process from the kitchen air to the refrigerated space is transient in nature since the thermal conditions in the kitchen and the refrigerator, in general, change with time. However, we would analyze this problem as a steady heat transfer problem under the worst anticipated conditions such as the lowest thermostat setting for the refrigerated space, and the anticipated highest temperature in the kitchen (the so-called design conditions). If the compressor is large enough to keep the refrigerated space at the desired temperature setting under the presumed worst conditions, then it is large enough to do so under all conditions by cycling on and off.

Heat transfer into the refrigerated space is three-dimensional in nature since heat will be entering through all six sides of the refrigerator. However, heat transfer through any wall or floor takes place in the direction normal to the surface, and thus it can be analyzed as being one-dimensional. Therefore, this problem can be simplified greatly by considering the heat transfer to be one-dimensional at each of the four sides as well as the top and bottom sections, and then by adding the calculated values of heat transfer at each surface.

EXAMPLE 2–2  Heat Generation in a Hair Dryer

The resistance wire of a 1200-W hair dryer is 80 cm long and has a diameter of \( D = 0.3 \text{ cm} \) (Fig. 2–12). Determine the rate of heat generation in the wire per unit volume, in W/cm\(^3\), and the heat flux on the outer surface of the wire as a result of this heat generation.

SOLUTION  The power consumed by the resistance wire of a hair dryer is given. The heat generation and the heat flux are to be determined.

Assumptions  Heat is generated uniformly in the resistance wire.

Analysis  A 1200-W hair dryer will convert electrical energy into heat in the wire at a rate of 1200 W. Therefore, the rate of heat generation in a resistance wire is equal to the power consumption of a resistance heater. Then the rate of heat generation in the wire per unit volume is determined by dividing the total rate of heat generation by the volume of the wire,

\[
\dot{q} = \frac{G}{V_{\text{wire}}} = \frac{G}{\pi D^2/4 \cdot L} = \frac{1200 \text{ W}}{\pi (0.3 \text{ cm})^2/4 \cdot (80 \text{ cm})} = 212 \text{ W/cm}^3
\]

Similarly, heat flux on the outer surface of the wire as a result of this heat generation is determined by dividing the total rate of heat generation by the surface area of the wire,

\[
\dot{q} = \frac{G}{A_{\text{wire}}} = \frac{G}{\pi D L} = \frac{1200 \text{ W}}{\pi (0.3 \text{ cm})(80 \text{ cm})} = 15.9 \text{ W/cm}^2
\]
Consider heat conduction through a large plane wall such as the wall of a house, the glass of a single pane window, the metal plate at the bottom of a pressing iron, a cast iron steam pipe, a cylindrical nuclear fuel element, an electrical resistance wire, the wall of a spherical container, or a spherical metal ball that is being quenched or tempered. Heat conduction in these and many other geometries can be approximated as being one-dimensional since heat conduction through these geometries will be dominant in one direction and negligible in other directions. Below we will develop the one-dimensional heat conduction equation in rectangular, cylindrical, and spherical coordinates.

Heat Conduction Equation in a Large Plane Wall
Consider a thin element of thickness \( x \) in a large plane wall, as shown in Figure 2–13. Assume the density of the wall is \( \rho \), the specific heat is \( C \), and the area of the wall normal to the direction of heat transfer is \( A \). An energy balance on this thin element during a small time interval \( \Delta t \) can be expressed as

\[
Q_x \cdot x - Q_{x+\Delta x} \cdot x + G_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}
\]

But the change in the energy content of the element and the rate of heat generation within the element can be expressed as

\[
\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = mC(T_{t+\Delta t} - T_t) = \rho C A \Delta x (T_{t+\Delta t} - T_t)
\]

\[
G_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} A \Delta x
\]

Substituting into Equation 2–6, we get

\[
\dot{Q}_x - \dot{Q}_{x+\Delta x} + \dot{g} A \Delta x = \rho C A \Delta x \frac{T_{t+\Delta t} - T_t}{\Delta t}
\]

Dividing by \( A \Delta x \) gives

\[
- \frac{1}{A} \frac{T_{x+\Delta x} - T_x}{\Delta x} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t}
\]
Taking the limit as $\Delta x \to 0$ and $\Delta t \to 0$ yields
\[ \frac{1}{A} \frac{\partial}{\partial x} \left( kA \frac{\partial T}{\partial x} \right) + \dot{g} = \rho C \frac{\partial T}{\partial t} \]  
(2-11)
since, from the definition of the derivative and Fourier’s law of heat conduction,
\[ \lim_{\Delta x \to 0} \frac{\bar{Q}_{x+\Delta x} - \bar{Q}_x}{\Delta x} = \frac{\partial \bar{Q}}{\partial x} = \frac{\partial}{\partial x} \left( -kA \frac{\partial T}{\partial x} \right) \]  
(2-12)
Noting that the area $A$ is constant for a plane wall, the one-dimensional transient heat conduction equation in a plane wall becomes

**Variable conductivity:**
\[ \frac{\partial}{\partial x} \left( k A \frac{\partial T}{\partial x} \right) + \dot{g} = \rho C \frac{\partial T}{\partial t} \]  
(2-13)

The thermal conductivity $k$ of a material, in general, depends on the temperature $T$ (and therefore $x$), and thus it cannot be taken out of the derivative. However, the thermal conductivity in most practical applications can be assumed to remain constant at some average value. The equation above in that case reduces to

**Constant conductivity:**
\[ \frac{1}{k} \frac{\partial^2 T}{\partial x^2} + \dot{g} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]  
(2-14)

where the property $\alpha = k/\rho C$ is the thermal diffusivity of the material and represents how fast heat propagates through a material. It reduces to the following forms under specified conditions (Fig. 2–14):

1. **Steady-state:**
   \[ \frac{\partial}{\partial t} = 0 \]
   \[ \frac{\partial^2 T}{\partial x^2} + \dot{g} = 0 \]  
   (2-15)
2. **Transient, no heat generation:**
   \[ \dot{g} = 0 \]
   \[ \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]  
   (2-16)
3. **Steady-state, no heat generation:**
   \[ \dot{g} = 0 \text{ and } \partial t = 0 \]
   \[ \frac{\partial^2 T}{\partial x^2} = 0 \]  
   (2-17)

Note that we replaced the partial derivatives by ordinary derivatives in the one-dimensional steady heat conduction case since the partial and ordinary derivatives of a function are identical when the function depends on a single variable only [$T = T(x)$ in this case].

**Heat Conduction Equation in a Long Cylinder**

Now consider a thin cylindrical shell element of thickness $\Delta r$ in a long cylinder, as shown in Figure 2–15. Assume the density of the cylinder is $\rho$, the specific heat is $C$, and the length is $L$. The area of the cylinder normal to the direction of heat transfer at any location is $A = 2\pi rL$ where $r$ is the value of the radius at that location. Note that the heat transfer area $A$ depends on $r$ in this case, and thus it varies with location. An energy balance on this thin cylindrical shell element during a small time interval $\Delta t$ can be expressed as

\[ \begin{align*}
\dot{Q}_r & \rightarrow \Delta Q_r \\
\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r A \frac{\partial T}{\partial r} \right) \right) + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r A \frac{\partial T}{\partial r} \right) \right) + \frac{\partial}{\partial t} \left( r A \frac{\partial T}{\partial t} \right) &= 0 \\
\end{align*} \]  
(2-18)

The simplification of the one-dimensional heat conduction equation in a plane wall for the case of constant conductivity for steady conduction with no heat generation.

**General, one dimensional:**

No Steady-state
\[ \frac{\partial^2 T}{\partial x^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]  

Steady, one-dimensional:
\[ \frac{\partial^2 T}{\partial x^2} = 0 \]  

**FIGURE 2–14**

One-dimensional heat conduction through a volume element in a long cylinder.
The change in the energy content of the element and the rate of heat generation within the element can be expressed as

\[
\Delta E_{\text{element}} = E_t + \Delta t - E_t = mC(T_t + \Delta t - T_t) = \rho C A \Delta r (T_t + \Delta r - T_t)
\]

(2-19)

\[
\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} A \Delta r
\]

(2-20)

Substituting into Eq. 2–18, we get

\[
\dot{Q}_r - \dot{Q}_r + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}
\]

(2-18)

or

\[
\dot{Q}_r - \dot{Q}_r + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}
\]

(2-18)

The change in the energy content of the element and the rate of heat generation within the element can be expressed as

\[
\Delta E_{\text{element}} = E_t + \Delta t - E_t = mC(T_t + \Delta t - T_t) = \rho C A \Delta r (T_t + \Delta r - T_t)
\]

(2-19)

\[
\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} A \Delta r
\]

(2-20)

Substituting into Eq. 2–18, we get

\[
\dot{Q}_r - \dot{Q}_r + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}
\]

(2-18)

or

\[
\dot{Q}_r - \dot{Q}_r + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}
\]

(2-18)

where \( A = 2\pi r L \). You may be tempted to express the area at the middle of the element using the average radius as \( A = 2\pi (r + \Delta r/2) L \). But there is nothing we can gain from this complication since later in the analysis we will take the limit as \( \Delta r \to 0 \) and thus the term \( \Delta r/2 \) will drop out. Now dividing the equation above by \( A \Delta r \) gives

\[
\frac{1}{A} \frac{\dot{Q}_r + \Delta r}{\Delta r} + \frac{\dot{g}}{\dot{g}} = \frac{C}{C} \frac{T_t + \Delta t - T_t}{\Delta t}
\]

(2-22)

Taking the limit as \( \Delta r \to 0 \) and \( \Delta t \to 0 \) yields

\[
\frac{1}{A} \frac{\partial}{\partial r} \left( k A \frac{\partial T}{\partial r} \right) + \frac{\dot{g}}{\dot{g}} = \frac{C}{C} \frac{\partial T}{\partial t}
\]

(2-23)

since, from the definition of the derivative and Fourier’s law of heat conduction,

\[
\lim_{\Delta r \to 0} \frac{Q_r + \Delta r}{\Delta r} = \frac{\partial Q}{\partial r} = \frac{\partial T}{\partial r} \left( -k A \frac{\partial T}{\partial r} \right)
\]

(2-24)

Noting that the heat transfer area in this case is \( A = 2\pi r L \), the one-dimensional transient heat conduction equation in a cylinder becomes

Variable conductivity:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\dot{g}}{\dot{g}} = \frac{C}{C} \frac{\partial T}{\partial t}
\]

(2-25)

For the case of constant thermal conductivity, the equation above reduces to

Constant conductivity:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\dot{g}}{\dot{g}} = \frac{1}{\alpha} \frac{\partial T}{\partial t}
\]

(2-26)
where again the property $\alpha = k/\rho C$ is the thermal diffusivity of the material. Equation 2–26 reduces to the following forms under specified conditions (Fig. 2–16):

1. **Steady-state:** $\frac{\partial}{\partial t} = 0$
   
   \[
   \frac{1}{r} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) + \frac{\dot{g}}{k} = 0 \tag{2-27}
   \]

2. **Transient, no heat generation:** $\dot{g} = 0$
   
   \[
   \frac{1}{r} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{2-28}
   \]

3. **Steady-state, no heat generation:** $\frac{\partial}{\partial t} = 0$ and $\dot{g} = 0$
   
   \[
   \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0 \tag{2-29}
   \]

Note that we again replaced the partial derivatives by ordinary derivatives in the one-dimensional steady heat conduction case since the partial and ordinary derivatives of a function are identical when the function depends on a single variable only $[T = T(r)$ in this case].

**Heat Conduction Equation in a Sphere**

Now consider a sphere with density $\rho$, specific heat $C$, and outer radius $R$. The area of the sphere normal to the direction of heat transfer at any location is $A = 4\pi r^2$, where $r$ is the value of the radius at that location. Note that the heat transfer area $A$ depends on $r$ in this case also, and thus it varies with location. By considering a thin spherical shell element of thickness $\Delta r$ and repeating the approach described above for the cylinder by using $A = 4\pi r^2$ instead of $A = 2\pi rL$, the one-dimensional transient heat conduction equation for a sphere is determined to be (Fig. 2–17)

Variable conductivity:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 k \frac{\partial T}{\partial r} \right) + \frac{\dot{g}}{k} = \rho C \frac{\partial T}{\partial t} \tag{2-30}
\]

which, in the case of constant thermal conductivity, reduces to

Constant conductivity:

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{\dot{g}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{2-31}
\]

where again the property $\alpha = k/\rho C$ is the thermal diffusivity of the material. It reduces to the following forms under specified conditions:

1. **Steady-state:** $\frac{\partial}{\partial t} = 0$
   
   \[
   \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) + \frac{\dot{g}}{k} = 0 \tag{2-32}
   \]

2. **Transient, no heat generation:** $\dot{g} = 0$
   
   \[
   \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{2-33}
   \]

3. **Steady-state, no heat generation:** $\frac{\partial}{\partial t} = 0$ and $\dot{g} = 0$
   
   \[
   \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0 \quad \text{or} \quad r \frac{d^2 T}{dr^2} + 2 \frac{dT}{dr} = 0 \tag{2-34}
   \]

where again we replaced the partial derivatives by ordinary derivatives in the one-dimensional steady heat conduction case.
Combined One-Dimensional Heat Conduction Equation

An examination of the one-dimensional transient heat conduction equations for the plane wall, cylinder, and sphere reveals that all three equations can be expressed in a compact form as

$$\frac{1}{r^n} \frac{\partial}{\partial r} \left( r^n k \frac{\partial T}{\partial r} \right) + \dot{g} = \rho C \frac{\partial T}{\partial t}$$  \hspace{1cm} (2-35)

where $n = 0$ for a plane wall, $n = 1$ for a cylinder, and $n = 2$ for a sphere. In the case of a plane wall, it is customary to replace the variable $r$ by $x$. This equation can be simplified for steady-state or no heat generation cases as described before.

**EXAMPLE 2–3  Heat Conduction through the Bottom of a Pan**

Consider a steel pan placed on top of an electric range to cook spaghetti (Fig. 2–18). The bottom section of the pan is $L = 0.4$ cm thick and has a diameter of $D = 18$ cm. The electric heating unit on the range top consumes 800 W of power during cooking, and 80 percent of the heat generated in the heating element is transferred uniformly to the pan. Assuming constant thermal conductivity, obtain the differential equation that describes the variation of the temperature in the bottom section of the pan during steady operation.

**SOLUTION**

The bottom section of the pan has a large surface area relative to its thickness and can be approximated as a large plane wall. Heat flux is applied to the bottom surface of the pan uniformly, and the conditions on the inner surface are also uniform. Therefore, we expect the heat transfer through the bottom section of the pan to be from the bottom surface toward the top, and heat transfer in this case can reasonably be approximated as being one-dimensional. Taking the direction normal to the bottom surface of the pan to be the $x$-axis, we will have $T = T(x)$ during steady operation since the temperature in this case will depend on $x$ only.

The thermal conductivity is given to be constant, and there is no heat generation in the medium (within the bottom section of the pan). Therefore, the differential equation governing the variation of temperature in the bottom section of the pan in this case is simply Eq. 2–17,

$$\frac{d^2 T}{dx^2} = 0$$

which is the steady one-dimensional heat conduction equation in rectangular coordinates under the conditions of constant thermal conductivity and no heat generation. Note that the conditions at the surface of the medium have no effect on the differential equation.

**EXAMPLE 2–4  Heat Conduction in a Resistance Heater**

A 2-kW resistance heater wire with thermal conductivity $k = 15$ W/m · °C, diameter $D = 0.4$ cm, and length $L = 50$ cm is used to boil water by immersing
it in water (Fig. 2–19). Assuming the variation of the thermal conductivity of the wire with temperature to be negligible, obtain the differential equation that describes the variation of the temperature in the wire during steady operation.

**SOLUTION** The resistance wire can be considered to be a very long cylinder since its length is more than 100 times its diameter. Also, heat is generated uniformly in the wire and the conditions on the outer surface of the wire are uniform. Therefore, it is reasonable to expect the temperature in the wire to vary in the radial $r$ direction only and thus the heat transfer to be one-dimensional. Then we will have $T = T(r)$ during steady operation since the temperature in this case will depend on $r$ only.

The resistance wire can be considered to be a very long cylinder since its length is more than 100 times its diameter. Also, heat is generated uniformly in the wire and the conditions on the outer surface of the wire are uniform. Therefore, it is reasonable to expect the temperature in the wire to vary in the radial $r$ direction only and thus the heat transfer to be one-dimensional. Then we will have $T = T(r)$ during steady operation since the temperature in this case will depend on $r$ only.

The rate of heat generation in the wire per unit volume can be determined from

$$\dot{g} = \frac{G}{V_{\text{wire}}} = \frac{G}{(\pi D^2/4)L} = \frac{2000 \text{ W}}{[\pi(0.004 \text{ m})^2/4](0.5 \text{ cm})} = 0.318 \times 10^3 \text{ W/m}^3$$

Noting that the thermal conductivity is given to be constant, the differential equation that governs the variation of temperature in the wire is simply Eq. 2–27,

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \frac{\dot{g}}{k} = 0$$

which is the steady one-dimensional heat conduction equation in cylindrical coordinates for the case of constant thermal conductivity. Note again that the conditions at the surface of the wire have no effect on the differential equation.

---

**EXAMPLE 2–5 Cooling of a Hot Metal Ball in Air**

A spherical metal ball of radius $R$ is heated in an oven to a temperature of 600°F throughout and is then taken out of the oven and allowed to cool in ambient air at $T_\infty = 75°F$ by convection and radiation (Fig. 2–20). The thermal conductivity of the ball material is known to vary linearly with temperature. Assuming the ball is cooled uniformly from the entire outer surface, obtain the differential equation that describes the variation of the temperature in the ball during cooling.

**SOLUTION** The ball is initially at a uniform temperature and is cooled uniformly from the entire outer surface. Also, the temperature at any point in the ball will change with time during cooling. Therefore, this is a one-dimensional transient heat conduction problem since the temperature within the ball will change with the radial distance $r$ and the time $t$. That is, $T = T(r, t)$.

The thermal conductivity is given to be variable, and there is no heat generation in the ball. Therefore, the differential equation that governs the variation of temperature in the ball in this case is obtained from Eq. 2–30 by setting the heat generation term equal to zero. We obtain

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 k \frac{\partial T}{\partial r} \right) = \rho C \frac{\partial T}{\partial t}$$
which is the one-dimensional transient heat conduction equation in spherical coordinates under the conditions of variable thermal conductivity and no heat generation. Note again that the conditions at the outer surface of the ball have no effect on the differential equation.

2–3 GENERAL HEAT CONDUCTION EQUATION

In the last section we considered one-dimensional heat conduction and assumed heat conduction in other directions to be negligible. Most heat transfer problems encountered in practice can be approximated as being one-dimensional, and we will mostly deal with such problems in this text. However, this is not always the case, and sometimes we need to consider heat transfer in other directions as well. In such cases heat conduction is said to be multidimensional, and in this section we will develop the governing differential equation in such systems in rectangular, cylindrical, and spherical coordinate systems.

Rectangular Coordinates

Consider a small rectangular element of length $\Delta x$, width $\Delta y$, and height $\Delta z$, as shown in Figure 2–21. Assume the density of the body is $\rho$ and the specific heat is $C$. An energy balance on this element during a small time interval $\Delta t$ can be expressed as

\[
\left( \frac{\text{Rate of heat conduction at } x, y, \text{ and } z}{\Delta x \Delta y \Delta z} \right) + \left( \frac{\text{Rate of heat generation inside the element}}{\Delta x \Delta y \Delta z} \right) = \left( \frac{\text{Rate of change of the energy content of the element}}{\Delta t} \right)
\]

or

\[
\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} (2-36)
\]

Noting that the volume of the element is $V_{\text{element}} = \Delta x \Delta y \Delta z$, the change in the energy content of the element and the rate of heat generation within the element can be expressed as

\[
\Delta E_{\text{element}} = E_{t+\Delta t} - E_t = \rho C \Delta x \Delta y \Delta z (T_{t+\Delta t} - T_t)
\]

\[
\dot{G}_{\text{element}} = \dot{g} V_{\text{element}} = \dot{g} \Delta x \Delta y \Delta z
\]

Substituting into Eq. 2–36, we get

\[
\dot{Q}_x + \dot{Q}_y + \dot{Q}_z - \dot{Q}_{x+\Delta x} - \dot{Q}_{y+\Delta y} - \dot{Q}_{z+\Delta z} + \dot{g} \Delta x \Delta y \Delta z = \rho C \Delta x \Delta y \Delta z \frac{T_{t+\Delta t} - T_t}{\Delta t}
\]

Dividing by $\Delta x \Delta y \Delta z$ gives

\[
-\frac{1}{\Delta y \Delta z} \frac{\dot{Q}_{x+\Delta x} - \dot{Q}_x}{\Delta x} - \frac{1}{\Delta x \Delta z} \frac{\dot{Q}_{y+\Delta y} - \dot{Q}_y}{\Delta y} - \frac{1}{\Delta x \Delta y} \frac{\dot{Q}_{z+\Delta z} - \dot{Q}_z}{\Delta z} + \dot{g} = \rho C \frac{T_{t+\Delta t} - T_t}{\Delta t} (2-37)
\]
Noting that the heat transfer areas of the element for heat conduction in the 
\(x, y,\) and \(z\) directions are \(A_x = \Delta y \Delta z,\) \(A_y = \Delta x \Delta z,\) and \(A_z = \Delta x \Delta y,\) respectively, 
and taking the limit as \(\Delta x, \Delta y, \Delta z\) and \(\Delta t \to 0\) yields

\[
\frac{\partial}{\partial x} \left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( k \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + \dot{g} = \rho C \frac{\partial T}{\partial t} \tag{2-38}
\]

since, from the definition of the derivative and Fourier’s law of heat conduction,

\[
\lim_{\Delta x \to 0} \frac{1}{\Delta y \Delta z} \left( \frac{\partial Q_x}{\partial x} - \frac{\partial Q_x}{\partial t} \right) = \frac{1}{\Delta y \Delta z} \frac{\partial Q_y}{\partial y} = \frac{1}{\Delta x \Delta z} \frac{\partial Q_z}{\partial z} = \frac{1}{\Delta x \Delta y} \frac{\partial Q_z}{\partial z} = \frac{1}{\Delta x \Delta y} \frac{\partial Q_z}{\partial t}
\]

Equation 2–38 is the general heat conduction equation in rectangular coordinates. In the case of constant thermal conductivity, it reduces to

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{2-39}
\]

where the property \(\alpha = k/\rho C\) is again the thermal diffusivity of the material.

Equation 2–39 is known as the Fourier-Biot equation, and it reduces to these forms under specified conditions:

1. **Steady-state:**
   (called the Poisson equation)
   \[
   \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} + \frac{\dot{g}}{k} = 0 \tag{2-40}
   \]
2. **Transient, no heat generation:**
   (called the diffusion equation)
   \[
   \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \tag{2-41}
   \]
3. **Steady-state, no heat generation:**
   (called the Laplace equation)
   \[
   \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{2-42}
   \]

Note that in the special case of one-dimensional heat transfer in the 
\(x\)-direction, the derivatives with respect to \(y\) and \(z\) drop out and the equations above reduce to the ones developed in the previous section for a plane wall (Fig. 2–22).

**Cylindrical Coordinates**

The general heat conduction equation in cylindrical coordinates can be 
obtained from an energy balance on a volume element in cylindrical coordinates, 
shown in Figure 2–23, by following the steps just outlined. It can also be 
obtained directly from Eq. 2–38 by coordinate transformation using the following relations between the coordinates of a point in rectangular and cylindrical coordinate systems:

\[x = r \cos \phi, \quad y = r \sin \phi, \quad \text{and} \quad z = z\]
After lengthy manipulations, we obtain
\[ \frac{1}{r} \frac{\partial}{\partial r} \left( kr \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( k \frac{\partial T}{\partial \theta} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + \dot{g} = \rho C \frac{\partial T}{\partial t} \] (2-43)

**Spherical Coordinates**

The general heat conduction equations in spherical coordinates can be obtained from an energy balance on a volume element in spherical coordinates, shown in Figure 2–24, by following the steps outlined above. It can also be obtained directly from Eq. 2–38 by coordinate transformation using the following relations between the coordinates of a point in rectangular and spherical coordinate systems:

\[ x = r \cos \phi \sin \theta, \quad y = r \sin \phi \sin \theta, \quad \text{and} \quad z = \cos \theta \]

Again after lengthy manipulations, we obtain
\[ \frac{1}{r} \frac{\partial}{\partial r} \left( kr \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left( k \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \phi} \left( k \sin \theta \frac{\partial T}{\partial \phi} \right) + \dot{g} = \rho C \frac{\partial T}{\partial t} \] (2-44)

Obtaining analytical solutions to these differential equations requires a knowledge of the solution techniques of partial differential equations, which is beyond the scope of this introductory text. Here we limit our consideration to one-dimensional steady-state cases or lumped systems, since they result in ordinary differential equations.

**EXAMPLE 2–6  Heat Conduction in a Short Cylinder**

A short cylindrical metal billet of radius \( R \) and height \( h \) is heated in an oven to a temperature of 600°F throughout and is then taken out of the oven and allowed to cool in ambient air at \( T_\infty = 65°F \) by convection and radiation. Assuming the billet is cooled uniformly from all outer surfaces and the variation of the thermal conductivity of the material with temperature is negligible, obtain the differential equation that describes the variation of the temperature in the billet during this cooling process.

**SOLUTION** The billet shown in Figure 2–25 is initially at a uniform temperature and is cooled uniformly from the top and bottom surfaces in the \( z \)-direction as well as the lateral surface in the radial \( r \)-direction. Also, the temperature at any point in the ball will change with time during cooling. Therefore, this is a two-dimensional transient heat conduction problem since the temperature within the billet will change with the radial and axial distances \( r \) and \( z \) and with time \( t \). That is, \( T = T(r, z, t) \).

The thermal conductivity is given to be constant, and there is no heat generation in the billet. Therefore, the differential equation that governs the variation of temperature in the billet in this case is obtained from Eq. 2–43 by setting the heat generation term and the derivatives with respect to \( \phi \) equal to zero. We obtain
\[ \frac{1}{r} \frac{\partial}{\partial r} \left( kr \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) = \rho C \frac{\partial T}{\partial t} \]
In the case of constant thermal conductivity, it reduces to
\[ \frac{1}{\tau} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t} \]
which is the desired equation.

2–4 • BOUNDARY AND INITIAL CONDITIONS

The heat conduction equations above were developed using an energy balance on a differential element inside the medium, and they remain the same regardless of the thermal conditions on the surfaces of the medium. That is, the differential equations do not incorporate any information related to the conditions on the surfaces such as the surface temperature or a specified heat flux. Yet we know that the heat flux and the temperature distribution in a medium depend on the conditions at the surfaces, and the description of a heat transfer problem in a medium is not complete without a full description of the thermal conditions at the bounding surfaces of the medium. The **mathematical expressions** of the thermal conditions at the boundaries are called the **boundary conditions**.

From a mathematical point of view, solving a differential equation is essentially a process of **removing derivatives**, or an **integration** process, and thus the solution of a differential equation typically involves arbitrary constants (Fig. 2–26). It follows that to obtain a unique solution to a problem, we need to specify more than just the governing differential equation. We need to specify some conditions (such as the value of the function or its derivatives at some value of the independent variable) so that forcing the solution to satisfy these conditions at specified points will result in unique values for the arbitrary constants and thus a **unique solution**. But since the differential equation has no place for the additional information or conditions, we need to supply them separately in the form of boundary or initial conditions.

Consider the variation of temperature along the wall of a brick house in winter. The temperature at any point in the wall depends on, among other things, the conditions at the two surfaces of the wall such as the air temperature of the house, the velocity and direction of the winds, and the solar energy incident on the outer surface. That is, the temperature distribution in a medium depends on the conditions at the boundaries of the medium as well as the heat transfer mechanism inside the medium. To describe a heat transfer problem completely, **two boundary conditions** must be given for each direction along the coordinate system along which heat transfer is significant (Fig. 2–27). Therefore, we need to specify **two boundary conditions** for one-dimensional problems, **four boundary conditions** for two-dimensional problems, and **six boundary conditions** for three-dimensional problems. In the case of the wall of a house, for example, we need to specify the conditions at two locations (the inner and the outer surfaces) of the wall since heat transfer in this case is one-dimensional. But in the case of a parallelepiped, we need to specify six boundary conditions (one at each face) when heat transfer in all three dimensions is significant.

\[ \frac{d^2 T}{dx^2} = 0 \]
\[ T(x) = C_1 x + C_2 \]
\[ T(x) = 2x + 5 \]
\[ T(x) = -x + 12 \]
\[ T(x) = -3 \]
\[ T(x) = 6.2x \]

FIGURE 2–26

The general solution of a typical differential equation involves arbitrary constants, and thus an infinite number of solutions.

\[ \frac{d^2 T}{dx^2} = 0 \]
\[ T(x) = 50^\circ C \]
\[ T(L) = 15^\circ C \]

FIGURE 2–27

To describe a heat transfer problem completely, two boundary conditions must be given for each direction along which heat transfer is significant.
The physical argument presented above is consistent with the mathematical nature of the problem since the heat conduction equation is second order (i.e., involves second derivatives with respect to the space variables) in all directions along which heat conduction is significant, and the general solution of a second-order linear differential equation involves two arbitrary constants for each direction. That is, the number of boundary conditions that needs to be specified in a direction is equal to the order of the differential equation in that direction.

Reconsider the brick wall already discussed. The temperature at any point on the wall at a specified time also depends on the condition of the wall at the beginning of the heat conduction process. Such a condition, which is usually specified at time \( t = 0 \), is called the initial condition, which is a mathematical expression for the temperature distribution of the medium initially. Note that we need only one initial condition for a heat conduction problem regardless of the dimension since the conduction equation is first order in time (it involves the first derivative of temperature with respect to time).

In rectangular coordinates, the initial condition can be specified in the general form as

\[
T(x, y, z, 0) = f(x, y, z) \tag{2-45}
\]

where the function \( f(x, y, z) \) represents the temperature distribution throughout the medium at time \( t = 0 \). When the medium is initially at a uniform temperature of \( T_r \), the initial condition of Eq. 2–45 can be expressed as \( T(x, y, z, 0) = T_r \). Note that under steady conditions, the heat conduction equation does not involve any time derivatives, and thus we do not need to specify an initial condition.

The heat conduction equation is first order in time, and thus the initial condition cannot involve any derivatives (it is limited to a specified temperature). However, the heat conduction equation is second order in space coordinates, and thus a boundary condition may involve first derivatives at the boundaries as well as specified values of temperature. Boundary conditions most commonly encountered in practice are the specified temperature, specified heat flux, convection, and radiation boundary conditions.

1 Specified Temperature Boundary Condition

The temperature of an exposed surface can usually be measured directly and easily. Therefore, one of the easiest ways to specify the thermal conditions on a surface is to specify the temperature. For one-dimensional heat transfer through a plane wall of thickness \( L \), for example, the specified temperature boundary conditions can be expressed as (Fig. 2–28)

\[
T(0, t) = T_1, \quad T(L, t) = T_2 \tag{2-46}
\]

where \( T_1 \) and \( T_2 \) are the specified temperatures at surfaces at \( x = 0 \) and \( x = L \), respectively. The specified temperatures can be constant, which is the case for steady heat conduction, or may vary with time.
2 Specified Heat Flux Boundary Condition

When there is sufficient information about energy interactions at a surface, it may be possible to determine the rate of heat transfer and thus the heat flux \( q \) (heat transfer rate per unit surface area, W/m²) on that surface, and this information can be used as one of the boundary conditions. The heat flux in the positive \( x \)-direction anywhere in the medium, including the boundaries, can be expressed by Fourier’s law of heat conduction as

\[
q = -k \frac{\partial T}{\partial x} \quad \text{(W/m}^2\text{)} 
\]  

Then the boundary condition at a boundary is obtained by setting the specified heat flux equal to \( -k \frac{\partial T}{\partial x} \) at that boundary. The sign of the specified heat flux is determined by inspection: positive if the heat flux is in the positive direction of the coordinate axis, and negative if it is in the opposite direction. Note that it is extremely important to have the correct sign for the specified heat flux since the wrong sign will invert the direction of heat transfer and cause the heat gain to be interpreted as heat loss (Fig. 2–29).

For a plate of thickness \( L \) subjected to heat flux of 50 W/m² into the medium from both sides, for example, the specified heat flux boundary conditions can be expressed as

\[
-k \frac{\partial T(0, t)}{\partial x} = 50 \quad \text{and} \quad -k \frac{\partial T(L, t)}{\partial x} = -50 
\]  

Note that the heat flux at the surface at \( x = L \) is in the negative \( x \)-direction, and thus it is \(-50 \) W/m².

Special Case: Insulated Boundary

Some surfaces are commonly insulated in practice in order to minimize heat loss (or heat gain) through them. Insulation reduces heat transfer but does not totally eliminate it unless its thickness is infinity. However, heat transfer through a properly insulated surface can be taken to be zero since adequate insulation reduces heat transfer through a surface to negligible levels. Therefore, a well-insulated surface can be modeled as a surface with a specified heat flux of zero. Then the boundary condition on a perfectly insulated surface (at \( x = 0 \), for example) can be expressed as (Fig. 2–30)

\[
k \frac{\partial T(0, t)}{\partial x} = 0 \quad \text{or} \quad \frac{\partial T(0, t)}{\partial x} = 0 
\]  

That is, on an insulated surface, the first derivative of temperature with respect to the space variable (the temperature gradient) in the direction normal to the insulated surface is zero. This also means that the temperature function must be perpendicular to an insulated surface since the slope of temperature at the surface must be zero.

Another Special Case: Thermal Symmetry

Some heat transfer problems possess thermal symmetry as a result of the symmetry in imposed thermal conditions. For example, the two surfaces of a large hot plate of thickness \( L \) suspended vertically in air will be subjected to
the same thermal conditions, and thus the temperature distribution in one half of the plate will be the same as that in the other half. That is, the heat transfer problem in this plate will possess thermal symmetry about the center plane at \( x = \frac{L}{2} \). Also, the direction of heat flow at any point in the plate will be toward the surface closer to the point, and there will be no heat flow across the center plane. Therefore, the center plane can be viewed as an insulated surface, and the thermal condition at this plane of symmetry can be expressed as (Fig. 2–31)

\[
\frac{\partial T(L/2, t)}{\partial x} = 0
\]  

which resembles the insulation or zero heat flux boundary condition. This result can also be deduced from a plot of temperature distribution with a maximum, and thus zero slope, at the center plane.

In the case of cylindrical (or spherical) bodies having thermal symmetry about the center line (or midpoint), the thermal symmetry boundary condition requires that the first derivative of temperature with respect to \( r \) (the radial variable) be zero at the centerline (or the midpoint).

**EXAMPLE 2–7**  
Heat Flux Boundary Condition

Consider an aluminum pan used to cook beef stew on top of an electric range. The bottom section of the pan is \( L = 0.3 \) cm thick and has a diameter of \( D = 20 \) cm. The electric heating unit on the range top consumes 800 W of power during cooking, and 90 percent of the heat generated in the heating element is transferred to the pan. During steady operation, the temperature of the inner surface of the pan is measured to be 110°C. Express the boundary conditions for the bottom section of the pan during this cooking process.

**SOLUTION**  
The heat transfer through the bottom section of the pan is from the bottom surface toward the top and can reasonably be approximated as being one-dimensional. We take the direction normal to the bottom surfaces of the pan as the \( x \) axis with the origin at the outer surface, as shown in Figure 2–32. Then the inner and outer surfaces of the bottom section of the pan can be represented by \( x = 0 \) and \( x = L \), respectively. During steady operation, the temperature will depend on \( x \) only and thus \( T = T(x) \).

The boundary condition on the outer surface of the bottom of the pan at \( x = 0 \) can be approximated as being specified heat flux since it is stated that 90 percent of the 800 W (i.e., 720 W) is transferred to the pan at that surface. Therefore,

\[
-k \frac{dT(0)}{dx} = q_0
\]

where

\[
q_0 = \frac{\text{Heat transfer rate}}{\text{Bottom surface area}} = \frac{0.720 \text{ kW}}{\pi(0.1 \text{ m})^2} = 22.9 \text{ kW/m}^2
\]
Convection Boundary Condition

Convection is probably the most common boundary condition encountered in practice since most heat transfer surfaces are exposed to an environment at a specified temperature. The convection boundary condition is based on a surface energy balance expressed as

\[ \text{Heat conduction at the surface in a selected direction} = \text{Heat convection at the surface in the same direction} \]

For one-dimensional heat transfer in the \( x \)-direction in a plate of thickness \( L \), the convection boundary conditions on both surfaces can be expressed as

\[ -k \frac{\partial T(0, t)}{\partial x} = h_1[T_{s1} - T(0, t)] \quad (2-51a) \]

and

\[ -k \frac{\partial T(L, t)}{\partial x} = h_2[T(L, t) - T_{s2}] \quad (2-51b) \]

where \( h_1 \) and \( h_2 \) are the convection heat transfer coefficients and \( T_{s1} \) and \( T_{s2} \) are the temperatures of the surrounding mediums on the two sides of the plate, as shown in Figure 2–33.

In writing Eqs. 2–51 for convection boundary conditions, we have selected the direction of heat transfer to be the positive \( x \)-direction at both surfaces. But those expressions are equally applicable when heat transfer is in the opposite direction at one or both surfaces since reversing the direction of heat transfer at a surface simply reverses the signs of both conduction and convection terms at that surface. This is equivalent to multiplying an equation by \(-1\), which has no effect on the equality (Fig. 2–34). Being able to select either direction as the direction of heat transfer is certainly a relief since often we do not know the surface temperature and thus the direction of heat transfer at a surface in advance. This argument is also valid for other boundary conditions such as the radiation and combined boundary conditions discussed shortly.

Note that a surface has zero thickness and thus no mass, and it cannot store any energy. Therefore, the entire net heat entering the surface from one side must leave the surface from the other side. The convection boundary condition simply states that heat continues to flow from a body to the surrounding medium at the same rate, and it just changes vehicles at the surface from conduction to convection (or vice versa in the other direction). This is analogous to people traveling on buses on land and transferring to the ships at the shore.

The temperature at the inner surface of the bottom of the pan is specified to be 110°C. Then the boundary condition on this surface can be expressed as

\[ T(L) = 110°C \]

where \( L = 0.003 \) m. Note that the determination of the boundary conditions may require some reasoning and approximations.
If the passengers are not allowed to wander around at the shore, then the rate at which the people are unloaded at the shore from the buses must equal the rate at which they board the ships. We may call this the conservation of “people” principle.

Also note that the surface temperatures \( T(0, t) \) and \( T(L, t) \) are not known (if they were known, we would simply use them as the specified temperature boundary condition and not bother with convection). But a surface temperature can be determined once the solution \( T(x, t) \) is obtained by substituting the value of \( x \) at that surface into the solution.

**EXAMPLE 2–8 Convection and Insulation Boundary Conditions**

Steam flows through a pipe shown in Figure 2–35 at an average temperature of \( T_\infty = 200^\circ \text{C} \). The inner and outer radii of the pipe are \( r_1 = 8 \text{ cm} \) and \( r_2 = 8.5 \text{ cm} \), respectively, and the outer surface of the pipe is heavily insulated. If the convection heat transfer coefficient on the inner surface of the pipe is \( h = 65 \text{ W/m}^2 \cdot ^\circ \text{C} \), express the boundary conditions on the inner and outer surfaces of the pipe during transient periods.

**SOLUTION** During initial transient periods, heat transfer through the pipe material will predominantly be in the radial direction, and thus can be approximated as being one-dimensional. Then the temperature within the pipe material will change with the radial distance \( r \) and the time \( t \). That is, \( T = T(r, t) \).

It is stated that heat transfer between the steam and the pipe at the inner surface is by convection. Then taking the direction of heat transfer to be the positive \( r \) direction, the boundary condition on that surface can be expressed as

\[
-k \frac{\partial T(r_1, t)}{\partial r} = h[T_\infty - T(r_1)]
\]

The pipe is said to be well insulated on the outside, and thus heat loss through the outer surface of the pipe can be assumed to be negligible. Then the boundary condition at the outer surface can be expressed as

\[
\frac{\partial T(r_2, t)}{\partial r} = 0
\]

That is, the temperature gradient must be zero on the outer surface of the pipe at all times.

**4 Radiation Boundary Condition**

In some cases, such as those encountered in space and cryogenic applications, a heat transfer surface is surrounded by an evacuated space and thus there is no convection heat transfer between a surface and the surrounding medium. In such cases, radiation becomes the only mechanism of heat transfer between the surface under consideration and the surroundings. Using an energy balance, the radiation boundary condition on a surface can be expressed as

\[
\text{Heat conduction at the surface in a selected direction} = \text{Radiation exchange at the surface in the same direction}
\]
For one-dimensional heat transfer in the \(x\)-direction in a plate of thickness \(L\), the radiation boundary conditions on both surfaces can be expressed as (Fig. 2–36)

\[
-k \frac{\partial T(x, t)}{\partial x} = \epsilon_1 \sigma [T^4_{\text{surr}, 1} - T(0, t)^4] \tag{2-52a}
\]

and

\[
-k \frac{\partial T(L, t)}{\partial x} = \epsilon_2 \sigma [T(L, t)^4 - T^4_{\text{surr}, 2}] \tag{2-52b}
\]

where \(\epsilon_1\) and \(\epsilon_2\) are the emissivities of the boundary surfaces, \(\sigma = 5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4\) is the Stefan–Boltzmann constant, and \(T_{\text{surr}, 1}\) and \(T_{\text{surr}, 2}\) are the average temperatures of the surfaces surrounding the two sides of the plate, respectively. Note that the temperatures in radiation calculations must be expressed in K or R (not in °C or °F).

The radiation boundary condition involves the fourth power of temperature, and thus it is a nonlinear condition. As a result, the application of this boundary condition results in powers of the unknown coefficients, which makes it difficult to determine them. Therefore, it is tempting to ignore radiation exchange at a surface during a heat transfer analysis in order to avoid the complications associated with nonlinearity. This is especially the case when heat transfer at the surface is dominated by convection, and the role of radiation is minor.

5 Interface Boundary Conditions

Some bodies are made up of layers of different materials, and the solution of a heat transfer problem in such a medium requires the solution of the heat transfer problem in each layer. This, in turn, requires the specification of the boundary conditions at each interface.

The boundary conditions at an interface are based on the requirements that (1) two bodies in contact must have the same temperature at the area of contact and (2) an interface (which is a surface) cannot store any energy, and thus the heat flux on the two sides of an interface must be the same. The boundary conditions at the interface of two bodies \(A\) and \(B\) in perfect contact at \(x = x_0\) can be expressed as (Fig. 2–37)

\[
T_A(x_0, t) = T_B(x_0, t) \tag{2-53}
\]

and

\[
-k_A \frac{\partial T_A(x_0, t)}{\partial x} = -k_B \frac{\partial T_B(x_0, t)}{\partial x} \tag{2-54}
\]

where \(k_A\) and \(k_B\) are the thermal conductivities of the layers \(A\) and \(B\), respectively. The case of imperfect contact results in thermal contact resistance, which is considered in the next chapter.
6 Generalized Boundary Conditions

So far we have considered surfaces subjected to single mode heat transfer, such as the specified heat flux, convection, or radiation for simplicity. In general, however, a surface may involve convection, radiation, and specified heat flux simultaneously. The boundary condition in such cases is again obtained from a surface energy balance, expressed as

\[
\begin{align*}
\text{Heat transfer} \\
\text{to the surface} \\
\text{in all modes} &= \text{Heat transfer} \\
\text{from the surface} \\
\text{in all modes}
\end{align*}
\]

(2-55)

This is illustrated in Examples 2–9 and 2–10.

**EXAMPLE 2–9 Combined Convection and Radiation Condition**

A spherical metal ball of radius \( r_0 \) is heated in an oven to a temperature of 600°F throughout and is then taken out of the oven and allowed to cool in ambient air at \( T_a = 78°F \), as shown in Figure 2–38. The thermal conductivity of the ball material is \( k = 8.3 \text{ Btu/h · ft · °F} \), and the average convection heat transfer coefficient on the outer surface of the ball is evaluated to be \( h = 4.5 \text{ Btu/h · ft}^2 · °F \). The emissivity of the outer surface of the ball is \( \varepsilon = 0.6 \), and the average temperature of the surrounding surfaces is \( T_{\text{surr}} = 525 \text{ R} \). Assuming the ball is cooled uniformly from the entire outer surface, express the initial and boundary conditions for the cooling process of the ball.

**SOLUTION** The ball is initially at a uniform temperature and is cooled uniformly from the entire outer surface. Therefore, this is a one-dimensional transient heat transfer problem since the temperature within the ball will change with the radial distance \( r \) and the time \( t \). That is, \( T = T(r, t) \). Taking the moment the ball is removed from the oven to be \( t = 0 \), the initial condition can be expressed as

\[
T(r, 0) = T_i = 600°F
\]

The problem possesses symmetry about the midpoint \(( r = 0)\) since the isotherms in this case will be concentric spheres, and thus no heat will be crossing the midpoint of the ball. Then the boundary condition at the midpoint can be expressed as

\[
\frac{\partial T(0, t)}{\partial r} = 0
\]

The heat conducted to the outer surface of the ball is lost to the environment by convection and radiation. Then taking the direction of heat transfer to be the positive \( r \) direction, the boundary condition on the outer surface can be expressed as

\[
-k \frac{\partial T(r_0, t)}{\partial r} = h[T(r_0) - T_a] + \varepsilon \sigma [T(r_0)^4 - T_{\text{surr}}^4]
\]

**FIGURE 2–38** Schematic for Example 2–9.
All the quantities in the above relations are known except the temperatures and their derivatives at $r = 0$ and $r_0$. Also, the radiation part of the boundary condition is often ignored for simplicity by modifying the convection heat transfer coefficient to account for the contribution of radiation. The convection coefficient $h$ in that case becomes the combined heat transfer coefficient.

**EXAMPLE 2–10** Combined Convection, Radiation, and Heat Flux

Consider the south wall of a house that is $L = 0.2$ m thick. The outer surface of the wall is exposed to solar radiation and has an absorptivity of $\alpha = 0.5$ for solar energy. The interior of the house is maintained at $T_{w1} = 20^\circ$C, while the ambient air temperature outside remains at $T_{w2} = 5^\circ$C. The sky, the ground, and the surfaces of the surrounding structures at this location can be modeled as a surface at an effective temperature of $T_{\text{sky}} = 255$ K for radiation exchange on the outer surface. The radiation exchange between the inner surface of the wall and the surfaces of the walls, floor, and ceiling it faces is negligible. The convection heat transfer coefficients on the inner and the outer surfaces of the wall are $h_1 = 6$ W/m$^2$ · °C and $h_2 = 25$ W/m$^2$ · °C, respectively. The thermal conductivity of the wall material is $k = 0.7$ W/m · °C, and the emissivity of the outer surface is $\varepsilon_2 = 0.9$. Assuming the heat transfer through the wall to be steady and one-dimensional, express the boundary conditions on the inner and the outer surfaces of the wall.

**SOLUTION** We take the direction normal to the wall surfaces as the $x$-axis with the origin at the inner surface of the wall, as shown in Figure 2–39. The heat transfer through the wall is given to be steady and one-dimensional, and thus the temperature depends on $x$ only and not on time. That is, $T = T(x)$.

The boundary condition on the inner surface of the wall at $x = 0$ is a typical convection condition since it does not involve any radiation or specified heat flux. Taking the direction of heat transfer to be the positive $x$-direction, the boundary condition on the inner surface can be expressed as

$$-k \frac{dT(0)}{dx} = h_1[T_{w1} - T(0)]$$

The boundary condition on the outer surface at $x = 0$ is quite general as it involves conduction, convection, radiation, and specified heat flux. Again taking the direction of heat transfer to be the positive $x$-direction, the boundary condition on the outer surface can be expressed as

$$-k \frac{dT(L)}{dx} = h_2[T_{w2}] + \varepsilon_2\sigma[T(L)^4 - T_{\text{sky}}^4] = -q_{\text{solar}}$$

where $q_{\text{solar}}$ is the incident solar heat flux. Assuming the opposite direction for heat transfer would give the same result multiplied by $-1$, which is equivalent to the relation here. All the quantities in these relations are known except the temperatures and their derivatives at the two boundaries.
Note that a heat transfer problem may involve different kinds of boundary conditions on different surfaces. For example, a plate may be subject to heat flux on one surface while losing or gaining heat by convection from the other surface. Also, the two boundary conditions in a direction may be specified at the same boundary, while no condition is imposed on the other boundary. For example, specifying the temperature and heat flux at \( x = 0 \) of a plate of thickness \( L \) will result in a unique solution for the one-dimensional steady temperature distribution in the plate, including the value of temperature at the surface \( x = L \). Although not necessary, there is nothing wrong with specifying more than two boundary conditions in a specified direction, provided that there is no contradiction. The extra conditions in this case can be used to verify the results.

2–5 SOLUTION OF STEADY ONE-DIMENSIONAL HEAT CONDUCTION PROBLEMS

So far we have derived the differential equations for heat conduction in various coordinate systems and discussed the possible boundary conditions. A heat conduction problem can be formulated by specifying the applicable differential equation and a set of proper boundary conditions.

In this section we will solve a wide range of heat conduction problems in rectangular, cylindrical, and spherical geometries. We will limit our attention to problems that result in ordinary differential equations such as the steady one-dimensional heat conduction problems. We will also assume constant thermal conductivity, but will consider variable conductivity later in this chapter. If you feel rusty on differential equations or haven’t taken differential equations yet, no need to panic. Simple integration is all you need to solve the steady one-dimensional heat conduction problems.

The solution procedure for solving heat conduction problems can be summarized as (1) formulate the problem by obtaining the applicable differential equation in its simplest form and specifying the boundary conditions, (2) obtain the general solution of the differential equation, and (3) apply the boundary conditions and determine the arbitrary constants in the general solution (Fig. 2–40). This is demonstrated below with examples.

EXAMPLE 2–11 Heat Conduction in a Plane Wall

Consider a large plane wall of thickness \( L = 0.2 \) m, thermal conductivity \( k = 1.2 \) W/m·°C, and surface area \( A = 15 \) m². The two sides of the wall are maintained at constant temperatures of \( T_1 = 120°C \) and \( T_2 = 50°C \), respectively, as shown in Figure 2–41. Determine (a) the variation of temperature within the wall and the value of temperature at \( x = 0.1 \) m and (b) the rate of heat conduction through the wall under steady conditions.

SOLUTION A plane wall with specified surface temperatures is given. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions 1 Heat conduction is steady. 2 Heat conduction is one-dimensional since the wall is large relative to its thickness and the thermal
conditions on both sides are uniform. 3 Thermal conductivity is constant. 4 There is no heat generation.

**Properties** The thermal conductivity is given to be \( k = 1.2 \) W/m \( \cdot \) °C.

**Analysis** (a) Taking the direction normal to the surface of the wall to be the \( x \)-direction, the differential equation for this problem can be expressed as

\[
\frac{d^2T}{dx^2} = 0
\]

with boundary conditions

\[
T(0) = T_1 = 120^\circ C \\
T(L) = T_2 = 50^\circ C
\]

The differential equation is linear and second order, and a quick inspection of it reveals that it has a single term involving derivatives and no terms involving the unknown function \( T \) as a factor. Thus, it can be solved by direct integration. Noting that an integration reduces the order of a derivative by one, the general solution of the differential equation above can be obtained by two simple successive integrations, each of which introduces an integration constant.

Integrating the differential equation once with respect to \( x \) yields

\[
\frac{dT}{dx} = C_1
\]

where \( C_1 \) is an arbitrary constant. Notice that the order of the derivative went down by one as a result of integration. As a check, if we take the derivative of this equation, we will obtain the original differential equation. This equation is not the solution yet since it involves a derivative.

Integrating one more time, we obtain

\[
T(x) = C_1x + C_2
\]

which is the general solution of the differential equation (Fig. 2–42). The general solution in this case resembles the general formula of a straight line whose slope is \( C_1 \) and whose value at \( x = 0 \) is \( C_2 \). This is not surprising since the second derivative represents the change in the slope of a function, and a zero second derivative indicates that the slope of the function remains constant. Therefore, *any straight line* is a solution of this differential equation.

The general solution contains two unknown constants \( C_1 \) and \( C_2 \), and thus we need two equations to determine them uniquely and obtain the specific solution. These equations are obtained by forcing the general solution to satisfy the specified boundary conditions. The application of each condition yields one equation, and thus we need to specify two conditions to determine the constants \( C_1 \) and \( C_2 \).

When applying a boundary condition to an equation, *all occurrences of the dependent and independent variables and any derivatives are replaced by the specified values*. Thus the only unknowns in the resulting equations are the arbitrary constants.

The first boundary condition can be interpreted as *in the general solution, replace all the x’s by zero and T(x) by T_1*. That is (Fig. 2–43),

\[
T(0) = C_1 \times 0 + C_2 \rightarrow C_2 = T_1
\]
The second boundary condition can be interpreted as in the general solution, replace all the x’s by L and T(x) by T. That is,

\[ T(L) = C_1L + C_2 \rightarrow T_2 = C_1L + T_1 \rightarrow C_1 = \frac{T_2 - T_1}{L} \]

Substituting the \( C_1 \) and \( C_2 \) expressions into the general solution, we obtain

\[ T(x) = \frac{T_2 - T_1}{L} x + T_1 \]  

(2-56)

which is the desired solution since it satisfies not only the differential equation but also the two specified boundary conditions. That is, differentiating Eq. 2–56 with respect to \( x \) twice will give \( \frac{d^2T}{dx^2} \), which is the given differential equation, and substituting \( x = 0 \) and \( x = L \) into Eq. 2–56 gives \( T(0) = T_1 \) and \( T(L) = T_2 \), respectively, which are the specified conditions at the boundaries.

Substituting the given information, the value of the temperature at \( x = 0.1 \) m is determined to be

\[ T(0.1 \text{ m}) = \frac{(50 - 120)^\circ \text{C}}{0.2 \text{ m}} (0.1 \text{ m}) + 120^\circ \text{C} = 85^\circ \text{C} \]

(b) The rate of heat conduction anywhere in the wall is determined from Fourier’s law to be

\[ \dot{Q}_{\text{wall}} = -kA \frac{dT}{dx} = -kAC_1 = -kA \frac{T_2 - T_1}{L} = kA \frac{T_1 - T_2}{L} \]  

(2-57)

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

\[ \dot{Q} = kA \frac{T_1 - T_2}{L} = (1.2 \text{ W/m} \cdot ^\circ \text{C})(15 \text{ m}^2) \frac{(120 - 50)^\circ \text{C}}{0.2 \text{ m}} = 6300 \text{ W} \]

Discussion Note that under steady conditions, the rate of heat conduction through a plane wall is constant.

**EXAMPLE 2–12  A Wall with Various Sets of Boundary Conditions**

Consider steady one-dimensional heat conduction in a large plane wall of thickness \( L \) and constant thermal conductivity \( k \) with no heat generation. Obtain expressions for the variation of temperature within the wall for the following pairs of boundary conditions (Fig. 2–44):

(a) \(-k \frac{dT(0)}{dx} = q_0 = 40 \text{ W/cm}^2\) and \( T(0) = T_0 = 15^\circ \text{C} \)

(b) \(-k \frac{dT(0)}{dx} = q_0 = 40 \text{ W/cm}^2\) and \(-k \frac{dT(L)}{dx} = q_L = -25 \text{ W/cm}^2\)

(c) \(-k \frac{dT(0)}{dx} = q_0 = 40 \text{ W/cm}^2\) and \(-k \frac{dT(L)}{dx} = q_0 = 40 \text{ W/cm}^2\)
SOLUTION  This is a steady one-dimensional heat conduction problem with constant thermal conductivity and no heat generation in the medium, and the heat conduction equation in this case can be expressed as (Eq. 2–17)

\[ \frac{d^2T}{dx^2} = 0 \]

whose general solution was determined in the previous example by direct integration to be

\[ T(x) = C_1x + C_2 \]

where \( C_1 \) and \( C_2 \) are two arbitrary integration constants. The specific solutions corresponding to each specified pair of boundary conditions are determined as follows.

(a) In this case, both boundary conditions are specified at the same boundary at \( x = 0 \), and no boundary condition is specified at the other boundary at \( x = L \). Noting that

\[ \frac{dT}{dx} = C_1 \]

the application of the boundary conditions gives

\[ -k \frac{dT(0)}{dx} = q_0 \quad \rightarrow \quad -kC_1 = q_0 \quad \rightarrow \quad C_1 = \frac{q_0}{k} \]

and

\[ T(0) = T_0 \quad \rightarrow \quad T_0 = C_1 \times 0 + C_2 \quad \rightarrow \quad C_2 = T_0 \]

Substituting, the specific solution in this case is determined to be

\[ T(x) = -\frac{q_0}{k} + T_0 \]

Therefore, the two boundary conditions can be specified at the same boundary, and it is not necessary to specify them at different locations. In fact, the fundamental theorem of linear ordinary differential equations guarantees that a
unique solution exists when both conditions are specified at the same location. But no such guarantee exists when the two conditions are specified at different boundaries, as you will see below.

(b) In this case different heat fluxes are specified at the two boundaries. The application of the boundary conditions gives

\[-k \frac{dT(0)}{dx} = \dot{q}_0 \rightarrow -kC_1 = \dot{q}_0 \rightarrow C_1 = -\frac{\dot{q}_0}{k}\]

and

\[-k \frac{dT(L)}{dx} = \dot{q}_L \rightarrow -kC_1 = \dot{q}_L \rightarrow C_1 = -\frac{\dot{q}_L}{k}\]

Since \(\dot{q}_0 \neq \dot{q}_L\) and the constant \(C_1\) cannot be equal to two different things at the same time, there is no solution in this case. This is not surprising since this case corresponds to supplying heat to the plane wall from both sides and expecting the temperature of the wall to remain steady (not to change with time). This is impossible.

(c) In this case, the same values for heat flux are specified at the two boundaries. The application of the boundary conditions gives

\[-k \frac{dT(0)}{dx} = \dot{q}_0 \rightarrow -kC_1 = \dot{q}_0 \rightarrow C_1 = -\frac{\dot{q}_0}{k}\]

and

\[-k \frac{dT(L)}{dx} = \dot{q}_0 \rightarrow -kC_1 = \dot{q}_0 \rightarrow C_1 = -\frac{\dot{q}_0}{k}\]

Thus, both conditions result in the same value for the constant \(C_1\), but no value for \(C_2\). Substituting, the specific solution in this case is determined to be

\[T(x) = -\frac{\dot{q}_0}{k}x + C_2\]

which is not a unique solution since \(C_2\) is arbitrary. This solution represents a family of straight lines whose slope is \(-\dot{q}_0/k\). Physically, this problem corresponds to requiring the rate of heat supplied to the wall at \(x = 0\) be equal to the rate of heat removal from the other side of the wall at \(x = L\). But this is a consequence of the heat conduction through the wall being steady, and thus the second boundary condition does not provide any new information. So it is not surprising that the solution of this problem is not unique. The three cases discussed above are summarized in Figure 2–45.

**EXAMPLE 2–13 Heat Conduction in the Base Plate of an Iron**

Consider the base plate of a 1200-W household iron that has a thickness of \(L = 0.5\) cm, base area of \(A = 300\) cm², and thermal conductivity of \(k = 15\) W/m · °C. The inner surface of the base plate is subjected to uniform heat flux generated by the resistance heaters inside, and the outer surface loses heat to the surroundings at \(T_e = 20°C\) by convection, as shown in Figure 2–46.
Taking the convection heat transfer coefficient to be $h = 80 \text{ W/m}^2 \cdot \degree \text{C}$ and disregarding heat loss by radiation, obtain an expression for the variation of temperature in the base plate, and evaluate the temperatures at the inner and the outer surfaces.

**SOLUTION**  The base plate of an iron is considered. The variation of temperature in the plate and the surface temperatures are to be determined.

**Assumptions**

1. Heat transfer is steady since there is no change with time.
2. Heat transfer is one-dimensional since the surface area of the base plate is large relative to its thickness, and the thermal conditions on both sides are uniform.
3. Thermal conductivity is constant.
4. There is no heat generation in the medium.
5. Heat transfer by radiation is negligible.
6. The upper part of the iron is well insulated so that the entire heat generated in the resistance wires is transferred to the base plate through its inner surface.

**Properties**  The thermal conductivity is given to be $k = 15 \text{ W/m} \cdot \degree \text{C}$.

**Analysis**  The inner surface of the base plate is subjected to uniform heat flux at a rate of

$$
q_0 = \frac{Q_0}{A_{\text{base}}} = \frac{1200 \text{ W}}{0.03 \text{ m}^2} = 40,000 \text{ W/m}^2
$$

The outer side of the plate is subjected to the convection condition. Taking the direction normal to the surface of the wall as the $x$-direction with its origin on the inner surface, the differential equation for this problem can be expressed as (Fig. 2–47)

$$
\frac{dT}{dx} = 0
$$

with the boundary conditions

$$
-k \frac{dT(0)}{dx} = q_0 = 40,000 \text{ W/m}^2
$$

$$
-k \frac{dT(L)}{dx} = h[T(L) - T_\infty]
$$

The general solution of the differential equation is again obtained by two successive integrations to be

$$
\frac{dT}{dx} = C_1
$$

and

$$
T(x) = C_1 x + C_2 \quad \text{(a)}
$$

where $C_1$ and $C_2$ are arbitrary constants. Applying the first boundary condition,

$$
-k \frac{dT(0)}{dx} = q_0 \rightarrow -kC_1 = q_0 \rightarrow C_1 = \frac{q_0}{k}
$$

Noting that $dT/dx = C_1$ and $T(L) = C_1 L + C_2$, the application of the second boundary condition gives

\[ hT(L) - T_\infty \]
\[-k \frac{dT(L)}{dx} = h[T(L) - T_\infty] \rightarrow -kC_1 = h(C_1L + C_2) - T_\infty\]

Substituting \(C_1 = -\dot{q}_s/k\) and solving for \(C_2\), we obtain

\[C_2 = T_\infty + \frac{\dot{q}_s}{h} + \frac{q_0}{k}L\]

Now substituting \(C_1\) and \(C_2\) into the general solution \((a)\) gives

\[T(x) = T_\infty + \frac{\dot{q}_s}{h} + \frac{L-x}{k} + \frac{1}{h}\]

which is the solution for the variation of the temperature in the plate. The temperatures at the inner and outer surfaces of the plate are determined by substituting \(x = 0\) and \(x = L\), respectively, into the relation \((b)\):

\[T(0) = T_\infty + \frac{\dot{q}_s}{h} \left( L + \frac{1}{h} \right) = 20^\circ C + (40,000 \text{ W/m}^2 \cdot \text{K}) \left( \frac{0.005 \text{ m}}{15 \text{ W/m} \cdot \text{K}} + \frac{1}{80 \text{ W/m}^2 \cdot \text{K}} \right) = 533^\circ C\]

and

\[T(L) = T_\infty + \frac{\dot{q}_s}{h} \left( 0 + \frac{1}{h} \right) = 20^\circ C + \frac{40,000 \text{ W/m}^2}{80 \text{ W/m}^2 \cdot \text{K}} = 520^\circ C\]

**Discussion** Note that the temperature of the inner surface of the base plate will be 13°C higher than the temperature of the outer surface when steady operating conditions are reached. Also note that this heat transfer analysis enables us to calculate the temperatures of surfaces that we cannot even reach. This example demonstrates how the heat flux and convection boundary conditions are applied to heat transfer problems.

**EXAMPLE 2–14 Heat Conduction in a Solar Heated Wall**

Consider a large plane wall of thickness \(L = 0.06 \text{ m}\) and thermal conductivity \(k = 1.2 \text{ W/m} \cdot \text{K}\) in space. The wall is covered with white porcelain tiles that have an emissivity of \(\varepsilon = 0.85\) and a solar absorptivity of \(\alpha = 0.26\), as shown in Figure 2–48. The inner surface of the wall is maintained at \(T_1 = 300 \text{ K}\) at all times, while the outer surface is exposed to solar radiation that is incident at a rate of \(q_{\text{solar}} = 800 \text{ W/m}^2\). The outer surface is also losing heat by radiation to deep space at 0 K. Determine the temperature of the outer surface of the wall and the rate of heat transfer through the wall when steady operating conditions are reached. What would your response be if no solar radiation was incident on the surface?

**SOLUTION** A plane wall in space is subjected to specified temperature on one side and solar radiation on the other side. The outer surface temperature and the rate of heat transfer are to be determined.
Assumptions 1 Heat transfer is steady since there is no change with time. 2 Heat transfer is one-dimensional since the wall is large relative to its thickness, and the thermal conditions on both sides are uniform. 3 Thermal conductivity is constant. 4 There is no heat generation.

Properties The thermal conductivity is given to be $k = 1.2 \text{ W/m} \cdot \text{°C}$.

Analysis Taking the direction normal to the surface of the wall as the $x$-direction with its origin on the inner surface, the differential equation for this problem can be expressed as

$$\frac{d^2T}{dx^2} = 0$$

with boundary conditions

$$T(0) = T_1 = 300 \text{ K}$$
$$-k\frac{dT(L)}{dx} = \varepsilon\sigma[T(L)^4 - T_{\text{space}}^4] - \alpha q_{\text{solar}}$$

where $T_{\text{space}} = 0$. The general solution of the differential equation is again obtained by two successive integrations to be

$$T(x) = C_1x + C_2$$ (a)

where $C_1$ and $C_2$ are arbitrary constants. Applying the first boundary condition yields

$$T(0) = C_1 \times 0 + C_2 \rightarrow C_2 = T_1$$

Noting that $dT/dx = C_1$ and $T(L) = C_1L + C_2 = C_1L + T_1$, the application of the second boundary conditions gives

$$-k\frac{dT(L)}{dx} = \varepsilon\sigma T(L)^4 - \alpha q_{\text{solar}} \rightarrow -kC_1 = \varepsilon\sigma(C_1L + T_1)^4 - \alpha q_{\text{solar}}$$

Although $C_1$ is the only unknown in this equation, we cannot get an explicit expression for it because the equation is nonlinear, and thus we cannot get a closed-form expression for the temperature distribution. This should explain why we do our best to avoid nonlinearities in the analysis, such as those associated with radiation.

Let us back up a little and denote the outer surface temperature by $T(L) = T_L$ instead of $T(L) = C_1L + T_1$. The application of the second boundary condition in this case gives

$$-k\frac{dT(L)}{dx} = \varepsilon\sigma T(L)^4 - \alpha q_{\text{solar}} \rightarrow -kC_1 = \varepsilon\sigma T_L^4 - \alpha q_{\text{solar}}$$

Solving for $C_1$ gives

$$C_1 = \frac{\alpha q_{\text{solar}} - \varepsilon\sigma T_L^4}{k}$$ (b)

Now substituting $C_1$ and $C_2$ into the general solution (a), we obtain

$$T(x) = \frac{\alpha q_{\text{solar}} - \varepsilon\sigma T_L^4}{k} x + T_1$$ (c)
(1) Rearrange the equation to be solved:

\[ T_L = 310.4 - 0.240975 \left( \frac{T_L}{100} \right)^4 \]

The equation is in the proper form since the left side consists of \( T_L \) only.

(2) Guess the value of \( T_L \), say 300 K, and substitute into the right side of the equation. It gives

\[ T_L = 290.2 \text{ K} \]

(3) Now substitute this value of \( T_L \) into the right side of the equation and get

\[ T_L = 293.1 \text{ K} \]

(4) Repeat step (3) until convergence to desired accuracy is achieved. The subsequent iterations give

\[ T_L = 292.6 \text{ K} \]
\[ T_L = 292.7 \text{ K} \]
\[ T_L = 292.7 \text{ K} \]

Therefore, the solution is \( T_L = 292.7 \text{ K} \). The result is independent of the initial guess.

**FIGURE 2–49**

A simple method of solving a nonlinear equation is to arrange the equation such that the unknown is alone on the left side while everything else is on the right side, and to iterate after an initial guess until convergence.

which is the solution for the variation of the temperature in the wall in terms of the unknown outer surface temperature \( T_L \). At \( x = L \) it becomes

\[ T_L = \frac{\alpha q_{solar} - \varepsilon \sigma T_L^4}{k} L + T_1 \]  

\[(d)\]

which is an implicit relation for the outer surface temperature \( T_L \). Substituting the given values, we get

\[ T_L = \frac{0.26 \times (800 \text{ W/m}^2) - 0.85 \times (5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4) T_L^4}{1.2 \text{ W/m} \cdot \text{K}} (0.06 \text{ m}) + 300 \text{ K} \]

which simplifies to

\[ T_L = 310.4 - 0.240975 \left( \frac{T_L}{100} \right)^4 \]

This equation can be solved by one of the several nonlinear equation solvers available (or by the old fashioned trial-and-error method) to give (Fig. 2–49)

\[ T_L = 292.7 \text{ K} \]

Knowing the outer surface temperature and knowing that it must remain constant under steady conditions, the temperature distribution in the wall can be determined by substituting the \( T_L \) value above into Eq. (c):

\[ T(x) = \frac{0.26 \times (800 \text{ W/m}^2) - 0.85 \times (5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)(292.7 \text{ K})^4}{1.2 \text{ W/m} \cdot \text{K}} x + 300 \text{ K} \]

which simplifies to

\[ T(x) = (-121.5 \text{ K/m}) x + 300 \text{ K} \]

Note that the outer surface temperature turned out to be lower than the inner surface temperature. Therefore, the heat transfer through the wall will be toward the outside despite the absorption of solar radiation by the outer surface. Knowing both the inner and outer surface temperatures of the wall, the steady rate of heat conduction through the wall can be determined from

\[ q = k \frac{T_0 - T_L}{L} = (1.2 \text{ W/m} \cdot \text{K}) \frac{(300 - 292.7) \text{ K}}{0.06 \text{ m}} = 146 \text{ W/m}^2 \]

**Discussion**  In the case of no incident solar radiation, the outer surface temperature, determined from Eq. (d) by setting \( q_{solar} = 0 \), will be \( T_L = 284.3 \text{ K} \). It is interesting to note that the solar energy incident on the surface causes the surface temperature to increase by about 8 K only when the inner surface temperature of the wall is maintained at 300 K.

**EXAMPLE 2–15  Heat Loss through a Steam Pipe**

Consider a steam pipe of length \( L = 20 \text{ m} \), inner radius \( r_1 = 6 \text{ cm} \), outer radius \( r_2 = 8 \text{ cm} \), and thermal conductivity \( k = 20 \text{ W/m} \cdot \text{°C} \), as shown in Figure 2–50. The inner and outer surfaces of the pipe are maintained at average temperatures of \( T_1 = 150^\circ \text{C} \) and \( T_2 = 60^\circ \text{C} \), respectively. Obtain a general relation
for the temperature distribution inside the pipe under steady conditions, and determine the rate of heat loss from the steam through the pipe.

**SOLUTION**  A steam pipe is subjected to specified temperatures on its surfaces. The variation of temperature and the rate of heat transfer are to be determined.

**Assumptions**  
1. Heat transfer is steady since there is no change with time. 
2. Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no variation in the axial direction, and thus $T = T(r)$. 
3. Thermal conductivity is constant. 
4. There is no heat generation.

**Properties**  The thermal conductivity is given to be $k = 20 \text{ W/m} \cdot \text{°C}$.

**Analysis**  The mathematical formulation of this problem can be expressed as

$$\frac{d}{dr} \left( r \frac{dT}{dr} \right) = 0$$

with boundary conditions

$$T(r_1) = T_1 = 150^\circ \text{C}$$
$$T(r_2) = T_2 = 60^\circ \text{C}$$

Integrating the differential equation once with respect to $r$ gives

$$r \frac{dT}{dr} = C_1$$

where $C_1$ is an arbitrary constant. We now divide both sides of this equation by $r$ to bring it to a readily integrable form,

$$\frac{dT}{dr} = \frac{C_1}{r}$$

Again integrating with respect to $r$ gives (Fig. 2–51)

$$T(r) = C_1 \ln r + C_2 \quad (a)$$

We now apply both boundary conditions by replacing all occurrences of $r$ and $T(r)$ in Eq. (a) with the specified values at the boundaries. We get

$$T(r_1) = T_1 \quad \rightarrow \quad C_1 \ln r_1 + C_2 = T_1$$
$$T(r_2) = T_2 \quad \rightarrow \quad C_1 \ln r_2 + C_2 = T_2$$

which are two equations in two unknowns, $C_1$ and $C_2$. Solving them simultaneously gives

$$C_1 = \frac{T_2 - T_1}{\ln(r_2/r_1)} \quad \text{and} \quad C_2 = T_1 - \frac{T_2 - T_1}{\ln(r_2/r_1)} \ln r_1$$

Substituting them into Eq. (a) and rearranging, the variation of temperature within the pipe is determined to be

$$T(r) = \left( \frac{\ln(r/r_1)}{\ln(r_2/r_1)} \right) (T_2 - T_1) + T_1 \quad (2-58)$$

The rate of heat loss from the steam is simply the total rate of heat conduction through the pipe, and is determined from Fourier’s law to be

$$\text{Rate of heat loss} = -k \frac{dT}{dr} \bigg|_{r=r_2}$$
HEAT TRANSFER

\[
\dot{Q}_{\text{cylinder}} = -kA \frac{dT}{dr} = -k(2\pi r L) \frac{C_1}{r} = -2\pi kL C_1 = 2\pi kL \frac{T_1 - T_2}{\ln(r_2/r_1)} \tag{2-59}
\]

The numerical value of the rate of heat conduction through the pipe is determined by substituting the given values

\[
\dot{Q} = 2\pi (20 \text{ W/m} \cdot \text{°C})(20 \text{ m}) \frac{(150 - 60)\text{°C}}{\ln(0.08/0.06)} = 786 \text{ kW}
\]

DISCUSSION  Note that the total rate of heat transfer through a pipe is constant, but the heat flux is not since it decreases in the direction of heat transfer with increasing radius since \( \dot{q} = \dot{Q} / (2\pi r L) \).

EXAMPLE 2–16  Heat Conduction through a Spherical Shell

Consider a spherical container of inner radius \( r_1 = 8 \text{ cm} \), outer radius \( r_2 = 10 \text{ cm} \), and thermal conductivity \( k = 45 \text{ W/m} \cdot \text{°C} \), as shown in Figure 2–52. The inner and outer surfaces of the container are maintained at constant temperatures of \( T_1 = 200\text{°C} \) and \( T_2 = 80\text{°C} \), respectively, as a result of some chemical reactions occurring inside. Obtain a general relation for the temperature distribution inside the shell under steady conditions, and determine the rate of heat loss from the container.

SOLUTION  A spherical container is subjected to specified temperatures on its surfaces. The variation of temperature and the rate of heat transfer are to be determined.

Assumptions  1 Heat transfer is steady since there is no change with time. 2 Heat transfer is one-dimensional since there is thermal symmetry about the midpoint, and thus \( T = T(r) \). 3 Thermal conductivity is constant. 4 There is no heat generation.

Properties  The thermal conductivity is given to be \( k = 45 \text{ W/m} \cdot \text{°C} \).

Analysis  The mathematical formulation of this problem can be expressed as

\[
\frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) = 0
\]

with boundary conditions

\[
T(r_1) = T_1 = 200\text{°C}
\]

\[
T(r_2) = T_2 = 80\text{°C}
\]

Integrating the differential equation once with respect to \( r \) yields

\[
r^2 \frac{dT}{dr} = C_1
\]

where \( C_1 \) is an arbitrary constant. We now divide both sides of this equation by \( r^2 \) to bring it to a readily integrable form,

\[
\frac{dT}{dr} = \frac{C_1}{r^2}
\]
Again integrating with respect to \( r \) gives

\[
T(r) = -\frac{C_1}{r} + C_2 \tag{a}
\]

We now apply both boundary conditions by replacing all occurrences of \( r \) and \( T(r) \) in the relation above by the specified values at the boundaries. We get

\[
T(r_1) = T_1 \quad \rightarrow \quad -\frac{C_1}{r_1} + C_2 = T_1
\]

\[
T(r_2) = T_2 \quad \rightarrow \quad -\frac{C_1}{r_2} + C_2 = T_2
\]

which are two equations in two unknowns, \( C_1 \) and \( C_2 \). Solving them simultaneously gives

\[
C_1 = r_2 T_2 - r_1 T_1 \quad \text{and} \quad C_2 = \frac{r_2 T_2 - r_1 T_1}{r_2 - r_1}
\]

Substituting into Eq. (a), the variation of temperature within the spherical shell is determined to be

\[
T(r) = \frac{r_1 r_2}{r (r_2 - r_1)} (T_1 - T_2) + \frac{r_2 T_2 - r_1 T_1}{r_2 - r_1} \tag{2-60}
\]

The rate of heat loss from the container is simply the total rate of heat conduction through the container wall and is determined from Fourier’s law

\[
\dot{Q}_\text{sphere} = -kA \frac{dT}{dr} = -k(4\pi r^2) \frac{C_1}{r^2} = -4\pi k C_1 = 4\pi k r_1 \frac{T_1 - T_2}{r_2 - r_1} \tag{2-61}
\]

The numerical value of the rate of heat conduction through the wall is determined by substituting the given values to be

\[
\dot{Q} = 4\pi(45 \text{ W/m} \cdot ^\circ\text{C})(0.08 \text{ m})(0.10 \text{ m})\frac{(200 - 80)^\circ\text{C}}{(0.10 - 0.08) \text{ m}} = 27,140 \text{ W}
\]

**Discussion** Note that the total rate of heat transfer through a spherical shell is constant, but the heat flux, \( \dot{q} = \dot{Q}/4\pi r^2 \), is not since it decreases in the direction of heat transfer with increasing radius as shown in Figure 2–53.

### 2–6 HEAT GENERATION IN A SOLID

Many practical heat transfer applications involve the conversion of some form of energy into thermal energy in the medium. Such mediums are said to involve internal heat generation, which manifests itself as a rise in temperature throughout the medium. Some examples of heat generation are resistance heating in wires, exothermic chemical reactions in a solid, and nuclear reactions in nuclear fuel rods where electrical, chemical, and nuclear energies are converted to heat, respectively (Fig. 2–54). The absorption of radiation throughout the volume of a semitransparent medium such as water can also be considered as heat generation within the medium, as explained earlier.
Heat generation is usually expressed per unit volume of the medium, and is denoted by \( \dot{g} \), whose unit is W/m\(^3\). For example, heat generation in an electrical wire of outer radius \( r_0 \) and length \( L \) can be expressed as

\[
\dot{g} = \frac{E_{\text{electric}}}{V_{\text{wire}}} = \frac{I^2 R_s}{\pi r_0^2 L} \quad \text{(W/m}^3\text{)}
\]  

(2-62)

where \( I \) is the electric current and \( R_s \) is the electrical resistance of the wire.

The temperature of a medium rises during heat generation as a result of the absorption of the generated heat by the medium during transient start-up period. As the temperature of the medium increases, so does the heat transfer from the medium to its surroundings. This continues until steady operating conditions are reached and the rate of heat generation equals the rate of heat transfer to the surroundings. Once steady operation has been established, the temperature of the medium at any point no longer changes.

The maximum temperature \( T_{\text{max}} \) in a solid that involves uniform heat generation will occur at a location farthest away from the outer surface when the outer surface of the solid is maintained at a constant temperature \( T_s \). For example, the maximum temperature occurs at the midplane in a plane wall, at the centerline in a long cylinder, and at the midpoint in a sphere. The temperature distribution within the solid in these cases will be symmetrical about the center of symmetry.

The quantities of major interest in a medium with heat generation are the surface temperature \( T_s \) and the maximum temperature \( T_{\text{max}} \) that occurs in the medium in steady operation. Below we develop expressions for these two quantities for common geometries for the case of uniform heat generation (\( \dot{g} = \) constant) within the medium.

Consider a solid medium of surface area \( A_s \), volume \( V \), and constant thermal conductivity \( k \), where heat is generated at a constant rate of \( \dot{g} \) per unit volume. Heat is transferred from the solid to the surrounding medium at \( T_s \), with a constant heat transfer coefficient of \( h \). All the surfaces of the solid are maintained at a common temperature \( T_s \). Under steady conditions, the energy balance for this solid can be expressed as (Fig. 2–55)

\[
\begin{pmatrix}
\text{Rate of energy generation within the solid} \\
\text{Rate of heat transfer from the solid}
\end{pmatrix} = \begin{pmatrix}
\text{Rate of energy generation within the solid} \\
\text{Rate of heat transfer from the solid}
\end{pmatrix}
\]

(2-63)

or

\[
\dot{Q} = \dot{g} V \quad \text{(W)}
\]  

(2-64)

Disregarding radiation (or incorporating it in the heat transfer coefficient \( h \)), the heat transfer rate can also be expressed from Newton’s law of cooling as

\[
\dot{Q} = hA_s (T_s - T_s) \quad \text{(W)}
\]  

(2-65)

Combining Eqs. 2–64 and 2–65 and solving for the surface temperature \( T_s \) gives

\[
T_s = T_s + \frac{\dot{g} V}{hA_s}
\]  

(2-66)
For a large plane wall of thickness $2L$ ($A_s = 2A_{wall}$ and $V = 2L_{wall}$), a long solid cylinder of radius $r_s$ ($A_s = 2\pi r_s L$ and $V = \pi r_s^2 L$), and a solid sphere of radius $r_0$ ($A_s = 4\pi r_0^2$ and $V = \frac{4}{3} \pi r_0^3$), Eq. 2–66 reduces to

$$T_s, \text{ plane wall} = T_0 + \frac{\dot{g} L}{h}$$  \hspace{1cm} (2-67)

$$T_s, \text{ cylinder} = T_0 + \frac{\dot{g} r_s}{2h}$$  \hspace{1cm} (2-68)

$$T_s, \text{ sphere} = T_0 + \frac{\dot{g} r_0}{3h}$$  \hspace{1cm} (2-69)

Note that the rise in surface temperature $T_s$ is due to heat generation in the solid.

Reconsider heat transfer from a long solid cylinder with heat generation. We mentioned above that, under steady conditions, the entire heat generated within the medium is conducted through the outer surface of the cylinder. Now consider an imaginary inner cylinder of radius $r$ within the cylinder (Fig. 2–56). Again the heat generated within this inner cylinder must be equal to the heat conducted through the outer surface of this inner cylinder. That is, from Fourier’s law of heat conduction,

$$-kA_r \frac{dT}{dr} = \dot{g} V_r$$  \hspace{1cm} (2-70)

where $A_r = 2\pi r L$ and $V_r = \pi r^2 L$ at any location $r$. Substituting these expressions into Eq. 2–70 and separating the variables, we get

$$-k(2\pi r L) \frac{dT}{dr} = \dot{g}(\pi r^2 L) \rightarrow dT = -\frac{\dot{g}}{2k} r^2 dr$$

Integrating from $r = 0$ where $T(0) = T_0$ to $r = r_o$ where $T(r_o) = T$, yields

$$\Delta T_{\text{max, cylinder}} = T_o - T = \frac{\dot{g} r_o^2}{4k}$$  \hspace{1cm} (2-71)

where $T_o$ is the centerline temperature of the cylinder, which is the maximum temperature, and $\Delta T_{\text{max}}$ is the difference between the centerline and the surface temperatures of the cylinder, which is the maximum temperature rise in the cylinder above the surface temperature. Once $\Delta T_{\text{max}}$ is available, the centerline temperature can easily be determined from (Fig. 2–57)

$$T_{\text{center}} = T_o = T_s + \Delta T_{\text{max}}$$  \hspace{1cm} (2-72)

The approach outlined above can also be used to determine the maximum temperature rise in a plane wall of thickness $2L$ and a solid sphere of radius $r_0$, with these results:

$$\Delta T_{\text{max, plane wall}} = \frac{\dot{g} L^2}{2k}$$  \hspace{1cm} (2-73)

$$\Delta T_{\text{max, sphere}} = \frac{\dot{g} r_o^2}{6k}$$  \hspace{1cm} (2-74)
Again the maximum temperature at the center can be determined from Eq. 2–72 by adding the maximum temperature rise to the surface temperature of the solid.

**EXAMPLE 2–17 Centerline Temperature of a Resistance Heater**

A 2-kW resistance heater wire whose thermal conductivity is \( k = 15 \text{ W/m } \cdot \degree \text{C} \) has a diameter of \( D = 4 \text{ mm} \) and a length of \( L = 0.5 \text{ m} \), and is used to boil water (Fig. 2–58). If the outer surface temperature of the resistance wire is \( T_s = 105^\circ \text{C} \), determine the temperature at the center of the wire.

**SOLUTION** The surface temperature of a resistance heater submerged in water is to be determined.

**Assumptions** 1 Heat transfer is steady since there is no change with time. 2 Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no change in the axial direction. 3 Thermal conductivity is constant. 4 Heat generation in the heater is uniform.

**Properties** The thermal conductivity is given to be \( k = 15 \text{ W/m } \cdot \degree \text{C} \).

**Analysis** The 2-kW resistance heater converts electric energy into heat at a rate of 2 kW. The heat generation per unit volume of the wire is

\[
\dot{q} = \frac{\dot{Q}_{\text{gen}}}{V_{\text{wire}}} = \frac{\dot{Q}_{\text{gen}}}{\pi r_0^2 L} = \frac{2000 \text{ W}}{\pi (0.002 \text{ m})^2 (0.5 \text{ m})} = 0.318 \times 10^6 \text{ W/m}^3
\]

Then the center temperature of the wire is determined from Eq. 2–71 to be

\[
T_o = T_s + \frac{\dot{q}r_0^2}{4k} = 105^\circ \text{C} + \frac{(0.318 \times 10^6 \text{ W/m}^3)(0.002 \text{ m})^2}{4 \times (15 \text{ W/m } \cdot \degree \text{C})} = 126^\circ \text{C}
\]

**Discussion** Note that the temperature difference between the center and the surface of the wire is 21°C.

We have developed these relations using the intuitive *energy balance* approach. However, we could have obtained the same relations by setting up the appropriate *differential equations* and solving them, as illustrated in Examples 2–18 and 2–19.

**EXAMPLE 2–18 Variation of Temperature in a Resistance Heater**

A long homogeneous resistance wire of radius \( r_0 = 0.2 \text{ in.} \) and thermal conductivity \( k = 7.8 \text{ Btu/h } \cdot \text{ft } \cdot \degree \text{F} \) is being used to boil water at atmospheric pressure by the passage of electric current, as shown in Figure 2–59. Heat is generated in the wire uniformly as a result of resistance heating at a rate of \( \dot{q} = 2400 \text{ Btu/h } \cdot \text{in}^3. \) If the outer surface temperature of the wire is measured to be \( T_s = 226^\circ \text{F} \), obtain a relation for the temperature distribution, and determine the temperature at the centerline of the wire when steady operating conditions are reached.
**SOLUTION** This heat transfer problem is similar to the problem in Example 2–17, except that we need to obtain a relation for the variation of temperature within the wire with \( r \). Differential equations are well suited for this purpose.

**Assumptions**
1. Heat transfer is steady since there is no change with time.
2. Heat transfer is one-dimensional since there is no thermal symmetry about the centerline and no change in the axial direction.
3. Thermal conductivity is constant.
4. Heat generation in the wire is uniform.

**Properties** The thermal conductivity is given to be \( k = 7.8 \) Btu/h · ft · °F.

**Analysis** The differential equation which governs the variation of temperature in the wire is simply Eq. 2–27,

\[
\frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) + \frac{\dot{g}}{k} = 0
\]

This is a second-order linear ordinary differential equation, and thus its general solution will contain two arbitrary constants. The determination of these constants requires the specification of two boundary conditions, which can be taken to be

\[ T(r_0) = T_s = 226^\circ F \]

and

\[ \frac{dT(0)}{dr} = 0 \]

The first boundary condition simply states that the temperature of the outer surface of the wire is 226°F. The second boundary condition is the symmetry condition at the centerline, and states that the maximum temperature in the wire will occur at the centerline, and thus the slope of the temperature at \( r = 0 \) must be zero (Fig. 2–60). This completes the mathematical formulation of the problem.

Although not immediately obvious, the differential equation is in a form that can be solved by direct integration. Multiplying both sides of the equation by \( r \) and rearranging, we obtain

\[
\frac{d}{dr} \left( r \frac{dT}{dr} \right) = -\frac{\dot{g}}{k} r
\]

Integrating with respect to \( r \) gives

\[
r \frac{dT}{dr} = -\frac{\dot{g}}{k} \frac{r^2}{2} + C_1 \quad (a)
\]

since the heat generation is constant, and the integral of a derivative of a function is the function itself. That is, integration removes a derivative. It is convenient at this point to apply the second boundary condition, since it is related to the first derivative of the temperature, by replacing all occurrences of \( r \) and \( dT/dr \) in Eq. (a) by zero. It yields

\[
0 \times \frac{dT(0)}{dr} = -\frac{\dot{g}}{2k} \times 0 + C_1 \quad \rightarrow \quad C_1 = 0
\]

**FIGURE 2–60** The thermal symmetry condition at the centerline of a wire in which heat is generated uniformly.
Thus $C_1$ cancels from the solution. We now divide Eq. (a) by $r$ to bring it to a readily integrable form,

$$\frac{dT}{dr} = -\frac{g}{2k} r$$

Again integrating with respect to $r$ gives

$$T(r) = -\frac{g}{4k} r^2 + C_2$$

We now apply the first boundary condition by replacing all occurrences of $r$ by $r_0$ and all occurrences of $T$ by $T_s$. We get

$$T_i = -\frac{g}{4k} r_0^2 + C_2 \quad \rightarrow \quad C_2 = T_s + \frac{g}{4k} r_0^2$$

Substituting this $C_2$ relation into Eq. (b) and rearranging give

$$T(r) = T_i + \frac{g}{4k} (r_0^2 - r^2)$$

which is the desired solution for the temperature distribution in the wire as a function of $r$. The temperature at the centerline ($r = 0$) is obtained by replacing $r$ in Eq. (c) by zero and substituting the known quantities,

$$T(0) = T_i + \frac{g}{4k} r_0^2 = 226^\circ F + \frac{2400 \text{ Btu/h} \cdot \text{in}^3}{4 \times (7.8 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ F) \left( 12 \text{ in.} \cdot 1 \text{ ft} \right)} (0.2 \text{ in.})^2 = 263^\circ F$$

**Discussion** The temperature of the centerline will be $37^\circ F$ above the temperature of the outer surface of the wire. Note that the expression above for the centerline temperature is identical to Eq. 2–71, which was obtained using an energy balance on a control volume.

---

**EXAMPLE 2–19** Heat Conduction in a Two-Layer Medium

Consider a long resistance wire of radius $r_1 = 0.2 \text{ cm}$ and thermal conductivity $k_{\text{wire}} = 15 \text{ W/m} \cdot ^\circ \text{C}$ in which heat is generated uniformly as a result of resistance heating at a constant rate of $\dot{g} = 50 \text{ W/cm}^3$ (Fig. 2–61). The wire is embedded in a 0.5-cm-thick layer of ceramic whose thermal conductivity is $k_{\text{ceramic}} = 1.2 \text{ W/m} \cdot ^\circ \text{C}$. If the outer surface temperature of the ceramic layer is measured to be $T_s = 45^\circ \text{C}$, determine the temperatures at the center of the resistance wire and the interface of the wire and the ceramic layer under steady conditions.

**SOLUTION** The surface and interface temperatures of a resistance wire covered with a ceramic layer are to be determined.

**Assumptions** 1 Heat transfer is steady since there is no change with time. 2 Heat transfer is one-dimensional since this two-layer heat transfer problem possesses symmetry about the centerline and involves no change in the axial direction, and thus $T = T(r)$. 3 Thermal conductivities are constant. 4 Heat generation in the wire is uniform.

**Properties** It is given that $k_{\text{wire}} = 15 \text{ W/m} \cdot ^\circ \text{C}$ and $k_{\text{ceramic}} = 1.2 \text{ W/m} \cdot ^\circ \text{C}$. 

![FIGURE 2–61](Schematic for Example 2–19.)
**Analysis**  Letting $T_I$ denote the unknown interface temperature, the heat transfer problem in the wire can be formulated as

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{dT_{\text{wire}}}{dr} \right) + \frac{g}{k} = 0$$

with

$$T_{\text{wire}}(r_1) = T_I$$
$$\frac{dT_{\text{wire}}(0)}{dr} = 0$$

This problem was solved in Example 2–18, and its solution was determined to be

$$T_{\text{wire}}(r) = T_I + \frac{g}{4k_{\text{wire}}} (r_1^2 - r^2) \quad (a)$$

Noting that the ceramic layer does not involve any heat generation and its outer surface temperature is specified, the heat conduction problem in that layer can be expressed as

$$\frac{d}{dr} \left( r \frac{dT_{\text{ceramic}}}{dr} \right) = 0$$

with

$$T_{\text{ceramic}}(r_1) = T_I$$
$$T_{\text{ceramic}}(r_2) = T_s = 45^\circ C$$

This problem was solved in Example 2–15, and its solution was determined to be

$$T_{\text{ceramic}}(r) = \frac{\ln(r/r_1)}{\ln(r_2/r_1)} (T_s - T_I) + T_I \quad (b)$$

We have already utilized the first interface condition by setting the wire and ceramic layer temperatures equal to $T_I$ at the interface $r = r_1$. The interface temperature $T_I$ is determined from the second interface condition that the heat flux in the wire and the ceramic layer at $r = r_1$ must be the same:

$$-k_{\text{wire}} \frac{dT_{\text{wire}}(r_1)}{dr} = -k_{\text{ceramic}} \frac{dT_{\text{ceramic}}(r_1)}{dr} \quad \rightarrow \quad \frac{gr_1}{2} = -k_{\text{ceramic}} \ln(r_2/r_1) \frac{T_s - T_I}{r_1} \left( \frac{1}{r_1^2} \right)$$

Solving for $T_I$ and substituting the given values, the interface temperature is determined to be

$$T_I = \frac{g r_1^2}{2k_{\text{ceramic}}} \ln \frac{r_2}{r_1} + T_s$$
$$= \frac{(50 \times 10^6 \text{ W/m}^3)(0.002 \text{ m})^2}{2(1.2 \text{ W/m} \cdot ^\circ C) \ln \frac{0.007 \text{ m}}{0.002 \text{ m}}} + 45^\circ C = 149.4^\circ C$$

Knowing the interface temperature, the temperature at the centerline ($r = 0$) is obtained by substituting the known quantities into Eq. (a),

$$T_{\text{wire}}(0) = T_I + \frac{g r_1^2}{4k_{\text{wire}}} = 149.4^\circ C + \frac{(50 \times 10^6 \text{ W/m}^3)(0.002 \text{ m})^2}{4 \times (15 \text{ W/m} \cdot ^\circ C)} = 152.7^\circ C$$
Thus the temperature of the centerline will be slightly above the interface temperature.

**Discussion** This example demonstrates how steady one-dimensional heat conduction problems in composite media can be solved. We could also solve this problem by determining the heat flux at the interface by dividing the total heat generated in the wire by the surface area of the wire, and then using this value as the specified heat flux boundary condition for both the wire and the ceramic layer. This way the two problems are decoupled and can be solved separately.

## 2–7 VARIABLE THERMAL CONDUCTIVITY, \( k(T) \)

You will recall from Chapter 1 that the thermal conductivity of a material, in general, varies with temperature (Fig. 2–62). However, this variation is mild for many materials in the range of practical interest and can be disregarded. In such cases, we can use an average value for the thermal conductivity and treat it as a constant, as we have been doing so far. This is also common practice for other temperature-dependent properties such as the density and specific heat.

When the variation of thermal conductivity with temperature in a specified temperature interval is large, however, it may be necessary to account for this variation to minimize the error. Accounting for the variation of the thermal conductivity with temperature, in general, complicates the analysis. But in the case of simple one-dimensional cases, we can obtain heat transfer relations in a straightforward manner.

When the variation of thermal conductivity with temperature \( k(T) \) is known, the average value of the thermal conductivity in the temperature range between \( T_1 \) and \( T_2 \) can be determined from

\[
 k_{\text{ave}} = \frac{\int_{T_1}^{T_2} k(T) dT}{T_2 - T_1} \tag{2-75}
\]

This relation is based on the requirement that the rate of heat transfer through a medium with constant average thermal conductivity \( k_{\text{ave}} \) equals the rate of heat transfer through the same medium with variable conductivity \( k(T) \). Note that in the case of constant thermal conductivity \( k(T) = k \), Eq. 2–75 reduces to \( k_{\text{ave}} = k \), as expected.

Then the rate of steady heat transfer through a plane wall, cylindrical layer, or spherical layer for the case of variable thermal conductivity can be determined by replacing the constant thermal conductivity \( k \) in Eqs. 2–57, 2–59, and 2–61 by the \( k_{\text{ave}} \) expression (or value) from Eq. 2–75:

\[
 \dot{Q}_{\text{plane wall}} = k_{\text{ave}} A \frac{T_1 - T_2}{L} = \frac{A}{L} \int_{T_1}^{T_2} k(T) dT
\]

\[
 \dot{Q}_{\text{cylinder}} = 2\pi k_{\text{ave}} L \ln\left(\frac{r_2}{r_1}\right) \frac{T_1 - T_2}{L} = 2\pi \frac{L}{\ln\left(\frac{r_2}{r_1}\right)} \int_{T_1}^{T_2} k(T) dT
\]

\[
 \dot{Q}_{\text{sphere}} = 4\pi k_{\text{ave}} \left(\frac{r_2}{r_1}\right) \frac{T_1 - T_2}{r_2 - r_1} = 4\pi \frac{r_2}{r_1} \int_{T_1}^{T_2} k(T) dT
\]
The variation in thermal conductivity of a material with temperature in the temperature range of interest can often be approximated as a linear function and expressed as

$$k(T) = k_0 (1 + \beta T) \quad (2-79)$$

where $\beta$ is called the **temperature coefficient of thermal conductivity**. The average value of thermal conductivity in the temperature range $T_1$ to $T_2$ in this case can be determined from

$$k_{ave} = \frac{\int_{T_1}^{T_2} k_0 (1 + \beta T) dT}{T_2 - T_1} = k_0 \left( 1 + \frac{\beta}{2} \frac{T_2 + T_1}{T_1} \right) = k(T_{ave}) \quad (2-80)$$

Note that the **average thermal conductivity** in this case is equal to the thermal conductivity value at the **average temperature**.

We have mentioned earlier that in a plane wall the temperature varies linearly during steady one-dimensional heat conduction when the thermal conductivity is constant. But this is no longer the case when the thermal conductivity changes with temperature, even linearly, as shown in Figure 2–63.

---

**EXAMPLE 2–20**  
Variation of Temperature in a Wall with $k(T)$

Consider a plane wall of thickness $L$ whose thermal conductivity varies linearly in a specified temperature range as $k(T) = k_0 (1 + \beta T)$ where $k_0$ and $\beta$ are constants. The wall surface at $x = 0$ is maintained at a constant temperature of $T_1$ while the surface at $x = L$ is maintained at $T_2$, as shown in Figure 2–64.

Assuming steady one-dimensional heat transfer, obtain a relation for (a) the heat transfer rate through the wall and (b) the temperature distribution $T(x)$ in the wall.

**SOLUTION**  
A plate with variable conductivity is subjected to specified temperatures on both sides. The variation of temperature and the rate of heat transfer are to be determined.

**Assumptions**  
1. Heat transfer is given to be steady and one-dimensional.  
2. Thermal conductivity varies linearly.  
3. There is no heat generation.

**Properties**  
The thermal conductivity is given to be $k(T) = k_0 (1 + \beta T)$.

**Analysis**  
(a) The rate of heat transfer through the wall can be determined from

$$\dot{Q} = k_{ave} A \frac{T_1 - T_2}{L}$$

where $A$ is the heat conduction area of the wall and

$$k_{ave} = k(T_{ave}) = k_0 \left( 1 + \frac{\beta}{2} \frac{T_2 + T_1}{T_1} \right)$$

is the average thermal conductivity (Eq. 2–80).

(b) To determine the temperature distribution in the wall, we begin with Fourier's law of heat conduction, expressed as

$$\dot{Q} = -k(T) A \frac{dT}{dx}$$
where the rate of conduction heat transfer $\dot{Q}$ and the area $A$ are constant. Separating variables and integrating from $x = 0$ where $T(0) = T_1$ to any $x$ where $T(x) = T_f$ we get

$$
\int_0^x \dot{Q} \, dx = -A \int_{T_1}^{T_f} k(T) \, dT
$$

Substituting $k(T) = k_0(1 + \beta T)$ and performing the integrations we obtain

$$
\dot{Q} \, x = -A k_0 [(T - T_1) + \beta(T^2 - T_f^2)/2]
$$

Substituting the $\dot{Q}$ expression from part (a) and rearranging give

$$
T^2 + \frac{2}{\beta} T + \frac{2k_{\text{ave}} x}{\beta k_0} (T_1 - T_2) - T_i^2 - \frac{2}{\beta} T_1 = 0
$$

which is a quadratic equation in the unknown temperature $T$. Using the quadratic formula, the temperature distribution $T(x)$ in the wall is determined to be

$$
T(x) = -\frac{1}{\beta} \pm \frac{1}{\sqrt{\beta^2 - \frac{2k_{\text{ave}} x}{\beta k_0} (T_1 - T_2) + T_i^2}} \frac{2}{\beta} T_1
$$

The proper sign of the square root term (+ or -) is determined from the requirement that the temperature at any point within the medium must remain between $T_1$ and $T_2$. This result explains why the temperature distribution in a plane wall is no longer a straight line when the thermal conductivity varies with temperature.

---

**EXAMPLE 2–21 Heat Conduction through a Wall with $k(T)$**

Consider a 2-m-high and 0.7-m-wide bronze plate whose thickness is 0.1 m. One side of the plate is maintained at a constant temperature of 600 K while the other side is maintained at 400 K, as shown in Figure 2–65. The thermal conductivity of the bronze plate can be assumed to vary linearly in that temperature range as $k(T) = k_0(1 + \beta T)$ where $k_0 = 38 \, \text{W/m} \cdot \text{K}$ and $\beta = 9.21 \times 10^{-4} \, \text{K}^{-1}$. Disregarding the edge effects and assuming steady one-dimensional heat transfer, determine the rate of heat conduction through the plate.

**SOLUTION** A plate with variable conductivity is subjected to specified temperatures on both sides. The rate of heat transfer is to be determined.

**Assumptions** 1 Heat transfer is given to be steady and one-dimensional. 2 Thermal conductivity varies linearly. 3 There is no heat generation.

**Properties** The thermal conductivity is given to be $k(T) = k_0(1 + \beta T)$.

**Analysis** The average thermal conductivity of the medium in this case is simply the value at the average temperature and is determined from

$$
k_{\text{ave}} = k(T_{\text{ave}}) = k_0 \left(1 + \beta \frac{T_2 + T_1}{2}\right)
$$

$$
= (38 \, \text{W/m} \cdot \text{K}) \left[1 + (9.21 \times 10^{-4} \, \text{K}^{-1}) \frac{(600 + 400) \, \text{K}}{2}\right]
$$

$$
= 55.5 \, \text{W/m} \cdot \text{K}
$$

![FIGURE 2–65 Schematic for Example 2–21.](image)
Then the rate of heat conduction through the plate can be determined from Eq. 2–76 to be

\[
\dot{Q} = \frac{k_{av} A}{\frac{T_1 - T_2}{L}} = (55.5 \text{ W/m} \cdot \text{K})(2 \text{ m} \times 0.7 \text{ m}) \frac{(600 - 400) \text{K}}{0.1 \text{ m}} = 155,400 \text{ W}
\]

**Discussion** We would have obtained the same result by substituting the given \( k(T) \) relation into the second part of Eq. 2–76 and performing the indicated integration.

---

**TOPIC OF SPECIAL INTEREST**

*A Brief Review of Differential Equations*

As we mentioned in Chapter 1, the description of most scientific problems involves relations that involve changes in some key variables with respect to each other. Usually the smaller the increment chosen in the changing variables, the more general and accurate the description. In the limiting case of infinitesimal or differential changes in variables, we obtain differential equations, which provide precise mathematical formulations for the physical principles and laws by representing the rates of change as derivatives. Therefore, differential equations are used to investigate a wide variety of problems in science and engineering, including heat transfer.

Differential equations arise when relevant physical laws and principles are applied to a problem by considering infinitesimal changes in the variables of interest. Therefore, obtaining the governing differential equation for a specific problem requires an adequate knowledge of the nature of the problem, the variables involved, appropriate simplifying assumptions, and the applicable physical laws and principles involved, as well as a careful analysis (Fig. 2–66).

An equation, in general, may involve one or more variables. As the name implies, a **variable** is a quantity that may assume various values during a study. A quantity whose value is fixed during a study is called a **constant**. Constants are usually denoted by the earlier letters of the alphabet such as \( a, b, c, \) and \( d \), whereas variables are usually denoted by the later ones such as \( t, x, y, \) and \( z \). A variable whose value can be changed arbitrarily is called an **independent variable** (or argument). A variable whose value depends on the value of other variables and thus cannot be varied independently is called a **dependent variable** (or a function).

A dependent variable \( y \) that depends on a variable \( x \) is usually denoted as \( y(x) \) for clarity. However, this notation becomes very inconvenient and cumbersome when \( y \) is repeated several times in an expression. In such cases it is desirable to denote \( y(x) \) simply as \( y \) when it is clear that \( y \) is a function of \( x \). This shortcut in notation improves the appearance and the

*This section can be skipped if desired without a loss in continuity.*
The derivative of a function at a point represents the slope of the tangent line of the function at that point.

**FIGURE 2–67**
The derivative of a function at a point represents the slope of the tangent line of the function at that point.

\[
y'(x) = \frac{dy(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x) - y(x)}{\Delta x}
\]

Here \( \Delta x \) represents a (small) change in the independent variable \( x \) and is called an *increment* of \( x \). The corresponding change in the function \( y \) is called an increment of \( y \) and is denoted by \( \Delta y \). Therefore, the derivative of a function can be viewed as the ratio of the increment \( \Delta y \) of the function to the increment \( \Delta x \) of the independent variable for very small \( \Delta x \). Note that \( \Delta y \) and thus \( y'(x) \) will be zero if the function \( y \) does not change with \( x \).

Most problems encountered in practice involve quantities that change with time \( t \), and their first derivatives with respect to time represent the rate of change of those quantities with time. For example, if \( N(t) \) denotes the population of a bacteria colony at time \( t \), then the first derivative \( N' = \frac{dN}{dt} \) represents the rate of change of the population, which is the amount the population increases or decreases per unit time.

The derivative of the first derivative of a function \( y \) is called the *second derivative* of \( y \), and is denoted by \( y'' \) or \( d^2y/dx^2 \). In general, the derivative of the \((n - 1)\)st derivative of \( y \) is called the \( n \)th derivative of \( y \) and is denoted by \( y^{(n)} \) or \( d^ny/dx^n \). Here, \( n \) is a positive integer and is called the *order* of the derivative. The order \( n \) should not be confused with the *degree* of a derivative. For example, \( y''' \) is the third-order derivative of \( y \), but \((y')^3\) is the third degree of the first derivative of \( y \). Note that the first derivative of a function represents the *slope* or the *rate of change* of the function with the independent variable, and the second derivative represents the *rate of change of the slope* of the function with the independent variable.

When a function \( y \) depends on two or more independent variables such as \( x \) and \( t \), it is sometimes of interest to examine the dependence of the function on one of the variables only. This is done by taking the derivative of the function with respect to that variable while holding the other variables constant. Such derivatives are called *partial derivatives*. The first partial derivatives of the function \( y(x, t) \) with respect to \( x \) and \( t \) are defined as (Fig. 2–68)

\[
\frac{\partial y}{\partial x} = \lim_{\Delta x \to 0} \frac{y(x + \Delta x, t) - y(x, t)}{\Delta x}
\]

\[
\frac{\partial y}{\partial t} = \lim_{\Delta t \to 0} \frac{y(x, t + \Delta t) - y(x, t)}{\Delta t}
\]

Note that when finding \( \partial y/\partial x \) we treat \( t \) as a constant and differentiate \( y \) with respect to \( x \). Likewise, when finding \( \partial y/\partial t \) we treat \( x \) as a constant and differentiate \( y \) with respect to \( t \).

Integration can be viewed as the inverse process of differentiation. Integration is commonly used in solving differential equations since solving a differential equation is essentially a process of removing the derivatives.
from the equation. Differentiation is the process of finding \( y'(x) \) when a function \( y(x) \) is given, whereas integration is the process of finding the function \( y(x) \) when its derivative \( y'(x) \) is given. The integral of this derivative is expressed as

\[
\int y'(x)\,dx = \int dy = y(x) + C
\]

(2-84)

since \( y'(x)\,dx = dy \) and the integral of the differential of a function is the function itself (plus a constant, of course). In Eq. 2–84, \( x \) is the integration variable and \( C \) is an arbitrary constant called the integration constant.

The derivative of \( y(x) + C \) is \( y'(x) \) no matter what the value of the constant \( C \) is. Therefore, two functions that differ by a constant have the same derivative, and we always add a constant \( C \) during integration to recover this constant that is lost during differentiation. The integral in Eq. 2–84 is called an indefinite integral since the value of the arbitrary constant \( C \) is indefinite. The described procedure can be extended to higher-order derivatives (Fig. 2–69). For example,

\[
\int y''(x)\,dx = y'(x) + C
\]

(2-85)

This can be proved by defining a new variable \( u(x) = y'(x) \), differentiating it to obtain \( u'(x) = y''(x) \), and then applying Eq. 2–84. Therefore, the order of a derivative decreases by one each time it is integrated.

**Classification of Differential Equations**

A differential equation that involves only ordinary derivatives is called an ordinary differential equation, and a differential equation that involves partial derivatives is called a partial differential equation. Then it follows that problems that involve a single independent variable result in ordinary differential equations, and problems that involve two or more independent variables result in partial differential equations. A differential equation may involve several derivatives of various orders of an unknown function. The order of the highest derivative in a differential equation is the order of the equation. For example, the order of \( y^{(m)} + (y')^4 = 7x^5 \) is 3 since it contains no fourth or higher order derivatives.

You will remember from algebra that the equation \( 3x - 5 = 0 \) is much easier to solve than the equation \( x^3 + 3x - 5 = 0 \) because the first equation is linear whereas the second one is nonlinear. This is also true for differential equations. Therefore, before we start solving a differential equation, we usually check for linearity. A differential equation is said to be linear if the dependent variable and all of its derivatives are of the first degree and their coefficients depend on the independent variable only. In other words, a differential equation is linear if it can be written in a form that does not involve (1) any powers of the dependent variable or its derivatives such as \( y^3 \) or \( (y')^2 \), (2) any products of the dependent variable or its derivatives such as \( yy' \) or \( y'y''' \), and (3) any other nonlinear functions of the dependent variable such as \( \sin y \) or \( e^y \). If any of these conditions apply, it is nonlinear (Fig. 2–70).
A linear differential equation, however, may contain (1) powers or nonlinear functions of the independent variable, such as $x^2$ and $\cos x$ and (2) products of the dependent variable (or its derivatives) and functions of the independent variable, such as $x^3y', x^2y$, and $e^{-2y}$. A linear differential equation of order $n$ can be expressed in the most general form as

$$y^{(n)} + f_1(x)y^{(n-1)} + \cdots + f_{n-1}(x)y' + f_n(x)y = R(x) \quad (2.86)$$

A differential equation that cannot be put into this form is nonlinear. A linear differential equation in $y$ is said to be **homogeneous** as well if $R(x) = 0$. Otherwise, it is nonhomogeneous. That is, each term in a linear homogeneous equation contains the dependent variable or one of its derivatives after the equation is cleared of any common factors. The term $R(x)$ is called the **nonhomogeneous term**.

Differential equations are also classified by the nature of the coefficients of the dependent variable and its derivatives. A differential equation is said to have **constant coefficients** if the coefficients of all the terms that involve the dependent variable or its derivatives are constants. If, after clearing any common factors, any of the terms with the dependent variable or its derivatives involves the independent variable as a coefficient, that equation is said to have **variable coefficients** (Fig. 2–71). Differential equations with constant coefficients are usually much easier to solve than those with variable coefficients.

### Solutions of Differential Equations

Solving a differential equation can be as easy as performing one or more integrations; but such simple differential equations are usually the exception rather than the rule. There is no single general solution method applicable to all differential equations. There are different solution techniques, each being applicable to different classes of differential equations. Sometimes solving a differential equation requires the use of two or more techniques as well as ingenuity and mastery of solution methods. Some differential equations can be solved only by using some very clever tricks. Some cannot be solved analytically at all.

In algebra, we usually seek discrete values that satisfy an algebraic equation such as $x^2 - 7x - 10 = 0$. When dealing with differential equations, however, we seek functions that satisfy the equation in a specified interval. For example, the algebraic equation $x^2 - 7x - 10 = 0$ is satisfied by two numbers only: 2 and 5. But the differential equation $y' - 7y = 0$ is satisfied by the function $e^{7x}$ for any value of $x$ (Fig. 2–72).

Consider the algebraic equation $x^3 - 6x^2 + 11x - 6 = 0$. Obviously, $x = 1$ satisfies this equation, and thus it is a solution. However, it is not the only solution of this equation. We can easily show by direct substitution that $x = 2$ and $x = 3$ also satisfy this equation, and thus they are solutions as well. But there are no other solutions to this equation. Therefore, we say that the set 1, 2, and 3 forms the complete solution to this algebraic equation.

The same line of reasoning also applies to differential equations. Typically, differential equations have multiple solutions that contain at least one arbitrary constant. Any function that satisfies the differential equation on an
interval is called a solution of that differential equation in that interval. A solution that involves one or more arbitrary constants represents a family of functions that satisfy the differential equation and is called a general solution of that equation. Not surprisingly, a differential equation may have more than one general solution. A general solution is usually referred to as the general solution or the complete solution if every solution of the equation can be obtained from it as a special case. A solution that can be obtained from a general solution by assigning particular values to the arbitrary constants is called a specific solution.

You will recall from algebra that a number is a solution of an algebraic equation if it satisfies the equation. For example, 2 is a solution of the equation $x^3 - 8 = 0$ because the substitution of 2 for $x$ yields identically zero. Likewise, a function is a solution of a differential equation if that function satisfies the differential equation. In other words, a solution function yields identity when substituted into the differential equation. For example, it can be shown by direct substitution that the function $3e^{-2x}$ is a solution of $y'' - 4y = 0$ (Fig. 2–73).

**SUMMARY**

In this chapter we have studied the heat conduction equation and its solutions. Heat conduction in a medium is said to be steady when the temperature does not vary with time and unsteady or transient when it does. Heat conduction in a medium is said to be one-dimensional when conduction is significant in one dimension only and negligible in the other two dimensions. It is said to be two-dimensional when conduction in the third dimension is negligible and three-dimensional when conduction in all dimensions is significant. In heat transfer analysis, the conversion of electrical, chemical, or nuclear energy into heat (or thermal) energy is characterized as heat generation.

The heat conduction equation can be derived by performing an energy balance on a differential volume element. The one-dimensional heat conduction equation in rectangular, cylindrical, and spherical coordinate systems for the case of constant thermal conductivities are expressed as

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\sigma}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\sigma}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}
\]

\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) + \frac{\sigma}{k} = \frac{1}{\alpha} \frac{\partial T}{\partial t}
\]

where the property $\alpha = k/\rho c$ is the thermal diffusivity of the material.

The solution of a heat conduction problem depends on the conditions at the surfaces, and the mathematical expressions for the thermal conditions at the boundaries are called the boundary conditions. The solution of transient heat conduction problems also depends on the condition of the medium at the beginning of the heat conduction process. Such a condition, which is usually specified at time $t = 0$, is called the initial condition, which is a mathematical expression for the temperature distribution of the medium initially. Complete mathematical description of a heat conduction problem requires the specification of two boundary conditions for each dimension along which heat conduction is significant, and an initial condition when the problem is transient. The most common boundary conditions are the specified temperature, specified heat flux, convection, and radiation boundary conditions. A boundary surface, in general, may involve specified heat flux, convection, and radiation at the same time.

For steady one-dimensional heat transfer through a plate of thickness $L$, the various types of boundary conditions at the surfaces at $x = 0$ and $x = L$ can be expressed as

**Specified temperature:**

\[ T(0) = T_1 \quad \text{and} \quad T(L) = T_2 \]

where $T_1$ and $T_2$ are the specified temperatures at surfaces at $x = 0$ and $x = L$.

**Specified heat flux:**

\[-k \frac{dT(0)}{dx} = q_0 \quad \text{and} \quad -k \frac{dT(L)}{dx} = q_L\]

where $q_0$ and $q_L$ are the specified heat fluxes at surfaces at $x = 0$ and $x = L$.
HEAT TRANSFER

Insulation or thermal symmetry:
\[
\frac{dT(0)}{dx} = 0 \quad \text{and} \quad \frac{dT(L)}{dx} = 0
\]

Convection:
\[
-k \frac{dT(0)}{dx} = h_1[T_{s1} - T(0)] \quad \text{and} \quad -k \frac{dT(L)}{dx} = h_2[T(L) - T_{s2}]
\]

where \(h_1\) and \(h_2\) are the convection heat transfer coefficients and \(T_{s1}\) and \(T_{s2}\) are the temperatures of the surrounding mediums on the two sides of the plate.

Radiation:
\[
-k \frac{dT(0)}{dx} = \varepsilon_1 \sigma [T_{s1}^4 - T(0)^4] \quad \text{and} \quad -k \frac{dT(L)}{dx} = \varepsilon_2 \sigma [T(L)^4 - T_{s2}^4]
\]

where \(\varepsilon_1\) and \(\varepsilon_2\) are the emissivities of the boundary surfaces, \(\sigma = 5.67 \times 10^{-8} \text{ W/m}^2\cdot\text{K}^4\), \(\varepsilon_1\) is the Stefan-Boltzmann constant, and \(T_{s1}\) and \(T_{s2}\) are the average temperatures of the surfaces surrounding the two sides of the plate. In radiation calculations, the temperatures must be in K or R.

Interface of two bodies A and B in perfect contact at \(x = x_0\):
\[
T_A(x_0) = T_B(x_0) \quad \text{and} \quad -k_A \frac{dT_A(x_0)}{dx} = -k_B \frac{dT_B(x_0)}{dx}
\]

where \(k_A\) and \(k_B\) are the thermal conductivities of the layers \(A\) and \(B\).

Heat generation is usually expressed per unit volume of the medium and is denoted by \(\dot{g}\), whose unit is W/m³. Under steady conditions, the surface temperature \(T_s\) of a plane wall of thickness \(2L\), a cylinder of outer radius \(r_c\), and a sphere of radius \(r_s\) in which heat is generated at a constant rate of \(\dot{g}\) per unit volume in a surrounding medium at \(T_e\) can be expressed as

\[
\begin{align*}
T_{s, \text{plane wall}} &= T_e + \frac{\dot{g}L}{h} \\
T_{s, \text{cylinder}} &= T_e + \frac{\dot{g}r_c}{2h} \\
T_{s, \text{sphere}} &= T_e + \frac{\dot{g}r_s}{3h}
\end{align*}
\]

where \(h\) is the convection heat transfer coefficient. The maximum temperature rise between the surface and the midsection of a medium is given by

\[
\Delta T_{\text{max, plane wall}} = \frac{\dot{g}L^2}{2k} \\
\Delta T_{\text{max, cylinder}} = \frac{\dot{g}r_c^2}{4k} \\
\Delta T_{\text{max, sphere}} = \frac{\dot{g}r_s^2}{6k}
\]

When the variation of thermal conductivity with temperature \(k(T)\) is known, the average value of the thermal conductivity in the temperature range between \(T_1\) and \(T_2\) can be determined from

\[
\bar{k} = \frac{\int_{T_1}^{T_2} k(T) dT}{T_2 - T_1}
\]

Then the rate of steady heat transfer through a plane wall, cylindrical layer, or spherical layer can be expressed as

\[
\begin{align*}
\dot{Q}_{\text{plane wall}} &= \bar{k}A \frac{T_1 - T_2}{L} = \frac{A}{L} \int_{T_1}^{T_2} k(T) dT \\
\dot{Q}_{\text{cylinder}} &= 2\pi\bar{k}L \frac{T_1 - T_2}{\ln(r_2/r_1)} = \frac{2\pi L}{\ln(r_2/r_1)} \int_{T_1}^{T_2} k(T) dT \\
\dot{Q}_{\text{sphere}} &= 4\pi\bar{k}r^2 \frac{T_1 - T_2}{r_2 - r_1} = \frac{4\pi r_1^2 r_2^2}{r_2 - r_1} \int_{T_1}^{T_2} k(T) dT
\end{align*}
\]

The variation of thermal conductivity of a material with temperature can often be approximated as a linear function and expressed as

\[
k(T) = k_0(1 + \beta T)
\]

where \(\beta\) is called the temperature coefficient of thermal conductivity.

REFERENCES AND SUGGESTED READING


PROBLEMS*

Introduction

2–1C Is heat transfer a scalar or vector quantity? Explain. Answer the same question for temperature.

2–2C How does transient heat transfer differ from steady heat transfer? How does one-dimensional heat transfer differ from two-dimensional heat transfer?

2–3C Consider a cold canned drink left on a dinner table. Would you model the heat transfer to the drink as one-, two-, or three-dimensional? Would the heat transfer be steady or transient? Also, which coordinate system would you use to analyze this heat transfer problem, and where would you place the origin? Explain.

2–4C Consider a round potato being baked in an oven. Would you model the heat transfer to the potato as one-, two-, or three-dimensional? Would the heat transfer be steady or transient? Also, which coordinate system would you use to solve this problem, and where would you place the origin? Explain.

2–5C Consider an egg being cooked in boiling water in a pan. Would you model the heat transfer to the egg as one-, two-, or three-dimensional? Would the heat transfer be steady or transient? Also, which coordinate system would you use to solve this problem, and where would you place the origin? Explain.

2–6C Consider a hot dog being cooked in boiling water in a pan. Would you model the heat transfer to the hot dog as one-, two-, or three-dimensional? Would the heat transfer be steady or transient? Also, which coordinate system would you use to solve this problem, and where would you place the origin? Explain.

2–7C Consider the cooking process of a roast beef in an oven. Would you consider this to be a steady or transient heat transfer problem? Also, would you consider this to be one-, two-, or three-dimensional? Explain.

2–8C Consider heat loss from a 200-L cylindrical hot water tank in a house to the surrounding medium. Would you consider this to be a steady or transient heat transfer problem? Also, would you consider this heat transfer problem to be one-, two-, or three-dimensional? Explain.

2–9C Does a heat flux vector at a point P on an isothermal surface of a medium have to be perpendicular to the surface at that point? Explain.

2–10C From a heat transfer point of view, what is the difference between isotropic and anisotropic materials?

2–11C What is heat generation in a solid? Give examples.

2–12C Heat generation is also referred to as energy generation or thermal energy generation. What do you think of these phrases?

2–13C In order to determine the size of the heating element of a new oven, it is desired to determine the rate of heat transfer through the walls, door, and the top and bottom sections of the oven. In your analysis, would you consider this to be a

*Problems designated by a “C” are concept questions, and students are encouraged to answer them all. Problems designated by an “E” are in English units, and the SI users can ignore them. Problems with an EES-CD icon are solved using EES, and complete solutions together with parametric studies are included on the enclosed CD. Problems with a computer-EES icon are comprehensive in nature, and are intended to be solved with a computer, preferably using the EES software that accompanies this text.
steady or transient heat transfer problem? Also, would you consider the heat transfer to be one-dimensional or multidimensional? Explain.

2–14E The resistance wire of a 1000-W iron is 15 in. long and has a diameter of \( D = 0.08 \) in. Determine the rate of heat generation in the wire per unit volume, in Btu/h·ft\(^3\), and the heat flux on the outer surface of the wire, in Btu/h·ft\(^2\), as a result of this heat generation.

2–15E Reconsider Problem 2–14E. Using EES (or other) software, evaluate and plot the surface heat flux as a function of wire diameter as the diameter varies from 0.02 to 0.20 in. Discuss the results.

2–16 In a nuclear reactor, heat is generated uniformly in the 5-cm-diameter cylindrical uranium rods at a rate of \( 7 \times 10^7 \) W/m\(^3\). If the length of the rods is 1 m, determine the rate of heat generation in each rod. \( \text{Answer: } 137 \text{ A kW} \)

2–17 In a solar pond, the absorption of solar energy can be modeled as heat generation and can be approximated by \( g = g_0 e^{-bx} \), where \( g_0 \) is the rate of heat absorption at the top surface per unit volume and \( b \) is a constant. Obtain a relation for the total rate of heat generation in a water layer of surface area \( A \) and thickness \( L \) at the top of the pond.

2–18 Consider a large 3-cm-thick stainless steel plate in which heat is generated uniformly at a rate of \( 5 \times 10^6 \) W/m\(^3\). Assuming the plate is losing heat from both sides, determine the heat flux on the surface of the plate during steady operation. \( \text{Answer: } 75,000 \) W/m\(^2\)

Heat Conduction Equation

2–19 Write down the one-dimensional transient heat conduction equation for a plane wall with constant thermal conductivity and heat generation in its simplest form, and indicate what each variable represents.

2–20 Write down the one-dimensional transient heat conduction equation for a long cylinder with constant thermal conductivity and heat generation, and indicate what each variable represents.

2–21 Starting with an energy balance on a rectangular volume element, derive the one-dimensional transient heat conduction equation for a plane wall with constant thermal conductivity and no heat generation.

2–22 Starting with an energy balance on a cylindrical shell volume element, derive the steady one-dimensional heat conduction equation for a long cylinder with constant thermal conductivity in which heat is generated at a rate of \( g \).

2–23 Starting with an energy balance on a spherical shell volume element, derive the one-dimensional transient heat conduction equation for a sphere with constant thermal conductivity and no heat generation.

2–24 Consider a medium in which the heat conduction equation is given in its simplest form as

\[
\frac{\partial^2 T}{\partial x^2} = \frac{\alpha}{\partial t}
\]
(a) Is heat transfer steady or transient?  
(b) Is heat transfer one-, two-, or three-dimensional?  
(c) Is there heat generation in the medium?  
(d) Is the thermal conductivity of the medium constant or variable?

2–25 Consider a medium in which the heat conduction equation is given in its simplest form as

$$\frac{1}{r} \frac{d}{dr} \left( r k \frac{dT}{dr} \right) + \dot{g} = 0$$

(a) Is heat transfer steady or transient?  
(b) Is heat transfer one-, two-, or three-dimensional?  
(c) Is there heat generation in the medium?  
(d) Is the thermal conductivity of the medium constant or variable?

2–26 Consider a medium in which the heat conduction equation is given in its simplest form as

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial T}{\partial r} \right) = \frac{1}{\alpha} \frac{\partial T}{\partial t}$$

(a) Is heat transfer steady or transient?  
(b) Is heat transfer one-, two-, or three-dimensional?  
(c) Is there heat generation in the medium?  
(d) Is the thermal conductivity of the medium constant or variable?

2–27 Consider a medium in which the heat conduction equation is given in its simplest form as

$$r \frac{d^2 T}{dr^2} + \frac{dT}{dr} = 0$$

(a) Is heat transfer steady or transient?  
(b) Is heat transfer one-, two-, or three-dimensional?  
(c) Is there heat generation in the medium?  
(d) Is the thermal conductivity of the medium constant or variable?

2–28 Starting with an energy balance on a volume element, derive the two-dimensional transient heat conduction equation in rectangular coordinates for \(T(x, y, t)\) for the case of constant thermal conductivity and no heat generation.

2–29 Starting with an energy balance on a ring-shaped volume element, derive the two-dimensional transient heat conduction equation in cylindrical coordinates for \(T(r, z)\) for the case of constant thermal conductivity and no heat generation.

2–30 Starting with an energy balance on a disk volume element, derive the one-dimensional transient heat conduction equation for \(T(z, t)\) in a cylinder of diameter \(D\) with an insulated side surface for the case of constant thermal conductivity with heat generation.

2–31 Consider a medium in which the heat conduction equation is given in its simplest form as

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r k \frac{\partial T}{\partial r} \right) + \frac{\partial}{\partial z} \left( k \frac{\partial T}{\partial z} \right) + \dot{g} = 0$$

(a) Is heat transfer steady or transient?  
(b) Is heat transfer one-, two-, or three-dimensional?  
(c) Is there heat generation in the medium?  
(d) Is the thermal conductivity of the medium constant or variable?
Boundary and Initial Conditions; Formulation of Heat Conduction Problems

2–34C What is a boundary condition? How many boundary conditions do we need to specify for a two-dimensional heat transfer problem?

2–35C What is an initial condition? How many initial conditions do we need to specify for a two-dimensional heat transfer problem?

2–36C What is a thermal symmetry boundary condition? How is it expressed mathematically?

2–37C How is the boundary condition on an insulated surface expressed mathematically?

2–38C It is claimed that the temperature profile in a medium must be perpendicular to an insulated surface. Is this a valid claim? Explain.

2–39C Why do we try to avoid the radiation boundary conditions in heat transfer analysis?

2–40 Consider a spherical container of inner radius \( r_1 \), outer radius \( r_2 \), and thermal conductivity \( k \). Express the boundary condition on the inner surface of the container for steady one-dimensional conduction for the following cases: (a) specified temperature of 50°C, (b) specified heat flux of 30 W/m² toward the center, (c) convection to a medium at \( T_w \) with a heat transfer coefficient of \( h \).

2–41 Heat is generated in a long wire of radius \( r_0 \) at a constant rate of \( g_0 \) per unit volume. The wire is covered with a plastic insulation layer. Express the heat flux boundary condition at the interface in terms of the heat generated.

2–42 Consider a long pipe of inner radius \( r_1 \), outer radius \( r_2 \), and thermal conductivity \( k \). The outer surface of the pipe is subjected to convection to a medium at \( T_w \) with a heat transfer coefficient of \( h \), but the direction of heat transfer is not known. Express the convection boundary condition on the outer surface of the pipe.

2–43 Consider a spherical shell of inner radius \( r_1 \), outer radius \( r_2 \), thermal conductivity \( k \), and emissivity \( e \). The outer surface of the shell is subjected to radiation to surrounding surfaces at \( T_{sur} \), but the direction of heat transfer is not known.

Express the radiation boundary condition on the outer surface of the shell.

2–44 A container consists of two spherical layers, A and B, that are in perfect contact. If the radius of the interface is \( r_0 \), express the boundary conditions at the interface.

2–45 Consider a steel pan used to boil water on top of an electric range. The bottom section of the pan is \( L = 0.5 \) cm thick and has a diameter of \( D = 20 \) cm. The electric heating unit on the range top consumes 1000 W of power during cooking, and 85 percent of the heat generated in the heating element is transferred uniformly to the pan. Heat transfer from the top surface of the bottom section to the water is by convection with a heat transfer coefficient of \( h \). Assuming constant thermal conductivity and one-dimensional heat transfer, express the mathematical formulation (the differential equation and the boundary conditions) of this heat conduction problem during steady operation. Do not solve.

2–46E A 2-kW resistance heater wire whose thermal conductivity is \( k = 10.4 \) Btu/h · ft · °F has a radius of \( r_0 = 0.06 \) in. and a length of \( L = 15 \) in., and is used for space heating. Assuming constant thermal conductivity and one-dimensional heat transfer, express the mathematical formulation (the differential equation and the boundary conditions) of this heat conduction problem during steady operation. Do not solve.

2–47 Consider an aluminum pan used to cook stew on top of an electric range. The bottom section of the pan is \( L = 0.25 \) cm thick and has a diameter of \( D = 18 \) cm. The electric heating unit on the range top consumes 900 W of power during cooking, and 90 percent of the heat generated in the heating element
is transferred to the pan. During steady operation, the temperature of the inner surface of the pan is measured to be 108°C. Assuming temperature-dependent thermal conductivity and one-dimensional heat transfer, express the mathematical formulation (the differential equation and the boundary conditions) of this heat conduction problem during steady operation. Do not solve.

2–48 Water flows through a pipe at an average temperature of $T_w = 50{°C}$. The inner and outer radii of the pipe are $r_1 = 6$ cm and $r_2 = 6.5$ cm, respectively. The outer surface of the pipe is wrapped with a thin electric heater that consumes 300 W per m length of the pipe. The exposed surface of the heater is heavily insulated so that the entire heat generated in the heater is transferred to the pipe. Heat is transferred from the inner surface of the pipe to the water by convection with a heat transfer coefficient of $h = 55$ W/m²·°C. Assuming constant thermal conductivity and one-dimensional heat transfer, express the mathematical formulation (the differential equation and the boundary conditions) of the heat conduction in the pipe during steady operation. Do not solve.

2–49 A spherical metal ball of radius $r_0$ is heated in an oven to a temperature of $T_0$ throughout and is then taken out of the oven and dropped into a large body of water at $T_w$, where it is cooled by convection with an average convection heat transfer coefficient of $h$. Assuming constant thermal conductivity and transient one-dimensional heat transfer, express the mathematical formulation (the differential equation and the boundary and initial conditions) of this heat conduction problem. Do not solve.

2–50 A spherical metal ball of radius $r_0$ is heated in an oven to a temperature of $T_0$ throughout and is then taken out of the oven and allowed to cool in ambient air at $T_{∞}$ by convection and radiation. The emissivity of the outer surface of the cylinder is $ε$, and the temperature of the surrounding surfaces is $T_{surr}$. The average convection heat transfer coefficient is estimated to be $h$. Assuming variable thermal conductivity and transient one-dimensional heat transfer, express the mathematical formulation (the differential equation and the boundary and initial conditions) of this heat conduction problem. Do not solve.

2–51 Consider the north wall of a house of thickness $L$. The outer surface of the wall exchanges heat by both convection and radiation. The interior of the house is maintained at $T_{in}$, while the ambient air temperature outside remains at $T_{∞}$. The sky, the ground, and the surfaces of the surrounding structures at this location can be modeled as a surface at an effective temperature of $T_{sky}$ for radiation exchange on the outer surface. The radiation exchange between the inner surface of the wall and the surfaces of the walls, floor, and ceiling it faces is negligible. The convection heat transfer coefficients on the inner and outer surfaces of the wall are $h_1$ and $h_2$, respectively. The thermal conductivity of the wall material is $k$ and the emissivity of the outer surface is $ε$. Assuming the heat transfer through the wall to be steady and one-dimensional, express the mathematical formulation (the differential equation and the boundary and initial conditions) of this heat conduction problem. Do not solve.
**Solution of Steady One-Dimensional Heat Conduction Problems**

**2–52C** Consider one-dimensional heat conduction through a large plane wall with no heat generation that is perfectly insulated on one side and is subjected to convection and radiation on the other side. It is claimed that under steady conditions, the temperature in a plane wall must be uniform (the same everywhere). Do you agree with this claim? Why?

**2–53C** It is stated that the temperature in a plane wall with constant thermal conductivity and no heat generation varies linearly during steady one-dimensional heat conduction. Will this still be the case when the wall loses heat by radiation from its surfaces?

**2–54C** Consider a solid cylindrical rod whose ends are maintained at constant but different temperatures while the side surface is perfectly insulated. There is no heat generation. It is claimed that the temperature along the axis of the rod varies linearly during steady heat conduction. Do you agree with this claim? Why?

**2–55C** Consider a solid cylindrical rod whose side surface is maintained at a constant temperature while the end surfaces are perfectly insulated. The thermal conductivity of the rod material is constant and there is no heat generation. It is claimed that the temperature in the radial direction within the rod will not vary during steady heat conduction. Do you agree with this claim? Why?

**2–56** Consider a large plane wall of thickness \( L = 0.4 \) m, thermal conductivity \( k = 2.3 \) W/m · °C, and surface area \( A = 20 \) m². The left side of the wall is maintained at a constant temperature of \( T_1 = 80 \) °C while the right side loses heat by convection to the surrounding air at \( T_c = 15 \) °C with a heat transfer coefficient of \( h = 24 \) W/m² · °C. Assuming constant thermal conductivity and no heat generation in the wall, (a) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the wall, (b) obtain a relation for the variation of temperature in the wall by solving the differential equation, and (c) evaluate the rate of heat transfer through the wall. **Answer:** (c) 6030 W

**2–57** Consider a solid cylindrical rod of length 0.15 m and diameter 0.05 m. The top and bottom surfaces of the rod are maintained at constant temperatures of 20°C and 95°C, respectively, while the side surface is perfectly insulated. Determine the rate of heat transfer through the rod if it is made of (a) copper, \( k = 380 \) W/m · °C, (b) steel, \( k = 18 \) W/m · °C, and (c) granite, \( k = 1.2 \) W/m · °C.

**2–58** Reconsider Problem 2–57. Using EES (or other) software, plot the rate of heat transfer as a function of the thermal conductivity of the rod in the range of 1 W/m · °C to 400 W/m · °C. Discuss the results.

**2–59** Consider the base plate of a 800-W household iron with a thickness of \( L = 0.6 \) cm, base area of \( A = 160 \) cm², and thermal conductivity of \( k = 20 \) W/m · °C. The inner surface of the base plate is subjected to uniform heat flux generated by the resistance heaters inside. When steady operating conditions are reached, the outer surface temperature of the plate is measured to be 85°C. Disregarding any heat loss through the upper part of the iron, (a) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the plate, (b) obtain a relation for the variation of temperature in the base plate by solving the differential equation, and (c) evaluate the inner surface temperature. **Answer:** (c) 100°C

**2–60** Repeat Problem 2–59 for a 1200-W iron.

**2–61** Reconsider Problem 2–59. Using the relation obtained for the variation of temperature in the base plate, plot the temperature as a function of the distance \( x \) in the range of \( x = 0 \) to \( x = L \), and discuss the results. Use the EES (or other) software.

**2–62E** Consider a steam pipe of length \( L = 15 \) ft, inner radius \( r_1 = 2 \) in., outer radius \( r_2 = 2.4 \) in., and thermal conductivity \( k = 7.2 \) Btu/h · ft · °F. Steam is flowing through the pipe at an average temperature of 250°F, and the average convection heat transfer coefficient on the inner surface is given to be \( h = 1.25 \) Btu/h · ft² · °F. If the average temperature on the outer...
surfaces of the pipe is $T_2 = 160^\circ F$. (a) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the pipe, (b) obtain a relation for the variation of temperature in the pipe by solving the differential equation, and (c) evaluate the rate of heat loss from the steam through the pipe. \textit{Answer: (c) 16,800 Btu/h}

2–63 A spherical container of inner radius $r_1 = 2$ m, outer radius $r_2 = 2.1$ m, and thermal conductivity $k = 30$ W/m $\cdot ^\circ C$ is filled with iced water at $0^\circ C$. The container is gaining heat by convection from the surrounding air at $T_a = 90^\circ F$ with a heat transfer coefficient of $h = 18$ W/m$^2 \cdot ^\circ C$. Assuming the inner surface temperature of the container to be $0^\circ C$, (a) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the container, (b) obtain a relation for the variation of temperature in the container by solving the differential equation, and (c) evaluate the rate of heat gain to the iced water.

2–64 Consider a large plane wall of thickness $L = 0.3$ m, thermal conductivity $k = 2.5$ W/m $\cdot ^\circ C$, and surface area $A = 12$ m$^2$. The left side of the wall at $x = 0$ is subjected to a net heat flux of $q_0 = 700$ W/m$^2$ while the temperature at that surface is measured to be $T_1 = 80^\circ C$. Assuming constant thermal conductivity and no heat generation in the wall, (a) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the wall, (b) obtain a relation for the variation of temperature in the wall by solving the differential equation, and (c) evaluate the temperature of the right surface of the wall at $x = L$. \textit{Answer: (c) }$-4^\circ C$

2–65 Repeat Problem 2–64 for a heat flux of 950 W/m$^2$ and a surface temperature of $85^\circ C$ at the left surface at $x = 0$.

2–66E A large steel plate having a thickness of $L = 4$ in., thermal conductivity of $k = 7.2$ Btu/h $\cdot$ ft $\cdot ^\circ F$, and an emissivity of $\varepsilon = 0.6$ is lying on the ground. The exposed surface of the plate at $x = L$ is known to exchange heat by convection with the ambient air at $T_a = 90^\circ F$ with an average heat transfer coefficient of $h = 12$ Btu/h $\cdot$ ft$^2 \cdot ^\circ F$ as well as by radiation with the open sky with an equivalent sky temperature of $T_{sky} = 510$ R. Also, the temperature of the upper surface of the plate is measured to be $75^\circ F$. Assuming steady one-dimensional heat transfer, (a) express the differential equation and the boundary conditions for heat conduction through the plate, (b) obtain a relation for the variation of temperature in the plate by solving the differential equation, and (c) determine the value of the lower surface temperature of the plate at $x = 0$.

2–67E Repeat Problem 2–66E by disregarding radiation heat transfer.

2–68 When a long section of a compressed air line passes through the outdoors, it is observed that the moisture in the compressed air freezes in cold weather, disrupting and even completely blocking the air flow in the pipe. To avoid this problem, the outer surface of the pipe is wrapped with electric strip heaters and then insulated.

Consider a compressed air pipe of length $L = 6$ m, inner radius $r_1 = 3.7$ cm, outer radius $r_2 = 4.0$ cm, and thermal conductivity $k = 14$ W/m $\cdot ^\circ C$ equipped with a 300-W strip heater. Air is flowing through the pipe at an average temperature of $-10^\circ C$, and the average convection heat transfer coefficient on the inner surface is $h = 30$ W/m$^2 \cdot ^\circ C$. Assuming 15 percent of the heat generated in the strip heater is lost through the insulation, (a) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the pipe, (b) obtain a relation for the variation of temperature in the pipe material by solving the differential equation, and (c) evaluate the inner and outer surface temperatures of the pipe. \textit{Answers: (c) }$-3.91^\circ C$, $-3.87^\circ C$

2–69 Reconsider Problem 2–68. Using the relation obtained for the variation of temperature in the pipe material, plot the temperature as a function of the radius $r$ in
the range of \( r = r_1 \) to \( r = r_2 \), and discuss the results. Use the EES (or other) software.

2–70 In a food processing facility, a spherical container of inner radius \( r_1 = 40 \) cm, outer radius \( r_2 = 41 \) cm, and thermal conductivity \( k = 1.5 \) W/m \( \cdot \) \( ^\circ \)C is used to store hot water and to keep it at 100\(^\circ\)C at all times. To accomplish this, the outer surface of the container is wrapped with a 500-W electric strip heater and then insulated. The temperature of the inner surface of the container is observed to be nearly 100\(^\circ\)C at all times. Assuming 10 percent of the heat generated in the heater is lost through the insulation, \((a)\) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the container, \((b)\) obtain a relation for the variation of temperature in the container material by solving the differential equation, and \((c)\) evaluate the outer surface temperature of the container. Also determine how much water at 100\(^\circ\)C this tank can supply steadily if the cold water enters at 20\(^\circ\)C.

2–71 Reconsider Problem 2–70. Using the relation obtained for the variation of temperature in the container material, plot the temperature as a function of the radius \( r \) in the range of \( r = r_1 \) to \( r = r_2 \), and discuss the results. Use the EES (or other) software.

Heat Generation in a Solid

2–72C Does heat generation in a solid violate the first law of thermodynamics, which states that energy cannot be created or destroyed? Explain.

2–73C What is heat generation? Give some examples.

2–74C An iron is left unattended and its base temperature rises as a result of resistance heating inside. When will the rate of heat generation inside the iron be equal to the rate of heat loss from the iron?

2–75C Consider the uniform heating of a plate in an environment at a constant temperature. Is it possible for part of the heat generated in the left half of the plate to leave the plate through the right surface? Explain.

2–76C Consider uniform heat generation in a cylinder and a sphere of equal radius made of the same material in the same environment. Which geometry will have a higher temperature at its center? Why?

2–77 A 2-kW resistance heater wire with thermal conductivity of \( k = 20 \) W/m \( \cdot \) \( ^\circ\)C, a diameter of \( D = 5 \) mm, and a length of \( L = 0.7 \) m is used to boil water. If the outer surface temperature of the resistance wire is \( T_s = 110\(^\circ\)C\), determine the temperature at the center of the wire.

2–78 Consider a long solid cylinder of radius \( r_0 = 4 \) cm and thermal conductivity \( k = 25 \) W/m \( \cdot \) \( ^\circ\)C. Heat is generated in the cylinder uniformly at a rate of \( g_0 = 35 \) W/cm\(^3\). The side surface of the cylinder is maintained at a constant temperature of \( T_s = 80\(^\circ\)C\). The variation of temperature in the cylinder is given by

\[
T(r) = \frac{g r^2}{k} \left[ 1 - \left( \frac{r}{r_0} \right)^2 \right] + T_s
\]

Based on this relation, determine \((a)\) if the heat conduction is steady or transient, \((b)\) if it is one-, two-, or three-dimensional, and \((c)\) the value of heat flux on the side surface of the cylinder at \( r = r_0 \).

2–79 Reconsider Problem 2–78. Using the relation obtained for the variation of temperature in the cylinder, plot the temperature as a function of the radius \( r \) in the range of \( r = 0 \) to \( r = r_0 \), and discuss the results. Use the EES (or other) software.

2–80E A long homogeneous resistance wire of radius \( r_0 = 0.25 \) in. and thermal conductivity \( k = 8.6 \) Btu/h \cdot ft \( \cdot \) \( ^\circ\)F is being used to boil water at atmospheric pressure by the passage of water through the insulation.
electric current. Heat is generated in the wire uniformly as a result of resistance heating at a rate of \( g = 1800 \text{ Btu/h \cdot in}^3 \). The heat generated is transferred to water at 212°F by convection with an average heat transfer coefficient of \( h = 820 \text{ Btu/h \cdot ft}^2 \cdot \text{°F} \). Assuming steady one-dimensional heat transfer, \((a)\) express the differential equation and the boundary conditions for heat conduction through the wire, \((b)\) obtain a relation for the variation of temperature in the wire by solving the differential equation, and \((c)\) determine the temperature at the centerline of the wire.  

2–85 Reconsider Problem 2–84. Using EES (or other) software, investigate the effect of the heat transfer coefficient on the highest and lowest temperatures in the plate. Let the heat transfer coefficient vary from 20 W/m² \cdot °C to 100 W/m² \cdot °C. Plot the highest and lowest temperatures as a function of the heat transfer coefficient, and discuss the results.

2–86 A 6-m-long 2-kW electrical resistance wire is made of 0.2-cm-diameter stainless steel \((k = 15.1 \text{ W/m \cdot °C})\). The resistance wire operates in an environment at 30°C with a heat transfer coefficient of 140 W/m² \cdot °C at the outer surface. Determine the surface temperature of the wire \((a)\) by using the applicable relation and \((b)\) by setting up the proper differential equation and solving it.  

Answers: \((a)\) 409°C, \((b)\) 409°C

2–87E Heat is generated uniformly at a rate of 3 kW per ft length in a 0.08-in.-diameter electric resistance wire made of nickel steel \((k = 5.8 \text{ Btu/h \cdot ft \cdot °F})\). Determine the temperature difference between the centerline and the surface of the wire.

2–88E Repeat Problem 2–87E for a manganese wire \((k = 4.5 \text{ Btu/h \cdot ft \cdot °F})\).

2–89 Consider a homogeneous spherical piece of radioactive material of radius \( r_0 = 0.04 \text{ m} \) that is generating heat at a constant rate of \( g = 4 \times 10^7 \text{ W/m}^3 \). The heat generated is dissipated to the environment steadily. The outer surface of the sphere is maintained at a uniform temperature of 80°C and the thermal conductivity of the sphere is \( k = 15 \text{ W/m \cdot °C} \). Assuming steady one-dimensional heat transfer, \((a)\) express the differential equation and the boundary conditions for heat conduction through the sphere, \((b)\) obtain a relation for the variation of temperature in the sphere by solving the differential equation, and \((c)\) determine the temperature at the center of the sphere.

2–90 Reconsider Problem 2–89. Using the relation obtained for the variation of temperature in the sphere, plot the temperature as a function of the radius \( r \) in the range of \( r = 0 \) to \( r = r_0 \). Also, plot the center temperature of the sphere as a function of the thermal conductivity in the range of 10 W/m \cdot °C to 400 W/m \cdot °C. Discuss the results. Use the EES (or other) software.
2–91 A long homogeneous resistance wire of radius \( r_0 = 5 \text{ mm} \) is being used to heat the air in a room by the passage of electric current. Heat is generated in the wire uniformly at a rate of \( g = 5 \times 10^6 \text{ W/m}^3 \) as a result of resistance heating. If the temperature of the outer surface of the wire remains at 180°C, determine the temperature at \( r = 2 \text{ mm} \) after steady operation conditions are reached. Take the thermal conductivity of the wire to be \( k = 8 \text{ W/m} \cdot \text{°C} \).  

\[ T = 212.8°C \]

2–92 Consider a large plane wall of thickness \( L = 0.05 \text{ m} \). The wall surface at \( x = 0 \) is insulated, while the surface at \( x = L \) is maintained at a temperature of 30°C. The thermal conductivity of the wall is \( k = 30 \text{ W/m} \cdot \text{°C} \), and heat is generated in the wall at a rate of \( g = g_0 e^{-0.5x/L} \text{ W/m}^3 \) where \( g_0 = 8 \times 10^6 \text{ W/m}^3 \). Assuming steady one-dimensional heat transfer, \( a \) express the differential equation and the boundary conditions for heat conduction through the wall, \( b \) obtain a relation for the variation of temperature in the wall by solving the differential equation, and \( c \) determine the temperature of the insulated surface of the wall.  

\[ T = 314°C \]

2–93 Consider Problem 2–92. Using the relation given for the heat generation in the wall, plot the heat generation as a function of the distance \( x \) in the range of \( x = 0 \) to \( x = L \), and discuss the results. Use the EES (or other) software.

**Variable Thermal Conductivity, \( k(T) \)**

2–94C Consider steady one-dimensional heat conduction in a plane wall, long cylinder, and sphere with constant thermal conductivity and no heat generation. Will the temperature in any of these mediums vary linearly? Explain.

2–95C Is the thermal conductivity of a medium, in general, constant or does it vary with temperature?

2–96C Consider steady one-dimensional heat conduction in a plane wall in which the thermal conductivity varies linearly. The error involved in heat transfer calculations by assuming constant thermal conductivity at the average temperature is \( a \) none, \( b \) small, or \( c \) significant.

2–97C The temperature of a plane wall during steady one-dimensional heat conduction varies linearly when the thermal conductivity is constant. Is this still the case when the thermal conductivity varies linearly with temperature?

2–98C When the thermal conductivity of a medium varies linearly with temperature, is the average thermal conductivity always equivalent to the conductivity value at the average temperature?

2–99 Consider a plane wall of thickness \( L \) whose thermal conductivity varies in a specified temperature range as \( k(T) = k_0(1 + \beta T) \) where \( k_0 \) and \( \beta \) are two specified constants. The wall surface at \( x = 0 \) is maintained at a constant temperature of \( T_1 \), while the surface at \( x = L \) is maintained at \( T_2 \). Assuming steady one-dimensional heat transfer, obtain a relation for the heat transfer rate through the wall.

2–100 Consider a cylindrical shell of length \( L \), inner radius \( r_1 \), and outer radius \( r_2 \) whose thermal conductivity varies linearly in a specified temperature range as \( k(T) = k_0(1 + \beta T) \) where \( k_0 \) and \( \beta \) are two specified constants. The inner surface of the shell is maintained at a constant temperature of \( T_1 \), while the outer surface is maintained at \( T_2 \). Assuming steady one-dimensional heat transfer, obtain a relation for \( a \) the heat transfer rate through the wall and \( b \) the temperature distribution \( T(r) \) in the shell.

2–101 Consider a spherical shell of inner radius \( r_1 \) and outer radius \( r_2 \) whose thermal conductivity varies linearly in a specified temperature range as \( k(T) = k_0(1 + \beta T) \) where \( k_0 \) and \( \beta \) are two specified constants. The inner surface of the shell is maintained at a constant temperature of \( T_1 \) while the outer surface is maintained at \( T_2 \). Assuming steady one-dimensional heat transfer, obtain a relation for \( a \) the heat transfer rate through the shell and \( b \) the temperature distribution \( T(r) \) in the shell.

2–102 Consider a 1.5-m-high and 0.6-m-wide plate whose thickness is 0.15 m. One side of the plate is maintained at a constant temperature of 500 K while the other side is maintained at 350 K. The thermal conductivity of the plate can be assumed to vary linearly in that temperature range as \( k(T) = k_0(1 + \beta T) \) where \( k_0 = 25 \text{ W/m} \cdot \text{K} \) and \( \beta = 8.7 \times 10^{-4} \text{ K}^{-1} \). Disregarding the edge effects and assuming steady one-dimensional heat transfer, determine the rate of heat conduction through the plate.  

\[ T = 30,800 \text{ W} \]
2–103 Reconsider Problem 2–102. Using EES (or other) software, plot the rate of heat conduction through the plate as a function of the temperature of the hot side of the plate in the range of 400 K to 700 K. Discuss the results.

Special Topic: Review of Differential Equations

2–104C Why do we often utilize simplifying assumptions when we derive differential equations?

2–105C What is a variable? How do you distinguish a dependent variable from an independent one in a problem?

2–106C Can a differential equation involve more than one independent variable? Can it involve more than one dependent variable? Give examples.

2–107C What is the geometrical interpretation of a derivative? What is the difference between partial derivatives and ordinary derivatives?

2–108C What is the difference between the degree and the order of a derivative?

2–109C Consider a function \( f(x, y) \) and its partial derivative \( \frac{\partial f}{\partial x} \). Under what conditions will this partial derivative be equal to the ordinary derivative \( \frac{df}{dx} \)?

2–110C Consider a function \( f(x) \) and its derivative \( \frac{df}{dx} \). Does this derivative have to be a function of \( x \)?

2–111C How is integration related to derivation?

2–112C What is the difference between an algebraic equation and a differential equation?

2–113C What is the difference between an ordinary differential equation and a partial differential equation?

2–114C How is the order of a differential equation determined?

2–115C How do you distinguish a linear differential equation from a nonlinear one?

2–116C How do you recognize a linear homogeneous differential equation? Give an example and explain why it is linear and homogeneous.

2–117C How do differential equations with constant coefficients differ from those with variable coefficients? Give an example for each type.

2–118C What kind of differential equations can be solved by direct integration?

2–119C Consider a third order linear and homogeneous differential equation. How many arbitrary constants will its general solution involve?

Review Problems

2–120 Consider a small hot metal object of mass \( m \) and specific heat \( C \) that is initially at a temperature of \( T_i \). Now the object is allowed to cool in an environment at \( T_w \) by convection with a heat transfer coefficient of \( h \). The temperature of the metal object is observed to vary uniformly with time during cooling. Writing an energy balance on the entire metal object, derive the differential equation that describes the variation of temperature of the ball with time, \( T(t) \). Assume constant thermal conductivity and no heat generation in the object. Do not solve.

2–121 Consider a long rectangular bar of length \( a \) in the \( x \)-direction and width \( b \) in the \( y \)-direction that is initially at a uniform temperature of \( T_0 \). The surfaces of the bar at \( x = 0 \) and \( y = 0 \) are insulated, while heat is lost from the other two surfaces by convection to the surrounding medium at temperature \( T_w \) with a heat transfer coefficient of \( h \). Assuming constant thermal conductivity and transient two-dimensional heat transfer with no heat generation, express the mathematical formulation (the differential equation and the boundary and initial conditions) of this heat conduction problem. Do not solve.

2–122 Consider a short cylinder of radius \( r_0 \) and height \( H \) in which heat is generated at a constant rate of \( q_g \). Heat is lost from the cylindrical surface at \( r = r_0 \) by convection to the surrounding medium at temperature \( T_w \) with a heat transfer coefficient of \( h \). The bottom surface of the cylinder at \( z = 0 \) is insulated, while the top surface at \( z = H \) is subjected to uniform heat flux \( q_h \). Assuming constant thermal conductivity and steady two-dimensional heat transfer, express the mathematical formulation (the differential equation and the boundary conditions) of this heat conduction problem. Do not solve.

2–123E Consider a large plane wall of thickness \( L = 0.5 \) ft and thermal conductivity \( k = 1.2 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F} \). The wall is covered with a material that has an emissivity of \( \varepsilon = 0.80 \) and a solar absorptivity of \( \alpha = 0.45 \). The inner surface of the wall is maintained at \( T_i = 520 \) R at all times, while the outer surface is exposed to solar radiation that is incident at a rate of \( q_{\text{Solar}} = 300 \text{ Btu/h} \cdot \text{ft}^2 \). The outer surface is also losing heat by
radiation to deep space at 0 K. Determine the temperature of the outer surface of the wall and the rate of heat transfer through the wall when steady operating conditions are reached.  

**Answers:** 530.9 R, 26.2 Btu/h · ft²

**2–124E** Repeat Problem 2–123E for the case of no solar radiation incident on the surface.

**2–125** Consider a steam pipe of length \( L \), inner radius \( r_1 \), outer radius \( r_2 \), and constant thermal conductivity \( k \). Steam flows inside the pipe at an average temperature of \( T_i \) with a convection heat transfer coefficient of \( h_c \). The outer surface of the pipe is exposed to convection to the surrounding air at a temperature of \( T_o \) with a heat transfer coefficient of \( h_o \). Assuming steady one-dimensional heat conduction through the pipe, \( (a) \) express the differential equation and the boundary conditions for heat conduction through the pipe material, \( (b) \) obtain a relation for the variation of temperature in the pipe material by solving the differential equation, and \( (c) \) obtain a relation for the temperature of the outer surface of the pipe.

**2–126** The boiling temperature of nitrogen at atmospheric pressure at sea level (1 atm pressure) is \(-196^\circ C\). Therefore, nitrogen is commonly used in low temperature scientific studies since the temperature of liquid nitrogen in a tank open to the atmosphere will remain constant at \(-196^\circ C\) until the liquid nitrogen in the tank is depleted. Any heat transfer to the tank will result in the evaporation of some liquid nitrogen, which has a heat of vaporization of 198 kJ/kg and a density of 810 kg/m³ at 1 atm.

Consider a thick-walled spherical tank of inner radius \( r_1 = 2 \) m, outer radius \( r_2 = 2.1 \) m, and constant thermal conductivity \( k = 18 \) W/m · °C. The tank is initially filled with liquid nitrogen at 1 atm and \(-196^\circ C\), and is exposed to ambient air at \( T_w = 20^\circ C \) with a heat transfer coefficient of \( h = 25 \) W/m² · °C. The inner surface temperature of the spherical tank is observed to be almost the same as the temperature of the nitrogen inside. Assuming steady one-dimensional heat transfer, \( (a) \) express the differential equation and the boundary conditions for heat conduction through the tank, \( (b) \) obtain a relation for the variation of temperature in the tank material by solving the differential equation, and \( (c) \) determine the rate of evaporation of the liquid nitrogen in the tank as a result of the heat transfer from the ambient air.  

**Answer:** \((c) 1.32 \) kg/s

**2–127** Repeat Problem 2–126 for liquid oxygen, which has a boiling temperature of \(-183^\circ C\), a heat of vaporization of 213 kJ/kg, and a density of 1140 kg/m³ at 1 atm.

**2–128** Consider a large plane wall of thickness \( L = 0.4 \) m and thermal conductivity \( k = 8.4 \) W/m · °C. There is no access to the inner side of the wall at \( x = 0 \) and thus the thermal conditions on that surface are not known. However, the outer surface of the wall at \( x = L \), whose emissivity is \( e = 0.7 \), is known to exchange heat by convection with ambient air at \( T_w = 25^\circ C \) with an average heat transfer coefficient of \( h = 14 \) W/m² · °C as well as by radiation with the surrounding surfaces at an average temperature of \( T_{\text{sur}} = 290 \) K. Further, the temperature of the outer surface is measured to be \( T_2 = 45^\circ C \). Assuming steady one-dimensional heat transfer, \( (a) \) express the differential equation and the boundary conditions for heat conduction through the plate, \( (b) \) obtain a relation for the temperature of the outer surface of the plate by solving the differential equation, and \( (c) \) evaluate the inner surface temperature of the wall at \( x = 0 \).  

**Answer:** \((c) 64.3^\circ C\)
2–129 A 1000-W iron is left on the iron board with its base exposed to ambient air at 20°F. The base plate of the iron has a thickness of $L = 0.5$ cm, base area of $A = 150$ cm$^2$, and thermal conductivity of $k = 18$ W/m · °C. The inner surface of the base plate is subjected to uniform heat flux generated by the resistance heaters inside. The outer surface of the base plate whose emissivity is $\varepsilon = 0.7$, loses heat by convection to ambient air at $T_a = 22^\circ$C with an average heat transfer coefficient of $h = 30$ W/m$^2$ · °C as well as by radiation to the surrounding surfaces at an average temperature of $T_{\text{sur}} = 290$ K. Disregarding any heat loss through the upper part of the iron, (a) express the differential equation and the boundary conditions for steady one-dimensional heat conduction through the plate, (b) obtain a relation for the temperature of the outer surface of the plate by solving the differential equation, and (c) evaluate the outer surface temperature.

![Figure P2–129](image)

**FIGURE P2–129**

2–130 Repeat Problem 2–129 for a 1500-W iron.

2–131E The roof of a house consists of a 0.8-ft-thick concrete slab ($k = 1.1$ Btu/h · ft · °F) that is 25 ft wide and 35 ft long. The emissivity of the outer surface of the roof is 0.8, and the convection heat transfer coefficient on that surface is estimated to be 3.2 Btu/h · ft$^2$ · °F. On a clear winter night, the ambient air is reported to be at 50°F, while the night sky temperature for radiation heat transfer is 310 R. If the inner surface temperature of the roof is $T_i = 62^\circ$F, determine the outer surface temperature of the roof and the rate of heat loss through the roof when steady operating conditions are reached.

2–132 Consider a long resistance wire of radius $r_1 = 0.3$ cm and thermal conductivity $k_{\text{wire}} = 18$ W/m · °C in which heat is generated uniformly at a constant rate of $g = 1.5$ W/cm$^3$ as a result of resistance heating. The wire is embedded in a 0.4-cm-thick layer of plastic whose thermal conductivity is $k_{\text{plastic}} = 1.8$ W/m · °C. The outer surface of the plastic cover loses heat by convection to the ambient air at $T_a = 25^\circ$C with an average combined heat transfer coefficient of $h = 14$ W/m$^2$ · °C. Assuming one-dimensional heat transfer, determine the temperatures at the center of the resistance wire and the wire-plastic layer interface under steady conditions.

*Answers: 97.1°C, 97.3°C*

2–133 Consider a cylindrical shell of length $L$, inner radius $r_1$, and outer radius $r_2$ whose thermal conductivity varies in a specified temperature range as $k(T) = k_0(1 + \beta T^2)$ where $k_0$ and $\beta$ are two specified constants. The inner surface of the shell is maintained at a constant temperature of $T_1$ while the outer surface is maintained at $T_2$. Assuming steady one-dimensional heat transfer, obtain a relation for the heat transfer rate through the shell.

2–134 In a nuclear reactor, heat is generated in 1-cm-diameter cylindrical uranium fuel rods at a rate of $4 \times 10^7$ W/m$^3$. Determine the temperature difference between the center and the surface of the fuel rod.

*Answer: 9.0°C*

2–135 Consider a 20-cm-thick large concrete plane wall ($k = 0.77$ W/m · °C) subjected to convection on both sides with $T_{x1} = 27^\circ$C and $h_1 = 5$ W/m$^2$ · °C on the inside, and $T_{x2} = 8^\circ$C and $h_2 = 12$ W/m$^2$ · °C on the outside. Assuming constant thermal conductivity with no heat generation and negligible...
radiation, (a) express the differential equations and the boundary conditions for steady one-dimensional heat conduction through the wall, (b) obtain a relation for the variation of temperature in the wall by solving the differential equation, and (c) evaluate the temperatures at the inner and outer surfaces of the wall.

2–136 Consider a water pipe of length \( L = 12 \) m, inner radius \( r_1 = 15 \) cm, outer radius \( r_2 = 20 \) cm, and thermal conductivity \( k = 20 \) W/m \( \cdot \) °C. Heat is generated in the pipe material uniformly by a 25-kW electric resistance heater. The inner and outer surfaces of the pipe are at \( T_1 = 60 \) °C and \( T_2 = 80 \) °C, respectively. Obtain a general relation for temperature distribution inside the pipe under steady conditions and determine the temperature at the center plane of the pipe.

2–137 Heat is generated uniformly at a rate of \( 2.6 \times 10^6 \) W/m\(^3\) in a spherical ball \( (k = 45 \) W/m \( \cdot \) °C\) of diameter 30 cm. The ball is exposed to iced-water at 0°C with a heat transfer coefficient of 1200 W/m\(^2\) \( \cdot \) °C. Determine the temperatures at the center and the surface of the ball.

Computer, Design, and Essay Problems

2–138 Write an essay on heat generation in nuclear fuel rods. Obtain information on the ranges of heat generation, the variation of heat generation with position in the rods, and the absorption of emitted radiation by the cooling medium.

2–139 Write an interactive computer program to calculate the heat transfer rate and the value of temperature anywhere in the medium for steady one-dimensional heat conduction in a long cylindrical shell for any combination of specified temperature, specified heat flux, and convection boundary conditions. Run the program for five different sets of specified boundary conditions.

2–140 Write an interactive computer program to calculate the heat transfer rate and the value of temperature anywhere in the medium for steady one-dimensional heat conduction in a spherical shell for any combination of specified temperature, specified heat flux, and convection boundary conditions. Run the program for five different sets of specified boundary conditions.

2–141 Write an interactive computer program to calculate the heat transfer rate and the value of temperature anywhere in the medium for steady one-dimensional heat conduction in a plane wall whose thermal conductivity varies linearly as \( k(T) = k_0(1 + \beta T) \) where the constants \( k_0 \) and \( \beta \) are specified by the user for specified temperature boundary conditions.
In heat transfer analysis, we are often interested in the rate of heat transfer through a medium under steady conditions and surface temperatures. Such problems can be solved easily without involving any differential equations by the introduction of thermal resistance concepts in an analogous manner to electrical circuit problems. In this case, the thermal resistance corresponds to electrical resistance, temperature difference corresponds to voltage, and the heat transfer rate corresponds to electric current.

We start this chapter with one-dimensional steady heat conduction in a plane wall, a cylinder, and a sphere, and develop relations for thermal resistances in these geometries. We also develop thermal resistance relations for convection and radiation conditions at the boundaries. We apply this concept to heat conduction problems in multilayer plane walls, cylinders, and spheres and generalize it to systems that involve heat transfer in two or three dimensions. We also discuss the thermal contact resistance and the overall heat transfer coefficient and develop relations for the critical radius of insulation for a cylinder and a sphere. Finally, we discuss steady heat transfer from finned surfaces and some complex geometries commonly encountered in practice through the use of conduction shape factors.
3–1 STEADY HEAT CONDUCTION IN PLANE WALLS

Consider steady heat conduction through the walls of a house during a winter day. We know that heat is continuously lost to the outdoors through the wall. We intuitively feel that heat transfer through the wall is in the normal direction to the wall surface, and no significant heat transfer takes place in the wall in other directions (Fig. 3–1).

Recall that heat transfer in a certain direction is driven by the temperature gradient in that direction. There will be no heat transfer in a direction in which there is no change in temperature. Temperature measurements at several locations on the inner or outer wall surface will confirm that a wall surface is nearly isothermal. That is, the temperatures at the top and bottom of a wall surface as well as at the right or left ends are almost the same. Therefore, there will be no heat transfer through the wall from the top to the bottom, or from left to right, but there will be considerable temperature difference between the inner and the outer surfaces of the wall, and thus significant heat transfer in the direction from the inner surface to the outer one.

The small thickness of the wall causes the temperature gradient in that direction to be large. Further, if the air temperatures in and outside the house remain constant, then heat transfer through the wall of a house can be modeled as steady and one-dimensional. The temperature of the wall in this case will depend on one direction only (say the x-direction) and can be expressed as \( T(x) \).

Noting that heat transfer is the only energy interaction involved in this case and there is no heat generation, the energy balance for the wall can be expressed as

\[
\left( \text{Rate of heat transfer into the wall} \right) - \left( \text{Rate of heat transfer out of the wall} \right) = \left( \text{Rate of change of the energy of the wall} \right)
\]

or

\[
\dot{Q}_{\text{in}} - \dot{Q}_{\text{out}} = \frac{dE_{\text{wall}}}{dt}
\]

But \( dE_{\text{wall}}/dt = 0 \) for steady operation, since there is no change in the temperature of the wall with time at any point. Therefore, the rate of heat transfer into the wall must be equal to the rate of heat transfer out of it. In other words, the rate of heat transfer through the wall must be constant, \( \dot{Q}_{\text{cond, wall}} = \text{constant} \).

Consider a plane wall of thickness \( L \) and average thermal conductivity \( k \). The two surfaces of the wall are maintained at constant temperatures of \( T_1 \) and \( T_2 \). For one-dimensional steady heat conduction through the wall, we have \( T(x) \). Then Fourier’s law of heat conduction for the wall can be expressed as

\[
\dot{Q}_{\text{cond, wall}} = -kA \frac{dT}{dx} \quad \text{(W)}
\]

where the rate of conduction heat transfer \( \dot{Q}_{\text{cond, wall}} \) and the wall area \( A \) are constant. Thus we have \( dT/dx = \text{constant} \), which means that the temperature...
through the wall varies linearly with $x$. That is, the temperature distribution in the wall under steady conditions is a straight line (Fig. 3–2).

Separating the variables in the above equation and integrating from $x = 0$, where $T(0) = T_1$, to $x = L$, where $T(L) = T_2$, we get

$$\int_{x=0}^{L} \dot{Q}_{\text{cond, wall}} \, dx = -\int_{T=T_1}^{T=T_2} k A \, dT$$

Performing the integrations and rearranging gives

$$\dot{Q}_{\text{cond, wall}} = k A \frac{T_1 - T_2}{L} \quad (\text{W})$$

which is identical to Eq. 3–1. Again, the rate of heat conduction through a plane wall is proportional to the average thermal conductivity, the wall area, and the temperature difference, but is inversely proportional to the wall thickness. Also, once the rate of heat conduction is available, the temperature $T(x)$ at any location $x$ can be determined by replacing $T_2$ in Eq. 3–3 by $T_1$ and $L$ by $x$.

**The Thermal Resistance Concept**

Equation 3–3 for heat conduction through a plane wall can be rearranged as

$$\dot{Q}_{\text{cond, wall}} = \frac{T_1 - T_2}{R_{\text{wall}}} \quad (\text{W})$$

where

$$R_{\text{wall}} = \frac{L}{k A} \quad (\text{°C/W})$$

is the thermal resistance of the wall against heat conduction or simply the conduction resistance of the wall. Note that the thermal resistance of a medium depends on the geometry and the thermal properties of the medium.

The equation above for heat flow is analogous to the relation for electric current flow $I$, expressed as

$$I = \frac{V_1 - V_2}{R_e}$$

where $R_e = L/\sigma A$ is the electric resistance and $V_1 - V_2$ is the voltage difference across the resistance ($\sigma_e$ is the electrical conductivity). Thus, the rate of heat transfer through a layer corresponds to the electric current, the thermal resistance corresponds to electrical resistance, and the temperature difference corresponds to voltage difference across the layer (Fig. 3–3).

Consider convection heat transfer from a solid surface of area $A_s$ and temperature $T_s$ to a fluid whose temperature sufficiently far from the surface is $T_a$, with a convection heat transfer coefficient $h$. Newton’s law of cooling for convection heat transfer rate $\dot{Q}_{\text{conv}} = h A_s (T_s - T_a)$ can be rearranged as

$$\dot{Q}_{\text{conv}} = \frac{T_s - T_a}{R_{\text{conv}}} \quad (\text{W})$$

**FIGURE 3–2**
Under steady conditions, the temperature distribution in a plane wall is a straight line.

**FIGURE 3–3**
Analogy between thermal and electrical resistance concepts.
where

\[ R_{\text{conv}} = \frac{1}{h} \] (°C/W) \hspace{1cm} (3-8)

is the thermal resistance of the surface against heat convection, or simply the convection resistance of the surface (Fig. 3–4). Note that when the convection heat transfer coefficient is very large \((h \to \infty)\), the convection resistance becomes zero and \(T_s = T_{\text{sur}}\). That is, the surface offers no resistance to convection, and thus it does not slow down the heat transfer process. This situation is approached in practice at surfaces where boiling and condensation occur. Also note that the surface does not have to be a plane surface. Equation 3–8 for convection resistance is valid for surfaces of any shape, provided that the assumption of \(h / \text{constant and uniform}\) is reasonable.

When the wall is surrounded by a gas, the radiation effects, which we have ignored so far, can be significant and may need to be considered. The rate of radiation heat transfer between a surface of emissivity \(\varepsilon\) and area \(A_s\) at temperature \(T_s\) and the surrounding surfaces at some average temperature \(T_{\text{sur}}\) can be expressed as

\[ \dot{Q}_{\text{rad}} = \varepsilon \sigma A_s (T_s^4 - T_{\text{sur}}^4) = h_{\text{rad}} A_s (T_s - T_{\text{sur}}) = \frac{T_s - T_{\text{sur}}}{R_{\text{rad}}} \] (W) \hspace{1cm} (3-9)

where

\[ R_{\text{rad}} = \frac{1}{h_{\text{rad}} A_s} \] (K/W) \hspace{1cm} (3-10)

is the thermal resistance of a surface against radiation, or the radiation resistance, and

\[ h_{\text{rad}} = \frac{\dot{Q}_{\text{rad}}}{A_s(T_s^4 - T_{\text{sur}}^4)} = \varepsilon \sigma (T_s^4 + T_{\text{sur}}^4)(T_s + T_{\text{sur}}) \] (W/m\(^2\) · K) \hspace{1cm} (3-11)

is the radiation heat transfer coefficient. Note that both \(T_s\) and \(T_{\text{sur}}\) must be in K in the evaluation of \(h_{\text{rad}}\). The definition of the radiation heat transfer coefficient enables us to express radiation conveniently in an analogous manner to convection in terms of a temperature difference. But \(h_{\text{rad}}\) depends strongly on temperature while \(h_{\text{conv}}\) usually does not.

A surface exposed to the surrounding air involves convection and radiation simultaneously, and the total heat transfer at the surface is determined by adding (or subtracting, if in the opposite direction) the radiation and convection components. The convection and radiation resistances are parallel to each other, as shown in Fig. 3–5, and may cause some complication in the thermal resistance network. When \(T_{\text{sur}} = T_{\text{ce}}\), the radiation effect can properly be accounted for by replacing \(h\) in the convection resistance relation by

\[ h_{\text{combined}} = h_{\text{conv}} + h_{\text{rad}} \] (W/m\(^2\) · K) \hspace{1cm} (3-12)

where \(h_{\text{combined}}\) is the combined heat transfer coefficient. This way all the complications associated with radiation are avoided.
Thermal Resistance Network

Now consider steady one-dimensional heat flow through a plane wall of thickness $L$, area $A$, and thermal conductivity $k$ that is exposed to convection on both sides to fluids at temperatures $T_1$ and $T_2$, respectively, as shown in Fig. 3–6. Assuming $T_{s2} < T_{s1}$, the variation of temperature will be as shown in the figure. Note that the temperature varies linearly in the wall, and asymptotically approaches $T_{s1}$ and $T_{s2}$ in the fluids as we move away from the wall.

Under steady conditions we have

$$
\dot{Q} = h_1 A (T_{s1} - T_1) = k A \frac{T_1 - T_2}{L} = h_2 A (T_2 - T_{s2})
$$

which can be rearranged as

$$
\dot{Q} = \frac{T_{s1} - T_1}{1/h_1 A} = \frac{T_1 - T_2}{L/kA} = \frac{T_2 - T_{s2}}{1/h_2 A}
$$

Adding the numerators and denominators yields (Fig. 3–7)

$$
\dot{Q} = \frac{T_{s1} - T_{s2}}{R_{\text{total}}} \quad \text{W}
$$
Note that the heat transfer area $A$ is constant for a plane wall, and the rate of heat transfer through a wall separating two mediums is equal to the temperature difference divided by the total thermal resistance between the mediums. Also note that the thermal resistances are in series, and the equivalent thermal resistance is determined by simply adding the individual resistances, just like the electrical resistances connected in series. Thus, the electrical analogy still applies. We summarize this as the rate of steady heat transfer between two surfaces is equal to the temperature difference divided by the total thermal resistance between those two surfaces.

Another observation that can be made from Eq. 3–15 is that the ratio of the temperature drop to the thermal resistance across any layer is constant, and thus the temperature drop across any layer is proportional to the thermal resistance of the layer. The larger the resistance, the larger the temperature drop. In fact, the equation $\frac{Q}{R}$ can be rearranged as

$$\Delta T = \frac{Q}{R}$$

which indicates that the temperature drop across any layer is equal to the rate of heat transfer times the thermal resistance across that layer (Fig. 3–8). You may recall that this is also true for voltage drop across an electrical resistance when the electric current is constant.

It is sometimes convenient to express heat transfer through a medium in an analogous manner to Newton’s law of cooling as

$$\dot{Q} = UA \Delta T$$

where $U$ is the overall heat transfer coefficient. A comparison of Eqs. 3–15 and 3–18 reveals that
Therefore, for a unit area, the overall heat transfer coefficient is equal to the inverse of the total thermal resistance.

Note that we do not need to know the surface temperatures of the wall in order to evaluate the rate of steady heat transfer through it. All we need to know is the convection heat transfer coefficients and the fluid temperatures on both sides of the wall. The surface temperature of the wall can be determined as described above using the thermal resistance concept, but by taking the surface at which the temperature is to be determined as one of the terminal surfaces. For example, once $\dot{Q}$ is evaluated, the surface temperature $T_1$ can be determined from

$$\dot{Q} = \frac{T_{s1} - T_1}{R_{conv, 1}} = \frac{T_{s1} - T_3}{1/h_1A}$$

(3-20)

**Multilayer Plane Walls**

In practice we often encounter plane walls that consist of several layers of different materials. The thermal resistance concept can still be used to determine the rate of steady heat transfer through such composite walls. As you may have already guessed, this is done by simply noting that the conduction resistance of each wall is $L/kA$ connected in series, and using the electrical analogy. That is, by dividing the temperature difference between two surfaces at known temperatures by the total thermal resistance between them.

Consider a plane wall that consists of two layers (such as a brick wall with a layer of insulation). The rate of steady heat transfer through this two-layer composite wall can be expressed as (Fig. 3–9)

$$\dot{Q} = \frac{T_{s1} - T_{s2}}{R_{total}}$$

(3-21)
where $R_{\text{total}}$ is the total thermal resistance, expressed as

$$R_{\text{total}} = R_{\text{conv, 1}} + R_{\text{wall, 1}} + R_{\text{wall, 2}} + R_{\text{conv, 2}}$$

$$= \frac{1}{\frac{L_1}{h_1 A} + \frac{L_2}{k_1 A} + \frac{1}{h_2 A}}$$

The subscripts 1 and 2 in the $R_{\text{wall}}$ relations above indicate the first and the second layers, respectively. We could also obtain this result by following the approach used above for the single-layer case by noting that the rate of steady heat transfer $\dot{Q}$ through a multilayer medium is constant, and thus it must be the same through each layer. Note from the thermal resistance network that the resistances are in series, and thus the total thermal resistance is simply the arithmetic sum of the individual thermal resistances in the path of heat flow.

This result for the two-layer case is analogous to the single-layer case, except that an additional resistance is added for the additional layer. This result can be extended to plane walls that consist of three or more layers by adding an additional resistance for each additional layer.

Once $\dot{Q}$ is known, an unknown surface temperature $T_j$ at any surface or interface $j$ can be determined from

$$\dot{Q} = \frac{T_j - T_i}{R_{\text{total, i-j}}}$$

where $T_i$ is a known temperature at location $i$ and $R_{\text{total, i-j}}$ is the total thermal resistance between locations $i$ and $j$. For example, when the fluid temperatures $T_{s1}$ and $T_{s2}$ for the two-layer case shown in Fig. 3–9 are available and $\dot{Q}$ is calculated from Eq. 3–21, the interface temperature $T_2$ between the two walls can be determined from (Fig. 3–10)

$$\dot{Q} = \frac{T_{s1} - T_{s2}}{R_{\text{conv, 1}} + R_{\text{wall, 1}}} = \frac{T_{s1} - T_{s2}}{\frac{1}{h_1 A} + \frac{L_1}{k_1 A}}$$

The temperature drop across a layer is easily determined from Eq. 3–17 by multiplying $\dot{Q}$ by the thermal resistance of that layer.

The thermal resistance concept is widely used in practice because it is intuitively easy to understand and it has proven to be a powerful tool in the solution of a wide range of heat transfer problems. But its use is limited to systems through which the rate of heat transfer $\dot{Q}$ remains constant; that is, to systems involving steady heat transfer with no heat generation (such as resistance heating or chemical reactions) within the medium.

**EXAMPLE 3–1  Heat Loss through a Wall**

Consider a 3-m-high, 5-m-wide, and 0.3-m-thick wall whose thermal conductivity is $k = 0.9$ W/m · °C (Fig. 3–11). On a certain day, the temperatures of the inner and the outer surfaces of the wall are measured to be 16°C and 2°C, respectively. Determine the rate of heat loss through the wall on that day.
SOLUTION The two surfaces of a wall are maintained at specified temperatures. The rate of heat loss through the wall is to be determined.

Assumptions 1 Heat transfer through the wall is steady since the surface temperatures remain constant at the specified values. 2 Heat transfer is one-dimensional since any significant temperature gradients will exist in the direction from the indoors to the outdoors. 3 Thermal conductivity is constant.

Properties The thermal conductivity is given to be \( k = 0.9 \text{ W/m} \cdot \text{°C} \).

Analysis Noting that the heat transfer through the wall is by conduction and the area of the wall is \( A = 3 \text{ m} \times 5 \text{ m} = 15 \text{ m}^2 \), the steady rate of heat transfer through the wall can be determined from Eq. 3–3 to be

\[
\dot{Q} = kA \frac{T_1 - T_2}{L} = (0.9 \text{ W/m} \cdot \text{°C})(15 \text{ m}^2) \frac{(16 - 2)\text{°C}}{0.3 \text{ m}} = 630 \text{ W}
\]

We could also determine the steady rate of heat transfer through the wall by making use of the thermal resistance concept from

\[
\dot{Q} = \frac{\Delta T_{\text{wall}}}{R_{\text{wall}}}
\]

where

\[
R_{\text{wall}} = \frac{L}{kA} = \frac{0.3 \text{ m}}{(0.9 \text{ W/m} \cdot \text{°C})(15 \text{ m}^2)} = 0.02222 ^\circ \text{C/W}
\]

Substituting, we get

\[
\dot{Q} = \frac{(16 - 2)\text{°C}}{0.02222 ^\circ \text{C/W}} = 630 \text{ W}
\]

Discussion This is the same result obtained earlier. Note that heat conduction through a plane wall with specified surface temperatures can be determined directly and easily without utilizing the thermal resistance concept. However, the thermal resistance concept serves as a valuable tool in more complex heat transfer problems, as you will see in the following examples.

EXAMPLE 3–2 Heat Loss through a Single-Pane Window

Consider a 0.8-m-high and 1.5-m-wide glass window with a thickness of 8 mm and a thermal conductivity of \( k = 0.78 \text{ W/m} \cdot \text{°C} \). Determine the steady rate of heat transfer through this glass window and the temperature of its inner surface for a day during which the room is maintained at 20°C while the temperature of the outdoors is –10°C. Take the heat transfer coefficients on the inner and outer surfaces of the window to be \( h_1 = 10 \text{ W/m}^2 \cdot \text{°C} \) and \( h_2 = 40 \text{ W/m}^2 \cdot \text{°C} \), which includes the effects of radiation.

SOLUTION Heat loss through a window glass is considered. The rate of heat transfer through the window and the inner surface temperature are to be determined.
**HEAT TRANSFER**

**Assumptions**
1. Heat transfer through the window is steady since the surface temperatures remain constant at the specified values.
2. Heat transfer through the wall is one-dimensional since any significant temperature gradients will exist in the direction from the indoors to the outdoors.
3. Thermal conductivity is constant.

**Properties**
The thermal conductivity is given to be $k = 0.78 \text{ W/m} \cdot \text{°C}$.

**Analysis**
This problem involves conduction through the glass window and convection at its surfaces, and can best be handled by making use of the thermal resistance concept and drawing the thermal resistance network, as shown in Fig. 3–12. Noting that the area of the window is $A = 0.8 \text{ m} \times 1.5 \text{ m} = 1.2 \text{ m}^2$, the individual resistances are evaluated from their definitions to be

- $R_i = R_{\text{conv},1} = \frac{1}{h_1 A} = \frac{1}{(10 \text{ W/m}^2 \cdot \text{°C})(1.2 \text{ m}^2)} = 0.08333\text{°C/W}$
- $R_{\text{glass}} = \frac{L}{k A} = \frac{0.008 \text{ m}}{(0.78 \text{ W/m} \cdot \text{°C})(1.2 \text{ m}^2)} = 0.00855\text{°C/W}$
- $R_o = R_{\text{conv},2} = \frac{1}{h_2 A} = \frac{1}{(40 \text{ W/m}^2 \cdot \text{°C})(1.2 \text{ m}^2)} = 0.02083\text{°C/W}$

Noting that all three resistances are in series, the total resistance is

$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{glass}} + R_{\text{conv},2} = 0.08333 + 0.00855 + 0.02083 = 0.1127\text{°C/W}$$

Then the steady rate of heat transfer through the window becomes

$$\dot{Q} = \frac{T_{\text{in}} - T_{\text{out}}}{R_{\text{total}}} = \frac{[20 - (-10)]\text{°C}}{0.1127\text{°C/W}} = 266 \text{ W}$$

Knowing the rate of heat transfer, the inner surface temperature of the window glass can be determined from

$$\dot{Q} = \frac{T_{\text{in}} - T_{\text{1}}}{R_{\text{conv},1}} \quad \rightarrow \quad T_{\text{1}} = T_{\text{in}} - \dot{Q} R_{\text{conv},1}$$

$$= 20\text{°C} - (266 \text{ W})(0.08333\text{°C/W})$$

$$= -2.2 \text{°C}$$

**Discussion**
Note that the inner surface temperature of the window glass will be $-2.2\text{°C}$ even though the temperature of the air in the room is maintained at $20\text{°C}$. Such low surface temperatures are highly undesirable since they cause the formation of fog or even frost on the inner surfaces of the glass when the humidity in the room is high.

---

**EXAMPLE 3–3**  Heat Loss through Double-Pane Windows

Consider a 0.8-m-high and 1.5-m-wide double-pane window consisting of two 4-mm-thick layers of glass ($k = 0.78 \text{ W/m} \cdot \text{°C}$) separated by a 10-mm-wide stagnant air space ($k = 0.026 \text{ W/m} \cdot \text{°C}$). Determine the steady rate of heat transfer.
transfer through this double-pane window and the temperature of its inner surface for a day during which the room is maintained at 20°C while the temperature of the outdoors is −10°C. Take the convection heat transfer coefficients on the inner and outer surfaces of the window to be \( h_1 = 10 \text{ W/m}^2 \cdot \text{°C} \) and \( h_2 = 40 \text{ W/m}^2 \cdot \text{°C} \), which includes the effects of radiation.

**SOLUTION** A double-pane window is considered. The rate of heat transfer through the window and the inner surface temperature are to be determined.

**Analysis** This example problem is identical to the previous one except that the single 8-mm-thick window glass is replaced by two 4-mm-thick glasses that enclose a 10-mm-wide stagnant air space. Therefore, the thermal resistance network of this problem will involve two additional conduction resistances corresponding to the two additional layers, as shown in Fig. 3–13. Noting that the area of the window is again \( A = 0.8 \text{ m} \times 1.5 \text{ m} = 1.2 \text{ m}^2 \), the individual resistances are evaluated from their definitions to be

\[
R_i = R_{\text{conv},1} = \frac{1}{h_1 A} = \frac{1}{(10 \text{ W/m}^2 \cdot \text{°C})(1.2 \text{ m}^2)} = 0.0833 \text{°C/W}
\]

\[
R_1 = R_3 = R_{\text{glass}} = \frac{L_1}{k_1 A} = \frac{0.004 \text{ m}}{(0.78 \text{ W/m} \cdot \text{°C})(1.2 \text{ m}^2)} = 0.00427 \text{°C/W}
\]

\[
R_2 = R_{\text{air}} = \frac{L_2}{k_2 A} = \frac{0.01 \text{ m}}{(0.026 \text{ W/m} \cdot \text{°C})(1.2 \text{ m}^2)} = 0.3205 \text{°C/W}
\]

\[
R_o = R_{\text{conv},2} = \frac{1}{h_2 A} = \frac{1}{(40 \text{ W/m}^2 \cdot \text{°C})(1.2 \text{ m}^2)} = 0.02083 \text{°C/W}
\]

Noting that all three resistances are in series, the total resistance is

\[
R_{\text{total}} = R_{\text{conv},1} + R_{\text{glass}} + R_{\text{air}} + R_{\text{glass}} + R_{\text{conv},2}
\]

\[
= 0.0833 + 0.00427 + 0.3205 + 0.00427 + 0.02083
\]

\[
= 0.4332 \text{°C/W}
\]

Then the steady rate of heat transfer through the window becomes

\[
\dot{Q} = \frac{T_{s1} - T_{s2}}{R_{\text{total}}} = \frac{[20 - (-10)]}{0.4332 \text{°C/W}} = 69.2 \text{ W}
\]

which is about one-fourth of the result obtained in the previous example. This explains the popularity of the double- and even triple-pane windows in cold climates. The drastic reduction in the heat transfer rate in this case is due to the large thermal resistance of the air layer between the glasses.

The inner surface temperature of the window in this case will be

\[
T_1 = T_{s1} - \dot{Q}R_{\text{conv},1} = 20°C - (69.2 \text{ W})(0.08333 \text{°C/W}) = 14.2°C
\]

which is considerably higher than the −2.2°C obtained in the previous example. Therefore, a double-pane window will rarely get fogged. A double-pane window will also reduce the heat gain in summer, and thus reduce the air-conditioning costs.
In the analysis of heat conduction through multilayer solids, we assumed “perfect contact” at the interface of two layers, and thus no temperature drop at the interface. This would be the case when the surfaces are perfectly smooth and they produce a perfect contact at each point. In reality, however, even flat surfaces that appear smooth to the eye turn out to be rather rough when examined under a microscope, as shown in Fig. 3–14, with numerous peaks and valleys. That is, a surface is microscopically rough no matter how smooth it appears to be.

When two such surfaces are pressed against each other, the peaks will form good material contact but the valleys will form voids filled with air. As a result, an interface will contain numerous air gaps of varying sizes that act as insulation because of the low thermal conductivity of air. Thus, an interface offers some resistance to heat transfer, and this resistance per unit interface area is called the thermal contact resistance, \( R_c \). The value of \( R_c \) is determined experimentally using a setup like the one shown in Fig. 3–15, and as expected, there is considerable scatter of data because of the difficulty in characterizing the surfaces.

Consider heat transfer through two metal rods of cross-sectional area \( A \) that are pressed against each other. Heat transfer through the interface of these two rods is the sum of the heat transfers through the solid contact spots and the gaps in the noncontact areas and can be expressed as

\[
\dot{Q} = \dot{Q}_{\text{contact}} + \dot{Q}_{\text{gap}} \tag{3-25}
\]

It can also be expressed in an analogous manner to Newton’s law of cooling as

\[
\dot{Q} = h_c A \Delta T_{\text{interface}} \tag{3-26}
\]
where $A$ is the apparent interface area (which is the same as the cross-sectional area of the rods) and $\Delta T_{\text{interface}}$ is the effective temperature difference at the interface. The quantity $h_c$, which corresponds to the convection heat transfer coefficient, is called the thermal contact conductance and is expressed as

$$h_c = \frac{\dot{Q}}{\Delta T_{\text{interface}}} \quad (\text{W/m}^2 \cdot ^\circ\text{C}) \quad (3-27)$$

It is related to thermal contact resistance by

$$R_c = \frac{1}{h_c} = \frac{\Delta T_{\text{interface}}}{\dot{Q}/A} \quad (\text{m}^2 \cdot ^\circ\text{C}/\text{W}) \quad (3-28)$$

That is, thermal contact resistance is the inverse of thermal contact conductance. Usually, thermal contact conductance is reported in the literature, but the concept of thermal contact resistance serves as a better vehicle for explaining the effect of interface on heat transfer. Note that $R_c$ represents thermal contact resistance per unit area. The thermal resistance for the entire interface is obtained by dividing $R_c$ by the apparent interface area $A$.

The thermal contact resistance can be determined from Eq. 3–28 by measuring the temperature drop at the interface and dividing it by the heat flux under steady conditions. The value of thermal contact resistance depends on the surface roughness and the material properties as well as the temperature and pressure at the interface and the type of fluid trapped at the interface. The situation becomes more complex when plates are fastened by bolts, screws, or rivets since the interface pressure in this case is nonuniform. The thermal contact resistance in that case also depends on the plate thickness, the bolt radius, and the size of the contact zone. Thermal contact resistance is observed to decrease with decreasing surface roughness and increasing interface pressure, as expected. Most experimentally determined values of the thermal contact resistance fall between 0.000005 and 0.0005 m$^2 \cdot ^\circ\text{C}/\text{W}$ (the corresponding range of thermal contact conductance is 2000 to 200,000 W/m$^2 \cdot ^\circ\text{C}$).

When we analyze heat transfer in a medium consisting of two or more layers, the first thing we need to know is whether the thermal contact resistance is significant or not. We can answer this question by comparing the magnitudes of the thermal resistances of the layers with typical values of thermal contact resistance. For example, the thermal resistance of a 1-cm-thick layer of an insulating material per unit surface area is

$$R_{c, \text{insulation}} = \frac{L}{k} = \frac{0.01 \text{ m}}{0.04 \text{ W/m} \cdot ^\circ\text{C}} = 0.25 \text{ m}^2 \cdot ^\circ\text{C}/\text{W}$$

whereas for a 1-cm-thick layer of copper, it is

$$R_{c, \text{copper}} = \frac{L}{k} = \frac{0.01 \text{ m}}{386 \text{ W/m} \cdot ^\circ\text{C}} = 0.000026 \text{ m}^2 \cdot ^\circ\text{C}/\text{W}$$

Comparing the values above with typical values of thermal contact resistance, we conclude that thermal contact resistance is significant and can even dominate the heat transfer for good heat conductors such as metals, but can be
disregarded for poor heat conductors such as insulations. This is not surprising since insulating materials consist mostly of air space just like the interface itself.

The thermal contact resistance can be minimized by applying a thermally conducting liquid called a thermal grease such as silicon oil on the surfaces before they are pressed against each other. This is commonly done when attaching electronic components such as power transistors to heat sinks. The thermal contact resistance can also be reduced by replacing the air at the interface by a better conducting gas such as helium or hydrogen, as shown in Table 3–1.

Another way to minimize the contact resistance is to insert a soft metallic foil such as tin, silver, copper, nickel, or aluminum between the two surfaces. Experimental studies show that the thermal contact resistance can be reduced by a factor of up to 7 by a metallic foil at the interface. For maximum effectiveness, the foils must be very thin. The effect of metallic coatings on thermal contact conductance is shown in Fig. 3–16 for various metal surfaces.

There is considerable uncertainty in the contact conductance data reported in the literature, and care should be exercised when using them. In Table 3–2 some experimental results are given for the contact conductance between similar and dissimilar metal surfaces for use in preliminary design calculations. Note that the thermal contact conductance is highest (and thus the contact resistance is lowest) for soft metals with smooth surfaces at high pressure.

---

**TABLE 3–1**

<table>
<thead>
<tr>
<th>Fluid at the Interface</th>
<th>Contact Conductance, $h_c$, W/m² °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air</td>
<td>3640</td>
</tr>
<tr>
<td>Helium</td>
<td>9520</td>
</tr>
<tr>
<td>Hydrogen</td>
<td>13,900</td>
</tr>
<tr>
<td>Silicone oil</td>
<td>19,000</td>
</tr>
<tr>
<td>Glycerin</td>
<td>37,700</td>
</tr>
</tbody>
</table>

---

**EXAMPLE 3–4 Equivalent Thickness for Contact Resistance**

The thermal contact conductance at the interface of two 1-cm-thick aluminum plates is measured to be 11,000 W/m² °C. Determine the thickness of the aluminum plate whose thermal resistance is equal to the thermal resistance of the interface between the plates (Fig. 3–17).

**SOLUTION**

The thickness of the aluminum plate whose thermal resistance is equal to the thermal contact resistance is to be determined.

**Properties**

The thermal conductivity of aluminum at room temperature is $k = 237$ W/m · °C (Table A-3).

**Analysis**

Noting that thermal contact resistance is the inverse of thermal contact conductance, the thermal contact resistance is

$$R_c = \frac{1}{h_c} = \frac{1}{11,000 \text{ W/m}^2 \cdot \degree \text{C}} = 0.909 \times 10^{-4} \text{ m}^2 \cdot \degree \text{C}/\text{W}$$

For a unit surface area, the thermal resistance of a flat plate is defined as

$$R = \frac{L}{k}$$

where $L$ is the thickness of the plate and $k$ is the thermal conductivity. Setting $R = R_c$, the equivalent thickness is determined from the relation above to be

$$L = kR_c = (237 \text{ W/m} \cdot \degree \text{C})(0.909 \times 10^{-4} \text{ m}^2 \cdot \degree \text{C}/\text{W}) = 0.0215 \text{ m} = 2.15 \text{ cm}$$
TABLE 3–2
Thermal contact conductance of some metal surfaces in air (from various sources)

<table>
<thead>
<tr>
<th>Material</th>
<th>Surface Condition</th>
<th>Roughness, μm</th>
<th>Temperature, °C</th>
<th>Pressure, MPa</th>
<th>( h_c ),* W/m² · °C</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Identical Metal Pairs</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>416 Stainless steel</td>
<td>Ground</td>
<td>2.54</td>
<td>90–200</td>
<td>0.3–2.5</td>
<td>3800</td>
</tr>
<tr>
<td>304 Stainless steel</td>
<td>Ground</td>
<td>1.14</td>
<td>20</td>
<td>4–7</td>
<td>1900</td>
</tr>
<tr>
<td>Aluminum</td>
<td>Ground</td>
<td>2.54</td>
<td>150</td>
<td>1.2–2.5</td>
<td>11,400</td>
</tr>
<tr>
<td>Copper</td>
<td>Ground</td>
<td>1.27</td>
<td>20</td>
<td>1.2–20</td>
<td>143,000</td>
</tr>
<tr>
<td>Copper</td>
<td>Milled</td>
<td>3.81</td>
<td>20</td>
<td>1–5</td>
<td>55,500</td>
</tr>
<tr>
<td>Copper (vacuum)</td>
<td>Milled</td>
<td>0.25</td>
<td>30</td>
<td>0.7–7</td>
<td>11,400</td>
</tr>
<tr>
<td><strong>Dissimilar Metal Pairs</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Stainless steel–Aluminum</td>
<td></td>
<td>20–30</td>
<td>20</td>
<td>10</td>
<td>2900</td>
</tr>
<tr>
<td>Stainless steel–Aluminum</td>
<td></td>
<td>1.0–2.0</td>
<td>20</td>
<td>10</td>
<td>16,400</td>
</tr>
<tr>
<td>Steel Ct-30–Aluminum</td>
<td>Ground</td>
<td>1.4–2.0</td>
<td>20</td>
<td>10</td>
<td>50,000</td>
</tr>
<tr>
<td>Steel Ct-30–Aluminum</td>
<td>Milled</td>
<td>4.5–7.2</td>
<td>20</td>
<td>15–35</td>
<td>59,000</td>
</tr>
<tr>
<td>Aluminum-Copper</td>
<td>Ground</td>
<td>1.3–1.4</td>
<td>20</td>
<td>15</td>
<td>42,000</td>
</tr>
<tr>
<td>Aluminum-Copper</td>
<td>Milled</td>
<td>4.4–4.5</td>
<td>20</td>
<td>20–35</td>
<td>12,000</td>
</tr>
</tbody>
</table>

*Divide the given values by 5.678 to convert to Btu/h · ft² · °F.

**Discussion** Note that the interface between the two plates offers as much resistance to heat transfer as a 2.3–cm-thick aluminum plate. It is interesting that the thermal contact resistance in this case is greater than the sum of the thermal resistances of both plates.

**EXAMPLE 3–5** Contact Resistance of Transistors

Four identical power transistors with aluminum casing are attached on one side of a 1-cm-thick 20-cm × 20-cm square copper plate (\( k = 386 \text{ W/m} \cdot \text{°C} \)) by screws that exert an average pressure of 6 MPa (Fig. 3–18). The base area of each transistor is 8 cm², and each transistor is placed at the center of a 10-cm × 10-cm quarter section of the plate. The interface roughness is estimated to be about 1.5 μm. All transistors are covered by a thick Plexiglas layer, which is a poor conductor of heat, and thus all the heat generated at the junction of the transistor must be dissipated to the ambient at 20°C through the back surface of the copper plate. The combined convection/radiation heat transfer coefficient at the back surface can be taken to be 25 W/m² · °C. If the case temperature of
the transistor is not to exceed 70°C, determine the maximum power each transistor can dissipate safely, and the temperature jump at the case-plate interface.

**SOLUTION** Four identical power transistors are attached on a copper plate. For a maximum case temperature of 70°C, the maximum power dissipation and the temperature jump at the interface are to be determined.

**Assumptions** 1 Steady operating conditions exist. 2 Heat transfer can be approximated as being one-dimensional, although it is recognized that heat conduction in some parts of the plate will be two-dimensional since the plate area is much larger than the base area of the transistor. But the large thermal conductivity of copper will minimize this effect. 3 All the heat generated at the junction is dissipated through the back surface of the plate since the transistors are covered by a thick Plexiglas layer. 4 Thermal conductivities are constant.

**Properties** The thermal conductivity of copper is given to be \( k = 386 \text{ W/m} \cdot \text{°C} \). The contact conductance is obtained from Table 3-2 to be \( h_c = 42,000 \text{ W/m}^2 \cdot \text{°C} \), which corresponds to copper-aluminum interface for the case of 1.3–1.4 μm roughness and 5 MPa pressure, which is sufficiently close to what we have.

**Analysis** The contact area between the case and the plate is given to be 8 cm², and the plate area for each transistor is 100 cm². The thermal resistance network of this problem consists of three resistances in series (interface, plate, and convection), which are determined to be

\[
R_{\text{interface}} = \frac{1}{h_c A_c} = \frac{1}{(42,000 \text{ W/m}^2 \cdot \text{°C})(8 \times 10^{-4} \text{ m}^2)} = 0.030\text{°C/W}
\]

\[
R_{\text{plate}} = \frac{L}{k A} = \frac{0.01 \text{ m}}{(386 \text{ W/m} \cdot \text{°C})(0.01 \text{ m}^2)} = 0.0026\text{°C/W}
\]

\[
R_{\text{conv}} = \frac{1}{h_c A} = \frac{1}{(25 \text{ W/m}^2 \cdot \text{°C})(0.01 \text{ m}^2)} = 4.0\text{°C/W}
\]

The total thermal resistance is then

\[
R_{\text{total}} = R_{\text{interface}} + R_{\text{plate}} + R_{\text{ambient}} = 0.030 + 0.0026 + 4.0 = 4.0326\text{°C/W}
\]

Note that the thermal resistance of a copper plate is very small and can be ignored altogether. Then the rate of heat transfer is determined to be

\[
\dot{Q} = \frac{\Delta T}{R_{\text{total}}} = \frac{(70 - 20)\text{°C}}{4.0326\text{°C/W}} = 12.4 \text{ W}
\]

Therefore, the power transistor should not be operated at power levels greater than 12.4 W if the case temperature is not to exceed 70°C.

The temperature jump at the interface is determined from

\[
\Delta T_{\text{interface}} = \frac{\dot{Q} R_{\text{interface}}}{(12.4 \text{ W})(0.030\text{°C/W})} = 0.37\text{°C}
\]

which is not very large. Therefore, even if we eliminate the thermal contact resistance at the interface completely, we will lower the operating temperature of the transistor in this case by less than 0.4°C.
3–3 • GENERALIZED THERMAL RESISTANCE NETWORKS

The thermal resistance concept or the electrical analogy can also be used to solve steady heat transfer problems that involve parallel layers or combined series-parallel arrangements. Although such problems are often two- or even three-dimensional, approximate solutions can be obtained by assuming one-dimensional heat transfer and using the thermal resistance network.

Consider the composite wall shown in Fig. 3–19, which consists of two parallel layers. The thermal resistance network, which consists of two parallel resistances, can be represented as shown in the figure. Noting that the total heat transfer is the sum of the heat transfers through each layer, we have

\[ \dot{Q} = \dot{Q}_1 + \dot{Q}_2 = \frac{T_1 - T_2}{R_1} + \frac{T_1 - T_2}{R_2} = (T_1 - T_2) \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \]  

(3-29)

Utilizing electrical analogy, we get

\[ \dot{Q} = \frac{T_1 - T_2}{R_{\text{total}}} \]  

(3-30)

where

\[ \frac{1}{R_{\text{total}}} = \frac{1}{R_1} + \frac{1}{R_2} \rightarrow R_{\text{total}} = \frac{R_1R_2}{R_1 + R_2} \]  

(3-31)

since the resistances are in parallel.

Now consider the combined series-parallel arrangement shown in Fig. 3–20. The total rate of heat transfer through this composite system can again be expressed as

\[ \dot{Q} = \frac{T_1 - T_\infty}{R_{\text{total}}} \]  

(3-32)

where

\[ R_{\text{total}} = R_{12} + R_3 + R_{\text{conv}} = \frac{R_1R_2}{R_1 + R_2} + R_3 + R_{\text{conv}} \]  

(3-33)

and

\[ R_1 = \frac{L_1}{k_1A_1}, \quad R_2 = \frac{L_2}{k_2A_2}, \quad R_3 = \frac{L_3}{k_3A_3}, \quad R_{\text{conv}} = \frac{1}{hA_3} \]  

(3-34)

Once the individual thermal resistances are evaluated, the total resistance and the total rate of heat transfer can easily be determined from the relations above.

The result obtained will be somewhat approximate, since the surfaces of the third layer will probably not be isothermal, and heat transfer between the first two layers is likely to occur.

Two assumptions commonly used in solving complex multidimensional heat transfer problems by treating them as one-dimensional (say, in the
x-direction) using the thermal resistance network are (1) any plane wall normal to the x-axis is isothermal (i.e., to assume the temperature to vary in the x-direction only) and (2) any plane parallel to the x-axis is adiabatic (i.e., to assume heat transfer to occur in the x-direction only). These two assumptions result in different resistance networks, and thus different (but usually close) values for the total thermal resistance and thus heat transfer. The actual result lies between these two values. In geometries in which heat transfer occurs predominantly in one direction, either approach gives satisfactory results.

**EXAMPLE 3–6 Heat Loss through a Composite Wall**

A 3-m-high and 5-m-wide wall consists of long 16-cm × 22-cm cross section horizontal bricks \( (k = 0.72 \text{ W/m} \cdot \text{°C}) \) separated by 3-cm-thick plaster layers \( (k = 0.22 \text{ W/m} \cdot \text{°C}) \). There are also 2-cm-thick plaster layers on each side of the brick and a 3-cm-thick rigid foam \( (k = 0.026 \text{ W/m} \cdot \text{°C}) \) on the inner side of the wall, as shown in Fig. 3–21. The indoor and the outdoor temperatures are 20°C and 10°C, and the convection heat transfer coefficients on the inner and the outer sides are \( h_1 = 10 \text{ W/m}^2 \cdot \text{°C} \) and \( h_2 = 25 \text{ W/m}^2 \cdot \text{°C} \), respectively. Assuming one-dimensional heat transfer and disregarding radiation, determine the rate of heat transfer through the wall.

**SOLUTION** The composition of a composite wall is given. The rate of heat transfer through the wall is to be determined.

**Assumptions**  
1. Heat transfer is steady since there is no indication of change with time.  
2. Heat transfer can be approximated as being one-dimensional since it is predominantly in the x-direction.  
3. Thermal conductivities are constant.  
4. Heat transfer by radiation is negligible.

**Properties** The thermal conductivities are given to be \( k = 0.72 \text{ W/m} \cdot \text{°C} \) for bricks, \( k = 0.22 \text{ W/m} \cdot \text{°C} \) for plaster layers, and \( k = 0.026 \text{ W/m} \cdot \text{°C} \) for the rigid foam.

**Analysis** There is a pattern in the construction of this wall that repeats itself every 25-cm distance in the vertical direction. There is no variation in the horizontal direction. Therefore, we consider a 1-m-deep and 0.25-m-high portion of the wall, since it is representative of the entire wall.

Assuming any cross section of the wall normal to the x-direction to be isothermal, the thermal resistance network for the representative section of the wall becomes as shown in Fig. 3–21. The individual resistances are evaluated as:

\[
R_1 = R_{\text{conv, 1}} = \frac{1}{h_1 A} = \frac{1}{(10 \text{ W/m}^2 \cdot \text{°C})(0.25 \times 1 \text{ m}^2)} = 0.4^\circ \text{C/W}
\]

\[
R_1 = R_{\text{foam}} = \frac{L}{kA} = \frac{0.03 \text{ m}}{(0.026 \text{ W/m} \cdot \text{°C})(0.25 \times 1 \text{ m}^2)} = 4.6^\circ \text{C/W}
\]

\[
R_2 = R_6 = R_{\text{plaster, side}} = \frac{L}{kA} = \frac{0.02 \text{ m}}{(0.22 \text{ W/m} \cdot \text{°C})(0.25 \times 1 \text{ m}^2)}
\]

\[
= 0.36^\circ \text{C/W}
\]

\[
R_3 = R_5 = R_{\text{plaster, center}} = \frac{L}{kA} = \frac{0.16 \text{ m}}{(0.22 \text{ W/m} \cdot \text{°C})(0.015 \times 1 \text{ m}^2)}
\]

\[
= 48.48^\circ \text{C/W}
\]
1.01 °C/W

\[ R_{39} = R_{\text{brick}} = \frac{L}{kA} = \frac{0.16 \text{ m}}{(0.72 \text{ W/m} \cdot \text{°C})(0.22 \times 1 \text{ m}^2)} = 1.01 \text{°C/W} \]

\[ R_{40} = R_{\text{conv.2}} = \frac{1}{h_2A} = \frac{1}{(25 \text{ W/m}^2 \cdot \text{°C})(0.25 \times 1 \text{ m}^2)} = 0.16 \text{°C/W} \]

The three resistances \( R_3, R_4, \) and \( R_5 \) in the middle are parallel, and their equivalent resistance is determined from

\[ \frac{1}{R_{\text{mid}}} = \frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5} = \frac{1}{48.48} + \frac{1}{1.01} + \frac{1}{48.48} = 1.03 \text{ W/°C} \]

which gives

\[ R_{\text{mid}} = 0.97 \text{°C/W} \]

Now all the resistances are in series, and the total resistance is

\[ R_{\text{total}} = R_1 + R_3 + R_4 + R_5 + R_6 + R_o \]

\[ = 0.4 + 4.6 + 0.36 + 0.97 + 0.36 + 0.16 \]

\[ = 6.85 \text{°C/W} \]

Then the steady rate of heat transfer through the wall becomes

\[ \dot{Q} = \frac{T_{x=1} - T_{x=2}}{R_{\text{total}}} = \frac{[20 - (-10)] \text{°C}}{6.85 \text{°C/W}} = 4.38 \text{ W} \] (per 0.25 m² surface area)

or 4.38/0.25 = 17.5 W per m² area. The total area of the wall is \( A = 3 \text{ m} \times 5 \text{ m} = 15 \text{ m}^2 \). Then the rate of heat transfer through the entire wall becomes

\[ \dot{Q}_{\text{total}} = (17.5 \text{ W/m}^2)(15 \text{ m}^2) = 263 \text{ W} \]

Of course, this result is approximate, since we assumed the temperature within the wall to vary in one direction only and ignored any temperature change (and thus heat transfer) in the other two directions.

**Discussion**  In the above solution, we assumed the temperature at any cross section of the wall normal to the x-direction to be *isothermal*. We could also solve this problem by going to the other extreme and assuming the surfaces parallel to the x-direction to be *adiabatic*. The thermal resistance network in this case will be as shown in Fig. 3–22. By following the approach outlined above, the total thermal resistance in this case is determined to be \( R_{\text{total}} = 6.97 \text{°C/W} \), which is very close to the value 6.85°C/W obtained before. Thus either approach would give roughly the same result in this case. This example demonstrates that either approach can be used in practice to obtain satisfactory results.
Consider steady heat conduction through a hot water pipe. Heat is continuously lost to the outdoors through the wall of the pipe, and we intuitively feel that heat transfer through the pipe is in the normal direction to the pipe surface and no significant heat transfer takes place in the pipe in other directions (Fig. 3–23). The wall of the pipe, whose thickness is rather small, separates two fluids at different temperatures, and thus the temperature gradient in the radial direction will be relatively large. Further, if the fluid temperatures inside and outside the pipe remain constant, then heat transfer through the pipe is steady. Thus heat transfer through the pipe can be modeled as steady and one-dimensional. The temperature of the pipe in this case will depend on one direction only (the radial \( r \)-direction) and can be expressed as

\[
T(r) = H_{11005}\left( T(r_1) - T(r_2) \right) \ln\left( \frac{r_2}{r_1} \right) / H_{20885} r_2 - T_2
\]

where \( A = 2\pi r L \) is the heat transfer area at location \( r \). Note that \( A \) depends on \( r \), and thus it varies in the direction of heat transfer. Separating the variables in the above equation and integrating from \( r = r_1 \), where \( T(r_1) = T_1 \), to \( r = r_2 \), where \( T(r_2) = T_2 \), gives

\[
\int_{r=r_1}^{r=r_2} \frac{\dot{Q}_{\text{cond, cyl}}}{A} dT = -\int_{T=T_1}^{T=T_2} k dT
\]

Substituting \( A = 2\pi r L \) and performing the integrations give

\[
\dot{Q}_{\text{cond, cyl}} = 2\pi L k \frac{T_1 - T_2}{\ln(r_2/r_1)} \quad (W)
\]

since \( \dot{Q}_{\text{cond, cyl}} = \text{constant} \). This equation can be rearranged as

\[
\dot{Q}_{\text{cond, cyl}} = \frac{T_1 - T_2}{R_{\text{cyl}}} \quad (W)
\]
where

\[ R_{\text{cyl}} = \frac{\ln(r_2/r_1)}{2\pi Lk} = \frac{\ln(\text{Outer radius}/\text{Inner radius})}{2\pi \times (\text{Length}) \times (\text{Thermal conductivity})} \tag{3-39} \]

is the thermal resistance of the cylindrical layer against heat conduction, or simply the conduction resistance of the cylinder layer.

We can repeat the analysis above for a spherical layer by taking \( A = 4\pi r^2 \) and performing the integrations in Eq. \( 3-36 \). The result can be expressed as

\[ \dot{Q}_{\text{cond, sph}} = \frac{T_1 - T_2}{R_{\text{sph}}} \tag{3-40} \]

where

\[ R_{\text{sph}} = \frac{r_2 - r_1}{4\pi r_1 r_2 k} = \frac{\text{Outer radius} - \text{Inner radius}}{4\pi (\text{Outer radius})(\text{Inner radius})(\text{Thermal conductivity})} \tag{3-41} \]

is the thermal resistance of the spherical layer against heat conduction, or simply the conduction resistance of the spherical layer.

Now consider steady one-dimensional heat flow through a cylindrical or spherical layer that is exposed to convection on both sides to fluids at temperatures \( T_{s1} \) and \( T_{s2} \) with heat transfer coefficients \( h_1 \) and \( h_2 \), respectively, as shown in Fig. 3–25. The thermal resistance network in this case consists of one conduction and two convection resistances in series, just like the one for the plane wall, and the rate of heat transfer under steady conditions can be expressed as

\[ \dot{Q} = \frac{T_{s1} - T_{s2}}{R_{\text{total}}} \tag{3-42} \]

**FIGURE 3–25**
The thermal resistance network for a cylindrical (or spherical) shell subjected to convection from both the inner and the outer sides.
where

\[ R_{\text{total}} = R_{\text{conv}, 1} + R_{\text{cyl}} + R_{\text{conv}, 2} \]
\[ = \frac{1}{(2\pi r_1 L)h_1} + \frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{(2\pi r_2 L)h_2} \tag{3-43} \]

for a cylindrical layer, and

\[ R_{\text{total}} = R_{\text{conv}, 1} + R_{\text{sph}} + R_{\text{conv}, 2} \]
\[ = \frac{1}{(4\pi r_1^2)h_1} + \frac{r_2 - r_1}{4\pi r_1 r_2 k} + \frac{1}{(4\pi r_2^2)h_2} \tag{3-44} \]

for a spherical layer. Note that \( A \) in the convection resistance relation \( R_{\text{conv}} = 1/hA \) is the surface area at which convection occurs. It is equal to \( A = 2\pi rL \) for a cylindrical surface and \( A = 4\pi r^2 \) for a spherical surface of radius \( r \). Also note that the thermal resistances are in series, and thus the total thermal resistance is determined by simply adding the individual resistances, just like the electrical resistances connected in series.

**Multilayered Cylinders and Spheres**

Steady heat transfer through multilayered cylindrical or spherical shells can be handled just like multilayered plane walls discussed earlier by simply adding an additional resistance in series for each additional layer. For example, the steady heat transfer rate through the three-layered composite cylinder of length \( L \) shown in Fig. 3–26 with convection on both sides can be expressed as

\[ \dot{Q} = \frac{T_{w1} - T_{w2}}{R_{\text{total}}} \tag{3-45} \]

**FIGURE 3–26**
The thermal resistance network for heat transfer through a three-layered composite cylinder subjected to convection on both sides.
where $R_{\text{total}}$ is the total thermal resistance, expressed as

$$R_{\text{total}} = R_{\text{conv,1}} + R_{\text{cyl,1}} + R_{\text{cyl,2}} + R_{\text{cyl,3}} + R_{\text{conv,2}}$$

$$= \frac{1}{h_1 A_1} + \frac{\ln(r_2/r_1)}{2\pi Lk_1} + \frac{\ln(r_4/r_3)}{2\pi Lk_2} + \frac{1}{h_2 A_4}$$

(3-46)

where $A_1 = 2\pi r_1 L$ and $A_4 = 2\pi r_4 L$. Equation 3–46 can also be used for a three-layered spherical shell by replacing the thermal resistances of cylindrical layers by the corresponding spherical ones. Again, note from the thermal resistance network that the resistances are in series, and thus the total thermal resistance is simply the arithmetic sum of the individual thermal resistances in the path of heat flow.

Once $\dot{Q}$ is known, we can determine any intermediate temperature $T_j$ by applying the relation $\dot{Q} = (T_i - T_j)R_{\text{total},i-j}$ across any layer or layers such that $T_j$ is a known temperature at location $i$ and $R_{\text{total},i-j}$ is the total thermal resistance between locations $i$ and $j$ (Fig. 3–27). For example, once $\dot{Q}$ has been calculated, the interface temperature $T_2$ between the first and second cylindrical layers can be determined from

$$\dot{Q} = \frac{T_{s1} - T_2}{R_{\text{conv,1}} + R_{\text{cyl,1}}} = \frac{T_{s1} - T_2}{\frac{1}{h_1(2\pi r_1 L)} + \frac{\ln(r_2/r_1)}{2\pi Lk_1}}$$

(3-47)

We could also calculate $T_2$ from

$$\dot{Q} = \frac{T_2 - T_{s2}}{R_2 + R_3 + R_{\text{conv,2}}} = \frac{T_2 - T_{s2}}{\frac{\ln(r_3/r_2)}{2\pi Lk_2} + \frac{\ln(r_4/r_3)}{2\pi Lk_3} + \frac{1}{h_4(2\pi r_4 L)}}$$

(3-48)

Although both relations will give the same result, we prefer the first one since it involves fewer terms and thus less work.

The thermal resistance concept can also be used for other geometries, provided that the proper conduction resistances and the proper surface areas in convection resistances are used.

**EXAMPLE 3–7 Heat Transfer to a Spherical Container**

A 3-m internal diameter spherical tank made of 2-cm-thick stainless steel ($k = 15$ W/m $\cdot$ °C) is used to store iced water at $T_{s1}$ = 0°C. The tank is located in a room whose temperature is $T_{r2}$ = 22°C. The walls of the room are also at 22°C. The outer surface of the tank is black and heat transfer between the outer surface of the tank and the surroundings is by natural convection and radiation. The convection heat transfer coefficients at the inner and the outer surfaces of the tank are $h_1 = 80$ W/m$^2$ $\cdot$ °C and $h_2 = 10$ W/m$^2$ $\cdot$ °C, respectively. Determine (a) the rate of heat transfer to the iced water in the tank and (b) the amount of ice at 0°C that melts during a 24-h period.

**SOLUTION** A spherical container filled with iced water is subjected to convection and radiation heat transfer at its outer surface. The rate of heat transfer and the amount of ice that melts per day are to be determined.
**HEAT TRANSFER**

**Assumptions** 1. Heat transfer is steady since the specified thermal conditions at the boundaries do not change with time. 2. Heat transfer is one-dimensional since there is thermal symmetry about the midpoint. 3. Thermal conductivity is constant.

**Properties** The thermal conductivity of steel is given to be $k = 15 \text{ W/m} \cdot \text{°C}$. The heat of fusion of water at atmospheric pressure is $h_f = 333.7 \text{ kJ/kg}$. The outer surface of the tank is black and thus its emissivity is $\varepsilon = 1$.

**Analysis**

(a) The thermal resistance network for this problem is given in Fig. 3–28. Noting that the inner diameter of the tank is $D_1 = 3 \text{ m}$ and the outer diameter is $D_2 = 3.04 \text{ m}$, the inner and the outer surface areas of the tank are

$$A_1 = \pi D_1^2 = \pi(3 \text{ m})^2 = 28.3 \text{ m}^2$$
$$A_2 = \pi D_2^2 = \pi(3.04 \text{ m})^2 = 29.0 \text{ m}^2$$

Also, the radiation heat transfer coefficient is given by

$$h_{\text{rad}} = \varepsilon \sigma (T_2^4 - T_1^4)$$

But we do not know the outer surface temperature $T_2$ of the tank, and thus we cannot calculate $h_{\text{rad}}$. Therefore, we need to assume a $T_2$ value now and check the accuracy of this assumption later. We will repeat the calculations if necessary using a revised value for $T_2$.

We note that $T_2$ must be between 0°C and 22°C, but it must be closer to 0°C, since the heat transfer coefficient inside the tank is much larger. Taking $T_2 = 5\text{°C} = 278 \text{ K}$, the radiation heat transfer coefficient is determined to be

$$h_{\text{rad}} = (1)(5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{°C})[(295 \text{ K})^2 + (278 \text{ K})^2][(295 + 278) \text{ K}] = 5.34 \text{ W/m}^2 \cdot \text{°C}$$

Then the individual thermal resistances become

$$R_i = R_{\text{conv,1}} = \frac{1}{h_1 A_1} = \frac{1}{(80 \text{ W/m}^2 \cdot \text{°C})(28.3 \text{ m}^2)} = 0.000442 \text{°C/W}$$

$$R_1 = R_{\text{sphere}} = \frac{r_2 - r_1}{4\pi kr_1r_2} = \frac{(1.52 - 1.50) \text{ m}}{4\pi (15 \text{ W/m} \cdot \text{°C})(1.52 \text{ m})(1.50 \text{ m})} = 0.000047 \text{°C/W}$$

$$R_2 = R_{\text{conv,2}} = \frac{1}{h_2 A_2} = \frac{1}{(10 \text{ W/m}^2 \cdot \text{°C})(29.0 \text{ m}^2)} = 0.00345 \text{°C/W}$$

$$R_{\text{rad}} = \frac{1}{h_{\text{rad}} A_2} = \frac{1}{(5.34 \text{ W/m}^2 \cdot \text{°C})(29.0 \text{ m}^2)} = 0.00646 \text{°C/W}$$

The two parallel resistances $R_o$ and $R_{\text{rad}}$ can be replaced by an equivalent resistance $R_{\text{equiv}}$ determined from

$$\frac{1}{R_{\text{equiv}}} = \frac{1}{R_o} + \frac{1}{R_{\text{rad}}} = \frac{1}{0.00345} + \frac{1}{0.00646} = 444.7 \text{ W/°C}$$

which gives

$$R_{\text{equiv}} = 0.00225 \text{°C/W}$$
Now all the resistances are in series, and the total resistance is determined to be

\[ R_{\text{total}} = R_1 + R_3 + R_{\text{equiv}} = 0.000442 + 0.000047 + 0.00225 = 0.00274 \text{°C/W} \]

Then the steady rate of heat transfer to the iced water becomes

\[ \dot{Q} = \frac{T_{s2} - T_{s1}}{R_{\text{total}}} = \frac{(22 - 0)\text{°C}}{0.00274 \text{°C/W}} = 8029 \text{ W} \quad \text{(or } \dot{Q} = 8.027 \text{ kJ/s)} \]

To check the validity of our original assumption, we now determine the outer surface temperature from

\[ \dot{Q} = \frac{T_{s2} - T_2}{R_{\text{equiv}}} \quad \rightarrow \quad T_2 = T_{s2} - \dot{Q} R_{\text{equiv}} \]

\[ = 22\text{°C} - (8029 \text{ W})(0.00225 \text{°C/W}) = 4\text{°C} \]

which is sufficiently close to the 5°C assumed in the determination of the radiation heat transfer coefficient. Therefore, there is no need to repeat the calculations using 4°C for \( T_2 \).

(b) The total amount of heat transfer during a 24-h period is

\[ Q = \dot{Q} \Delta t = (8.029 \text{ kJ/s})(24 \times 3600 \text{ s}) = 673,700 \text{ kJ} \]

Noting that it takes 333.7 kJ of energy to melt 1 kg of ice at 0°C, the amount of ice that will melt during a 24-h period is

\[ m_{\text{ice}} = \frac{Q}{h_f} = \frac{673,700 \text{ kJ}}{333.7 \text{ kJ/kg}} = 2079 \text{ kg} \]

Therefore, about 2 metric tons of ice will melt in the tank every day.

**Discussion** An easier way to deal with combined convection and radiation at a surface when the surrounding medium and surfaces are at the same temperature is to add the radiation and convection heat transfer coefficients and to treat the result as the convection heat transfer coefficient. That is, to take \( h = 10 + 5.34 = 15.34 \text{ W/m}^2 \cdot \text{°C} \) in this case. This way, we can ignore radiation since its contribution is accounted for in the convection heat transfer coefficient. The convection resistance of the outer surface in this case would be

\[ R_{\text{combined}} = \frac{1}{h_{\text{combined}} A_2} = \frac{1}{(15.34 \text{ W/m}^2 \cdot \text{°C})(29.0 \text{ m}^2)} = 0.00225 \text{°C/W} \]

which is identical to the value obtained for the equivalent resistance for the parallel convection and the radiation resistances.

---

**EXAMPLE 3–8 Heat Loss through an Insulated Steam Pipe**

Steam at \( T_{1a} = 320\text{°C} \) flows in a cast iron pipe \((k = 80 \text{ W/m} \cdot \text{°C})\) whose inner and outer diameters are \( D_1 = 5 \text{ cm} \) and \( D_2 = 5.5 \text{ cm} \), respectively. The pipe is covered with 3-cm-thick glass wool insulation with \( k = 0.05 \text{ W/m} \cdot \text{°C} \). Heat is lost to the surroundings at \( T_{s2} = 5\text{°C} \) by natural convection and radiation, with
a combined heat transfer coefficient of \( h_2 = 18 \text{ W/m}^2 \cdot \text{°C} \). Taking the heat transfer coefficient inside the pipe to be \( h_1 = 60 \text{ W/m}^2 \cdot \text{°C} \), determine the rate of heat loss from the steam per unit length of the pipe. Also determine the temperature drops across the pipe shell and the insulation.

**SOLUTION**  
A steam pipe covered with glass wool insulation is subjected to convection on its surfaces. The rate of heat transfer per unit length and the temperature drops across the pipe and the insulation are to be determined.

**Assumptions**
1. Heat transfer is steady since there is no indication of any change with time.
2. Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no variation in the axial direction.
3. Thermal conductivities are constant.
4. The thermal contact resistance at the interface is negligible.

**Properties**
The thermal conductivities are given to be \( k_1 = 80 \text{ W/m} \cdot \text{°C} \) for cast iron and \( k_2 = 0.05 \text{ W/m} \cdot \text{°C} \) for glass wool insulation.

**Analysis**
The thermal resistance network for this problem involves four resistances in series and is given in Fig. 3–29. Taking \( L = 1 \text{ m} \), the areas of the surfaces exposed to convection are determined to be

\[
A_1 = 2\pi r_1 L = 2\pi(0.025 \text{ m})(1 \text{ m}) = 0.157 \text{ m}^2 \\
A_3 = 2\pi r_3 L = 2\pi(0.0575 \text{ m})(1 \text{ m}) = 0.361 \text{ m}^2 
\]

Then the individual thermal resistances become

\[
R_1 = R_{\text{conv,1}} = \frac{1}{h_1 A_1} = \frac{1}{(60 \text{ W/m}^2 \cdot \text{°C})(0.157 \text{ m}^2)} = 0.106\text{°C/W} \\
R_2 = R_{\text{insulation}} = \frac{1}{h_2 A_3} = \frac{1}{(18 \text{ W/m}^2 \cdot \text{°C})(0.361 \text{ m}^2)} = 0.154\text{°C/W} \\
R_3 = R_{\text{conv,2}} = \frac{1}{h_2 A_3} = \frac{1}{(18 \text{ W/m}^2 \cdot \text{°C})(0.361 \text{ m}^2)} = 0.154\text{°C/W} \\
R_0 = R_{\text{conv,2}} = \frac{1}{h_2 A_3} = \frac{1}{(18 \text{ W/m}^2 \cdot \text{°C})(0.361 \text{ m}^2)} = 0.154\text{°C/W} \\
R_{\text{pipe}} = \frac{1}{2\pi k_1 L} = \frac{1}{2\pi(80 \text{ W/m} \cdot \text{°C})(1 \text{ m})} = 0.0002\text{°C/W} \\
R_{\text{total}} = R_1 + R_2 + R_3 + R_0 = 0.106 + 0.0002 + 2.35 + 0.154 = 2.61\text{°C/W} \\
\]

Noting that all resistances are in series, the total resistance is determined to be

\[
R_{\text{total}} = R_1 + R_2 + R_3 + R_0 = 0.106 + 0.0002 + 2.35 + 0.154 = 2.61\text{°C/W} \\
\]

Then the steady rate of heat loss from the steam becomes

\[
\dot{Q} = \frac{T_{\text{in}} - T_{\text{out}}}{R_{\text{total}}} = \frac{(320 - 5)\text{°C}}{2.61\text{°C/W}} = 121 \text{ W} \\
\]

The heat loss for a given pipe length can be determined by multiplying the above quantity by the pipe length \( L \). The temperature drops across the pipe and the insulation are determined from Eq. 3–17 to be

\[
\Delta T_{\text{pipe}} = \dot{Q} R_{\text{pipe}} = (121 \text{ W})(0.0002\text{°C/W}) = 0.02\text{°C} \\
\Delta T_{\text{insulation}} = \dot{Q} R_{\text{insulation}} = (121 \text{ W})(2.35\text{°C/W}) = 284\text{°C} \\
\]

That is, the temperatures between the inner and the outer surfaces of the pipe differ by 0.02° C, whereas the temperatures between the inner and the outer surfaces of the insulation differ by 284° C.
3–5 CRITICAL RADIUS OF INSULATION

We know that adding more insulation to a wall or to the attic always decreases heat transfer. The thicker the insulation, the lower the heat transfer rate. This is expected, since the heat transfer area $A$ is constant, and adding insulation always increases the thermal resistance of the wall without increasing the convection resistance.

Adding insulation to a cylindrical pipe or a spherical shell, however, is a different matter. The additional insulation increases the conduction resistance of the insulation layer but decreases the convection resistance of the surface because of the increase in the outer surface area for convection. The heat transfer from the pipe may increase or decrease, depending on which effect dominates.

Consider a cylindrical pipe of outer radius $r_1$ whose outer surface temperature $T_1$ is maintained constant (Fig. 3–30). The pipe is now insulated with a material whose thermal conductivity is $k$ and outer radius is $r_2$. Heat is lost from the pipe to the surrounding medium at temperature $T_e$, with a convection heat transfer coefficient $h$. The rate of heat transfer from the insulated pipe to the surrounding air can be expressed as (Fig. 3–31)

$$\dot{Q} = \frac{T_1 - T_e}{R_{\text{ins}} + R_{\text{conv}}} = \frac{T_1 - T_e}{\frac{\ln(r_2/r_1)}{2\pi L k} + \frac{1}{h(2\pi r_2 L)}}$$

(3-49)

The variation of $\dot{Q}$ with the outer radius of the insulation $r_2$ is plotted in Fig. 3–31. The value of $r_2$ at which $\dot{Q}$ reaches a maximum is determined from the requirement that $d\dot{Q}/dr_2 = 0$ (zero slope). Performing the differentiation and solving for $r_2$ yields the critical radius of insulation for a cylindrical body to be

$$r_{\text{cr, cylinder}} = \frac{k}{h} \text{ (m)}$$

(3-50)

Note that the critical radius of insulation depends on the thermal conductivity of the insulation $k$ and the external convection heat transfer coefficient $h$. The rate of heat transfer from the cylinder increases with the addition of insulation for $r_2 < r_{\text{cr}}$, reaches a maximum when $r_2 = r_{\text{cr}}$, and starts to decrease for $r_2 > r_{\text{cr}}$. Thus, insulating the pipe may actually increase the rate of heat transfer from the pipe instead of decreasing it when $r_2 < r_{\text{cr}}$.

The important question to answer at this point is whether we need to be concerned about the critical radius of insulation when insulating hot water pipes or even hot water tanks. Should we always check and make sure that the outer
radius of insulation exceeds the critical radius before we install any insulation? Probably not, as explained here.

The value of the critical radius $r_{cr}$ will be the largest when $k$ is large and $h$ is small. Noting that the lowest value of $h$ encountered in practice is about 5 W/m$^2$·°C for the case of natural convection of gases, and that the thermal conductivity of common insulating materials is about 0.05 W/m$^2$·°C, the largest value of the critical radius we are likely to encounter is

$$r_{cr, max} = \frac{k_{max, insulation}}{h_{min}} \approx \frac{0.05 \text{ W/m} \cdot \text{°C}}{5 \text{ W/m}^2 \cdot \text{°C}} = 0.01 \text{ m} = 1 \text{ cm}$$

This value would be even smaller when the radiation effects are considered. The critical radius would be much less in forced convection, often less than 1 mm, because of much larger $h$ values associated with forced convection. Therefore, we can insulate hot water or steam pipes freely without worrying about the possibility of increasing the heat transfer by insulating the pipes.

The radius of electric wires may be smaller than the critical radius. Therefore, the plastic electrical insulation may actually enhance the heat transfer from electric wires and thus keep their steady operating temperatures at lower and thus safer levels.

The discussions above can be repeated for a sphere, and it can be shown in a similar manner that the critical radius of insulation for a spherical shell is

$$r_{cr, sphere} = \frac{2k}{h}$$

where $k$ is the thermal conductivity of the insulation and $h$ is the convection heat transfer coefficient on the outer surface.

**EXAMPLE 3-9  Heat Loss from an Insulated Electric Wire**

A 3-mm-diameter and 5-m-long electric wire is tightly wrapped with a 2-mm-thick plastic cover whose thermal conductivity is $k = 0.15 \text{ W/m} \cdot \text{°C}$. Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at $T_\infty = 30^\circ\text{C}$ with a heat transfer coefficient of $h = 12 \text{ W/m}^2 \cdot \text{°C}$, determine the temperature at the interface of the wire and the plastic cover in steady operation. Also determine whether doubling the thickness of the plastic cover will increase or decrease this interface temperature.

**SOLUTION** An electric wire is tightly wrapped with a plastic cover. The interface temperature and the effect of doubling the thickness of the plastic cover on the interface temperature are to be determined.

**Assumptions** 1 Heat transfer is steady since there is no indication of any change with time, 2 Heat transfer is one-dimensional since there is thermal symmetry about the centerline and no variation in the axial direction, 3 Thermal conductivities are constant, 4 The thermal contact resistance at the interface is negligible, 5 Heat transfer coefficient incorporates the radiation effects, if any.

**Properties** The thermal conductivity of plastic is given to be $k = 0.15 \text{ W/m} \cdot \text{°C}$.  
Analysis

Heat is generated in the wire and its temperature rises as a result of resistance heating. We assume heat is generated uniformly throughout the wire and is transferred to the surrounding medium in the radial direction. In steady operation, the rate of heat transfer becomes equal to the heat generated within the wire, which is determined to be

$$\dot{Q} = \dot{W} = VI = (8 \text{ V})(10 \text{ A}) = 80 \text{ W}$$

The thermal resistance network for this problem involves a conduction resistance for the plastic cover and a convection resistance for the outer surface in series, as shown in Fig. 3–32. The values of these two resistances are determined to be

$$A_2 = (2\pi r_2)L = 2\pi(0.0035 \text{ m})(5 \text{ m}) = 0.110 \text{ m}^2$$

$$R_{\text{conv}} = \frac{1}{hA_2} = \frac{1}{(12 \text{ W/m}^2 \cdot ^\circ\text{C})(0.110 \text{ m}^2)} = 0.76^\circ\text{C/W}$$

$$R_{\text{plastic}} = \frac{\ln(r_2/r_1)}{2\pi kl} = \frac{\ln(3.5/1.5)}{2\pi(0.15 \text{ W/m} \cdot ^\circ\text{C})(5 \text{ m})} = 0.18^\circ\text{C/W}$$

and therefore

$$R_{\text{total}} = R_{\text{plastic}} + R_{\text{conv}} = 0.76 + 0.18 = 0.94^\circ\text{C/W}$$

Then the interface temperature can be determined from

$$\dot{Q} = \frac{T_1 - T_\infty}{R_{\text{total}}} \quad \longrightarrow \quad T_1 = T_\infty + \dot{Q}R_{\text{total}}$$

$$= 30^\circ\text{C} + (80 \text{ W})(0.94^\circ\text{C/W}) = 105^\circ\text{C}$$

Note that we did not involve the electrical wire directly in the thermal resistance network, since the wire involves heat generation.

To answer the second part of the question, we need to know the critical radius of insulation of the plastic cover. It is determined from Eq. 3–50 to be

$$r_c = \frac{k}{h} = \frac{0.15 \text{ W/m} \cdot ^\circ\text{C}}{12 \text{ W/m}^2 \cdot ^\circ\text{C}} = 0.0125 \text{ m} = 12.5 \text{ mm}$$

which is larger than the radius of the plastic cover. Therefore, increasing the thickness of the plastic cover will enhance heat transfer until the outer radius of the cover reaches 12.5 mm. As a result, the rate of heat transfer \(\dot{Q}\) will increase when the interface temperature \(T_1\) is held constant, or \(T_1\) will decrease when \(\dot{Q}\) is held constant, which is the case here.

Discussion

It can be shown by repeating the calculations above for a 4-mm-thick plastic cover that the interface temperature drops to 90.6°C when the thickness of the plastic cover is doubled. It can also be shown in a similar manner that the interface reaches a minimum temperature of 83°C when the outer radius of the plastic cover equals the critical radius.
The rate of heat transfer from a surface at a temperature $T_s$ to the surrounding medium at $T_a$ is given by Newton's law of cooling as

$$\dot{Q}_{\text{conv}} = hA_s(T_s - T_a)$$

where $A_s$ is the heat transfer surface area and $h$ is the convection heat transfer coefficient. When the temperatures $T_s$ and $T_a$ are fixed by design considerations, as is often the case, there are two ways to increase the rate of heat transfer: to increase the convection heat transfer coefficient $h$ or to increase the surface area $A_s$. Increasing $h$ may require the installation of a pump or fan, or replacing the existing one with a larger one, but this approach may or may not be practical. Besides, it may not be adequate. The alternative is to increase the surface area by attaching to the surface extended surfaces called fins made of highly conductive materials such as aluminum. Finned surfaces are manufactured by extruding, welding, or wrapping a thin metal sheet on a surface. Fins enhance heat transfer from a surface by exposing a larger surface area to convection and radiation.

Finned surfaces are commonly used in practice to enhance heat transfer, and they often increase the rate of heat transfer from a surface severalfold. The car radiator shown in Fig. 3–33 is an example of a finned surface. The closely packed thin metal sheets attached to the hot water tubes increase the surface area for convection and thus the rate of convection heat transfer from the tubes to the air many times. There are a variety of innovative fin designs available in the market, and they seem to be limited only by imagination (Fig. 3–34).

In the analysis of fins, we consider steady operation with no heat generation in the fin, and we assume the thermal conductivity $k$ of the material to remain constant. We also assume the convection heat transfer coefficient $h$ to be constant and uniform over the entire surface of the fin for convenience in the analysis. We recognize that the convection heat transfer coefficient $h$, in general, varies along the fin as well as its circumference, and its value at a point is a strong function of the fluid motion at that point. The value of $h$ is usually much lower at the fin base than it is at the fin tip because the fluid is surrounded by solid surfaces near the base, which seriously disrupt its motion to...
the point of “suffocating” it, while the fluid near the fin tip has little contact with a solid surface and thus encounters little resistance to flow. Therefore, adding too many fins on a surface may actually decrease the overall heat transfer when the decrease in $h$ offsets any gain resulting from the increase in the surface area.

**Fin Equation**

Consider a volume element of a fin at location $x$ having a length of $\Delta x$, cross-sectional area of $A_x$, and a perimeter of $p_x$, as shown in Fig. 3–35. Under steady conditions, the energy balance on this volume element can be expressed as

$$\begin{align*}
\text{Rate of heat conduction into the element at } x &= \text{Rate of heat conduction from the element at } x + \Delta x + \text{Rate of heat convection from the element}
\end{align*}$$

or

$$\dot{Q}_{\text{cond}, x} = \dot{Q}_{\text{cond}, x + \Delta x} + \dot{Q}_{\text{conv}}$$

where

$$\dot{Q}_{\text{conv}} = h(p \Delta x)(T - T_w)$$

Substituting and dividing by $\Delta x$, we obtain

$$\frac{\dot{Q}_{\text{cond}, x + \Delta x} - \dot{Q}_{\text{cond}, x}}{\Delta x} + hp(T - T_w) = 0 \quad (3-52)$$

Taking the limit as $\Delta x \to 0$ gives

$$\frac{d\dot{Q}_{\text{cond}}}{dx} + hp(T - T_w) = 0 \quad (3-53)$$

From Fourier’s law of heat conduction we have

$$\dot{Q}_{\text{cond}} = -kA_x \frac{dT}{dx} \quad (3-54)$$

where $A_x$ is the cross-sectional area of the fin at location $x$. Substitution of this relation into Eq. 3–53 gives the differential equation governing heat transfer in fins,

$$\frac{d}{dx} \left( kA_x \frac{dT}{dx} \right) - hp(T - T_w) = 0 \quad (3-55)$$

In general, the cross-sectional area $A_x$ and the perimeter $p_x$ of a fin vary with $x$, which makes this differential equation difficult to solve. In the special case of **constant cross section** and **constant thermal conductivity**, the differential equation 3–55 reduces to

$$\frac{d^2 \theta}{dx^2} - \alpha^2 \theta = 0 \quad (3-56)$$
where

\[ a^2 = \frac{h_p}{kA} \]  

(3-57)

and \( \theta = T - T_\infty \) is the temperature excess. At the fin base we have \( \theta_b = T_b - T_\infty \).

Equation 3–56 is a linear, homogeneous, second-order differential equation with constant coefficients. A fundamental theory of differential equations states that such an equation has two linearly independent solution functions, and its general solution is the linear combination of those two solution functions. A careful examination of the differential equation reveals that subtracting a constant multiple of the solution function \( \theta \) from its second derivative yields zero. Thus we conclude that the function \( \theta \) and its second derivative must be constant multiples of each other. The only functions whose derivatives are constant multiples of the functions themselves are the exponential functions (or a linear combination of exponential functions such as sine and cosine hyperbolic functions). Therefore, the solution functions of the differential equation above are the exponential functions \( e^{-ax} \) or \( e^{ax} \) or constant multiples of them. This can be verified by direct substitution. For example, the second derivative of \( e^{-ax} \) is \( a^2 e^{-ax} \), and its substitution into Eq. 3–56 yields zero. Therefore, the general solution of the differential equation Eq. 3–56 is

\[ \theta(x) = C_1 e^{ax} + C_2 e^{-ax} \]  

(3-58)

where \( C_1 \) and \( C_2 \) are arbitrary constants whose values are to be determined from the boundary conditions at the base and at the tip of the fin. Note that we need only two conditions to determine \( C_1 \) and \( C_2 \) uniquely.

The temperature of the plate to which the fins are attached is normally known in advance. Therefore, at the fin base we have a specified temperature boundary condition, expressed as

Boundary condition at fin base: \[ \theta(0) = \theta_b = T_b - T_\infty \]  

(3-59)

At the fin tip we have several possibilities, including specified temperature, negligible heat loss (idealized as an insulated tip), convection, and combined convection and radiation (Fig. 3–36). Next, we consider each case separately.

1 Infinitely Long Fin (\( T_{\text{fin tip}} = T_\infty \))

For a sufficiently long fin of uniform cross section \( (A_x = \text{constant}) \), the temperature of the fin at the fin tip will approach the environment temperature \( T_\infty \) and thus \( \theta \) will approach zero. That is,

Boundary condition at fin tip: \[ \theta(L) = T(L) - T_\infty = 0 \quad \text{as} \quad L \to \infty \]

This condition will be satisfied by the function \( e^{-ax} \), but not by the other prospective solution function \( e^{ax} \) since it tends to infinity as \( x \) gets larger. Therefore, the general solution in this case will consist of a constant multiple of \( e^{-ax} \). The value of the constant multiple is determined from the requirement that at the fin base where \( x = 0 \) the value of \( \theta \) will be \( \theta_b \). Noting that
\(e^{-\alpha x} = e^{0} = 1\), the proper value of the constant is \(\theta_b\), and the solution function we are looking for is \(\theta(x) = \theta_b e^{-\alpha x}\). This function satisfies the differential equation as well as the requirements that the solution reduce to \(\theta_b\) at the fin base and approach zero at the fin tip for large \(x\). Noting that \(\theta = T - T_{in}\) and \(\alpha = \sqrt{h pk/A}\), the variation of temperature along the fin in this case can be expressed as

**Very long fin:**
\[
\frac{T(x) - T_{in}}{T_b - T_{in}} = e^{-\alpha x} = e^{-\alpha x} \sqrt{h pk/A}.
\]

(3-60)

Note that the temperature along the fin in this case decreases exponentially from \(T_b\) to \(T_{in}\), as shown in Fig. 3-37. The steady rate of heat transfer from the entire fin can be determined from Fourier’s law of heat conduction

**Very long fin:**
\[
\dot{Q}_{\text{long fin}} = -kA \left. \frac{dT}{dx} \right|_{x=0} = \sqrt{h pk/A} \left( T_b - T_{in} \right)
\]

(3-61)

where \(p\) is the perimeter, \(A_c\) is the cross-sectional area of the fin, and \(x\) is the distance from the fin base. Alternatively, the rate of heat transfer from the fin could also be determined by considering heat transfer from a differential volume element of the fin and integrating it over the entire surface of the fin. That is,

\[
\dot{Q}_{\text{fin}} = \int_{A_in} h [T(x) - T_{in}] \, dA_{fin} = \int_{A_in} h \theta(x) \, dA_{fin}
\]

(3-62)

The two approaches described are equivalent and give the same result since, under steady conditions, the heat transfer from the exposed surfaces of the fin is equal to the heat transfer to the fin at the base (Fig. 3-38).

### 2 Negligible Heat Loss from the Fin Tip

**Insulated fin tip, \(\dot{Q}_{\text{fin tip}} = 0\)**

Fins are not likely to be so long that their temperature approaches the surrounding temperature at the tip. A more realistic situation is for heat transfer from the fin tip to be negligible since the heat transfer from the fin is proportional to its surface area, and the surface area of the fin tip is usually a negligible fraction of the total fin area. Then the fin tip can be assumed to be insulated, and the condition at the fin tip can be expressed as

**Boundary condition at fin tip:**
\[
\frac{d\theta}{dx} \bigg|_{x=L} = 0
\]

(3-63)

The condition at the fin base remains the same as expressed in Eq. 3-59. The application of these two conditions on the general solution (Eq. 3-58) yields, after some manipulations, this relation for the temperature distribution:

**Adiabatic fin tip:**
\[
\frac{T(x) - T_{in}}{T_b - T_{in}} = \frac{\cosh a(L - x)}{\cosh aL}
\]

(3-64)
The rate of heat transfer from the fin can be determined again from Fourier’s law of heat conduction:

\[
\dot{Q}_{\text{ insulated tip}} = -kA_e \frac{dT}{dx}
\]

\[
= \sqrt{\text{hpk}A_e} (T_b - T_\infty) \tanh aL
\] (3-65)

Note that the heat transfer relations for the very long fin and the fin with negligible heat loss at the tip differ by the factor \( \tanh aL \), which approaches 1 as \( L \) becomes very large.

3 Convection (or Combined Convection and Radiation) from Fin Tip

The fin tips, in practice, are exposed to the surroundings, and thus the proper boundary condition for the fin tip is convection that also includes the effects of radiation. The fin equation can still be solved in this case using the convection at the fin tip as the second boundary condition, but the analysis becomes more involved, and it results in rather lengthy expressions for the temperature distribution and the heat transfer. Yet, in general, the fin tip area is a small fraction of the total fin surface area, and thus the complexities involved can hardly justify the improvement in accuracy.

A practical way of accounting for the heat loss from the fin tip is to replace the length \( L \) in the relation for the insulated tip case by a corrected length defined as (Fig. 3–39)

\[
L_c = L + \frac{A_e}{p}
\] (3-66)

where \( A_e \) is the cross-sectional area and \( p \) is the perimeter of the fin at the tip. Multiplying the relation above by the perimeter gives \( A_{\text{corrected}} = A_{\text{fin (lateral)}} + A_{\text{tip}} \), which indicates that the fin area determined using the corrected length is equivalent to the sum of the lateral fin area plus the fin tip area.

The corrected length approximation gives very good results when the variation of temperature near the fin tip is small (which is the case when \( aL \leq 1 \)) and the heat transfer coefficient at the fin tip is about the same as that at the lateral surface of the fin. Therefore, fins subjected to convection at their tips can be treated as fins with insulated tips by replacing the actual fin length by the corrected length in Eqs. 3–64 and 3–65.

Using the proper relations for \( A_e \) and \( p \), the corrected lengths for rectangular and cylindrical fins are easily determined to be

\[
L_{c, \text{ rectangular fin}} = L + \frac{t}{2} \quad \text{and} \quad L_{c, \text{ cylindrical fin}} = L + \frac{D}{4}
\]

where \( t \) is the thickness of the rectangular fins and \( D \) is the diameter of the cylindrical fins.

Fin Efficiency

Consider the surface of a plane wall at temperature \( T_b \) exposed to a medium at temperature \( T_\infty \). Heat is lost from the surface to the surrounding medium by
convection with a heat transfer coefficient of \(h\). Disregarding radiation or accounting for its contribution in the convection coefficient \(h\), heat transfer from a surface area \(A_s\) is expressed as \(Q = hA_s(T_s - T_a)\).

Now let us consider a fin of constant cross-sectional area \(A_c = A_b\) and length \(L\) that is attached to the surface with a perfect contact (Fig. 3–40). This time heat will flow from the surface to the fin by conduction and from the fin to the surrounding medium by convection with the same heat transfer coefficient \(h\). The temperature of the fin will be \(T_b\) at the fin base and gradually decrease toward the fin tip. Convection from the fin surface causes the temperature at any cross section to drop somewhat from the midsection toward the outer surfaces. However, the cross-sectional area of the fins is usually very small, and thus the temperature at any cross section can be considered to be uniform. Also, the fin tip can be assumed for convenience and simplicity to be insulated by using the corrected length for the fin instead of the actual length.

In the limiting case of zero thermal resistance or infinite thermal conductivity \((k \rightarrow \infty)\), the temperature of the fin will be uniform at the base value of \(T_b\). The heat transfer from the fin will be maximum in this case and can be expressed as

\[
\dot{Q}_{\text{fin, max}} = hA_{\text{fin}}(T_b - T_a)
\]  

(3-67)

In reality, however, the temperature of the fin will drop along the fin, and thus the heat transfer from the fin will be less because of the decreasing temperature difference \(T(x) - T_a\) toward the fin tip, as shown in Fig. 3–41. To account for the effect of this decrease in temperature on heat transfer, we define a fin efficiency as

\[
\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal heat transfer rate from the fin if the entire fin were at base temperature}}
\]  

(3-68)

or

\[
\dot{Q}_{\text{fin}} = \eta_{\text{fin}} \dot{Q}_{\text{fin, max}} = \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_a)
\]  

(3-69)

where \(A_{\text{fin}}\) is the total surface area of the fin. This relation enables us to determine the heat transfer from a fin when its efficiency is known. For the cases of constant cross section of very long fins and fins with insulated tips, the fin efficiency can be expressed as

\[
\eta_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_{\text{fin}}(T_b - T_a)}}{hA_{\text{fin}}(T_b - T_a)} = \frac{1}{L} \sqrt{\frac{kA_{\text{fin}}}{h}} = \frac{1}{aL}
\]  

(3-70)

and

\[
\eta_{\text{insulated tip}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\sqrt{hpkA_{\text{fin}}(T_b - T_a) \tanh aL}}{hA_{\text{fin}}(T_b - T_a)} = \frac{\tanh aL}{aL}
\]  

(3-71)

since \(A_{\text{fin}} = pL\) for fins with constant cross section. Equation 3–71 can also be used for fins subjected to convection provided that the fin length \(L\) is replaced by the corrected length \(L_c\).
Fin efficiency relations are developed for fins of various profiles and are plotted in Fig. 3–42 for fins on a plain surface and in Fig. 3–43 for circular fins of constant thickness. The fin surface area associated with each profile is also given on each figure. For most fins of constant thickness encountered in practice, the fin thickness $t$ is too small relative to the fin length $L$, and thus the fin tip area is negligible.

**Figure 3–42**
Efficiency of circular, rectangular, and triangular fins on a plain surface of width $w$ (from Gardner, Ref. 6).

**Figure 3–43**
Efficiency of circular fins of length $L$ and constant thickness $t$ (from Gardner, Ref. 6).
Note that fins with triangular and parabolic profiles contain less material and are more efficient than the ones with rectangular profiles, and thus are more suitable for applications requiring minimum weight such as space applications.

An important consideration in the design of finned surfaces is the selection of the proper fin length $L$. Normally the longer the fin, the larger the heat transfer area and thus the higher the rate of heat transfer from the fin. But also the larger the fin, the bigger the mass, the higher the price, and the larger the fluid friction. Therefore, increasing the length of the fin beyond a certain value cannot be justified unless the added benefits outweigh the added cost. Also, the fin efficiency decreases with increasing fin length because of the decrease in fin temperature with length. Fin lengths that cause the fin efficiency to drop below 60 percent usually cannot be justified economically and should be avoided. The efficiency of most fins used in practice is above 90 percent.

**Fin Effectiveness**

Fins are used to enhance heat transfer, and the use of fins on a surface cannot be recommended unless the enhancement in heat transfer justifies the added cost and complexity associated with the fins. In fact, there is no assurance that adding fins on a surface will enhance heat transfer. The performance of the fins is judged on the basis of the enhancement in heat transfer relative to the no-fin case. The performance of fins expressed in terms of the fin effectiveness $\epsilon_{\text{fin}}$ is defined as (Fig. 3–44)

$$\epsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\frac{\dot{Q}_{\text{fin}}}{hA_b(T_b - T_s)}}{\frac{\dot{Q}_{\text{no fin}}}{hA_b(T_b - T_s)}}$$  \hspace{1cm} (3-72)

Here, $A_b$ is the cross-sectional area of the fin at the base and $\dot{Q}_{\text{no fin}}$ represents the rate of heat transfer from this area if no fins are attached to the surface. An effectiveness of $\epsilon_{\text{fin}} = 1$ indicates that the addition of fins to the surface does not affect heat transfer at all. That is, heat conducted to the fin through the base area $A_b$ is equal to the heat transferred from the same area $A_b$ to the surrounding medium. An effectiveness of $\epsilon_{\text{fin}} < 1$ indicates that the fin actually acts as insulation, slowing down the heat transfer from the surface. This situation can occur when fins made of low thermal conductivity materials are used. An effectiveness of $\epsilon_{\text{fin}} > 1$ indicates that fins are enhancing heat transfer from the surface, as they should. However, the use of fins cannot be justified unless $\epsilon_{\text{fin}}$ is sufficiently larger than 1. Finned surfaces are designed on the basis of maximizing effectiveness for a specified cost or minimizing cost for a desired effectiveness.

Note that both the fin efficiency and fin effectiveness are related to the performance of the fin, but they are different quantities. However, they are related to each other by

$$\epsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\frac{\dot{Q}_{\text{fin}}}{hA_b(T_b - T_s)}}{\frac{\dot{Q}_{\text{no fin}}}{hA_b(T_b - T_s)}} = \frac{A_{\text{fin}}}{A_b} \eta_{\text{fin}}$$  \hspace{1cm} (3-73)

**FIGURE 3–44**

The effectiveness of a fin.
Therefore, the fin effectiveness can be determined easily when the fin efficiency is known, or vice versa.

The rate of heat transfer from a sufficiently long fin of uniform cross section under steady conditions is given by Eq. 3–61. Substituting this relation into Eq. 3–72, the effectiveness of such a long fin is determined to be

\[ e_{\text{long fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} = \frac{\sqrt{h} pk A_c (T_b - T_a)}{h A_b (T_b - T_a)} = \frac{\sqrt{k p}}{h A_c} \]  

(3-74)

since \( A_c = A_b \) in this case. We can draw several important conclusions from the fin effectiveness relation above for consideration in the design and selection of the fins:

- The thermal conductivity \( k \) of the fin material should be as high as possible. Thus it is no coincidence that fins are made from metals, with copper, aluminum, and iron being the most common ones. Perhaps the most widely used fins are made of aluminum because of its low cost and weight and its resistance to corrosion.

- The ratio of the perimeter to the cross-sectional area of the fin \( p/A_c \) should be as high as possible. This criterion is satisfied by thin plate fins and slender pin fins.

- The use of fins is most effective in applications involving a low convection heat transfer coefficient. Thus, the use of fins is more easily justified when the medium is a gas instead of a liquid and the heat transfer is by natural convection instead of by forced convection. Therefore, it is no coincidence that in liquid-to-gas heat exchangers such as the car radiator, fins are placed on the gas side.

When determining the rate of heat transfer from a finned surface, we must consider the unfinned portion of the surface as well as the fins. Therefore, the rate of heat transfer for a surface containing \( n \) fins can be expressed as

\[ \dot{Q}_{\text{total, fin}} = \dot{Q}_{\text{unfin}} + \dot{Q}_{\text{fin}} \]

\[ = h A_{\text{unfin}} (T_b - T_a) + \eta_{\text{fin}} h A_{\text{fin}} (T_b - T_a) \]

\[ = h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_a) \] 

(3-75)

We can also define an overall effectiveness for a finned surface as the ratio of the total heat transfer from the finned surface to the heat transfer from the same surface if there were no fins,

\[ \varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{h(A_{\text{unfin}} + \eta_{\text{fin}} A_{\text{fin}})(T_b - T_a)}{h A_{\text{no fin}} (T_b - T_a)} \] 

(3-76)

where \( A_{\text{no fin}} \) is the area of the surface when there are no fins, \( A_{\text{fin}} \) is the total surface area of all the fins on the surface, and \( A_{\text{unfin}} \) is the area of the unfinned portion of the surface (Fig. 3–45). Note that the overall fin effectiveness depends on the fin density (number of fins per unit length) as well as the effectiveness of the individual fins. The overall effectiveness is a better measure of the performance of a finned surface than the effectiveness of the individual fins.
Proper Length of a Fin

An important step in the design of a fin is the determination of the appropriate length of the fin once the fin material and the fin cross section are specified. You may be tempted to think that the longer the fin, the larger the surface area and thus the higher the rate of heat transfer. Therefore, for maximum heat transfer, the fin should be infinitely long. However, the temperature drops along the fin exponentially and reaches the environment temperature at some length. The part of the fin beyond this length does not contribute to heat transfer since it is at the temperature of the environment, as shown in Fig. 3–46. Therefore, designing such an “extra long” fin is out of the question since it results in material waste, excessive weight, and increased size and thus increased cost with no benefit in return (in fact, such a long fin will hurt performance since it will suppress fluid motion and thus reduce the convection heat transfer coefficient). Fins that are so long that the temperature approaches the environment temperature cannot be recommended either since the little increase in heat transfer at the tip region cannot justify the large increase in the weight and cost.

To get a sense of the proper length of a fin, we compare heat transfer from a fin of finite length to heat transfer from an infinitely long fin under the same conditions. The ratio of these two heat transfers is

\[
\frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{long fin}}} = \frac{\sqrt{hpk} \alpha (T_b - T_e) \tanh aL}{\sqrt{hpk} \alpha (T_b - T_e)} = \tanh aL \tag{3-77}
\]

Using a hand calculator, the values of \( \tanh aL \) are evaluated for some values of \( aL \) and the results are given in Table 3–3. We observe from the table that heat transfer from a fin increases with \( aL \) almost linearly at first, but the curve reaches a plateau later and reaches a value for the infinitely long fin at about \( aL = 5 \). Therefore, a fin whose length is \( L = \frac{1}{3} aL \) can be considered to be an infinitely long fin. We also observe that reducing the fin length by half in that case (from \( aL = 5 \) to \( aL = 2.5 \)) causes a drop of just 1 percent in heat transfer. We certainly would not hesitate sacrificing 1 percent in heat transfer performance in return for 50 percent reduction in the size and possibly the cost of the fin. In practice, a fin length that corresponds to about \( aL = 1 \) will transfer 76.2 percent of the heat that can be transferred by an infinitely long fin, and thus it should offer a good compromise between heat transfer performance and the fin size.

A common approximation used in the analysis of fins is to assume the fin temperature varies in one direction only (along the fin length) and the temperature variation along other directions is negligible. Perhaps you are wondering if this one-dimensional approximation is a reasonable one. This is certainly the case for fins made of thin metal sheets such as the fins on a car radiator, but we wouldn’t be so sure for fins made of thick materials. Studies have shown that the error involved in one-dimensional fin analysis is negligible (less than about 1 percent) when

\[
\frac{h\delta}{k} < 0.2
\]
where \( \delta \) is the characteristic thickness of the fin, which is taken to be the plate thickness \( t \) for rectangular fins and the diameter \( D \) for cylindrical ones.

Specially designed finned surfaces called heat sinks, which are commonly used in the cooling of electronic equipment, involve one-of-a-kind complex geometries, as shown in Table 3–4. The heat transfer performance of heat sinks is usually expressed in terms of their thermal resistances \( R \) in °C/W, which is defined as

\[
\dot{Q}_{\text{fin}} = \frac{T_b - T_s}{R} = hA_{\text{fin}} \eta_{\text{fin}} (T_b - T_s)
\]  

(3-78)

A small value of thermal resistance indicates a small temperature drop across the heat sink, and thus a high fin efficiency.

**EXAMPLE 3–10  Maximum Power Dissipation of a Transistor**

Power transistors that are commonly used in electronic devices consume large amounts of electric power. The failure rate of electronic components increases almost exponentially with operating temperature. As a rule of thumb, the failure rate of electronic components is halved for each 10°C reduction in the junction operating temperature. Therefore, the operating temperature of electronic components is kept below a safe level to minimize the risk of failure.

The sensitive electronic circuitry of a power transistor at the junction is protected by its case, which is a rigid metal enclosure. Heat transfer characteristics of a power transistor are usually specified by the manufacturer in terms of the case-to-ambient thermal resistance, which accounts for both the natural convection and radiation heat transfers.

The case-to-ambient thermal resistance of a power transistor that has a maximum power rating of 10 W is given to be 20°C/W. If the case temperature of the transistor is not to exceed 85°C, determine the power at which this transistor can be operated safely in an environment at 25°C.

**SOLUTION**  The maximum power rating of a transistor whose case temperature is not to exceed 85°C is to be determined.

**Assumptions** 1 Steady operating conditions exist. 2 The transistor case is isothermal at 85°C.

**Properties**  The case-to-ambient thermal resistance is given to be 20°C/W.

**Analysis**  The power transistor and the thermal resistance network associated with it are shown in Fig. 3–47. We notice from the thermal resistance network that there is a single resistance of 20°C/W between the case at \( T_c = 85°C \) and the ambient at \( T_a = 25°C \), and thus the rate of heat transfer is

\[
\dot{Q} = \left( \frac{\Delta T}{R} \right)_{\text{case-ambient}} = \frac{\Delta T}{R_{\text{case-ambient}}} = \frac{(85 - 25)\, \text{°C}}{20\, \text{°C/W}} = 3 \, \text{W}
\]

Therefore, this power transistor should not be operated at power levels above 3 W if its case temperature is not to exceed 85°C.

**Discussion**  This transistor can be used at higher power levels by attaching it to a heat sink (which lowers the thermal resistance by increasing the heat transfer surface area, as discussed in the next example) or by using a fan (which lowers the thermal resistance by increasing the convection heat transfer coefficient).
TABLE 3–4

Combined natural convection and radiation thermal resistance of various heat sinks used in the cooling of electronic devices between the heat sink and the surroundings. All fins are made of aluminum 6063T-5, are black anodized, and are 76 mm (3 in.) long (courtesy of Vemaline Products, Inc.).

<table>
<thead>
<tr>
<th>Heat Sink</th>
<th>Thermal Resistance (°C/W)</th>
<th>Dimensions</th>
<th>Surface Area</th>
</tr>
</thead>
<tbody>
<tr>
<td>HS 5030</td>
<td>$R = 0.9°C/W$ (vertical)</td>
<td>76 mm x 105 mm x 44 mm</td>
<td>677 cm²</td>
</tr>
<tr>
<td></td>
<td>$R = 1.2°C/W$ (horizontal)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HS 6065</td>
<td>$R = 5°C/W$</td>
<td>76 mm x 38 mm x 24 mm</td>
<td>387 cm²</td>
</tr>
<tr>
<td>HS 6071</td>
<td>$R = 1.4°C/W$ (vertical)</td>
<td>76 mm x 92 mm x 26 mm</td>
<td>968 cm²</td>
</tr>
<tr>
<td></td>
<td>$R = 1.8°C/W$ (horizontal)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HS 6105</td>
<td>$R = 1.8°C/W$ (vertical)</td>
<td>76 mm x 127 mm x 91 mm</td>
<td>677 cm²</td>
</tr>
<tr>
<td></td>
<td>$R = 2.1°C/W$ (horizontal)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HS 6115</td>
<td>$R = 1.1°C/W$ (vertical)</td>
<td>76 mm x 102 mm x 25 mm</td>
<td>929 cm²</td>
</tr>
<tr>
<td></td>
<td>$R = 1.3°C/W$ (horizontal)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>HS 7030</td>
<td>$R = 2.9°C/W$ (vertical)</td>
<td>76 mm x 97 mm x 19 mm</td>
<td>290 cm²</td>
</tr>
<tr>
<td></td>
<td>$R = 3.1°C/W$ (horizontal)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

EXAMPLE 3–11 Selecting a Heat Sink for a Transistor

A 60-W power transistor is to be cooled by attaching it to one of the commercially available heat sinks shown in Table 3–4. Select a heat sink that will allow the case temperature of the transistor not to exceed 90°C in the ambient air at 30°C.
SOLUTION A commercially available heat sink from Table 3–4 is to be selected to keep the case temperature of a transistor below 90°C.

Assumptions 1 Steady operating conditions exist. 2 The transistor case is isothermal at 90°C. 3 The contact resistance between the transistor and the heat sink is negligible.

Analysis The rate of heat transfer from a 60-W transistor at full power is $Q_{\text{no fin}} = 60$ W. The thermal resistance between the transistor attached to the heat sink and the ambient air for the specified temperature difference is determined to be

$$ R = \frac{\Delta T}{Q} = \frac{(90 - 30)\,^\circ\text{C}}{60\,\text{W}} = 1.0\,^\circ\text{C}/\text{W} $$

Therefore, the thermal resistance of the heat sink should be below 1.0°C/W. An examination of Table 3–4 reveals that the HS 5030, whose thermal resistance is 0.9°C/W in the vertical position, is the only heat sink that will meet this requirement.

EXAMPLE 3–12 Effect of Fins on Heat Transfer from Steam Pipes

Steam in a heating system flows through tubes whose outer diameter is $D_1 = 3$ cm and whose walls are maintained at a temperature of 120°C. Circular aluminum fins ($k = 180$ W/m·°C) of outer diameter $D_2 = 6$ cm and constant thickness $t = 2$ mm are attached to the tube, as shown in Fig. 3–48. The space between the fins is 3 mm, and thus there are 200 fins per meter length of the tube. Heat is transferred to the surrounding air at $T_a = 25°C$, with a combined heat transfer coefficient of $h = 60$ W/m²·°C. Determine the increase in heat transfer from the tube per meter of its length as a result of adding fins.

SOLUTION Circular aluminum fins are to be attached to the tubes of a heating system. The increase in heat transfer from the tubes per unit length as a result of adding fins is to be determined.

Assumptions 1 Steady operating conditions exist. 2 The heat transfer coefficient is uniform over the entire fin surfaces. 3 Thermal conductivity is constant. 4 Heat transfer by radiation is negligible.

Properties The thermal conductivity of the fins is given to be $k = 180$ W/m·°C.

Analysis In the case of no fins, heat transfer from the tube per meter of its length is determined from Newton’s law of cooling to be

$$ A_{\text{no fin}} = \pi D_1 L = \pi(0.03\,\text{m})(1\,\text{m}) = 0.0942\,\text{m}^2 $$

$$ \dot{Q}_{\text{no fin}} = hA_{\text{no fin}}(T_b - T_a) $$

$$ = (60\,\text{W/m}^2\cdot\circ\text{C})(0.0942\,\text{m}^2)(120 - 25)\,^\circ\text{C} $$

$$ = 537\,\text{W} $$

The efficiency of the circular fins attached to a circular tube is plotted in Fig. 3–43. Noting that $L = \frac{1}{2}(D_2 - D_1) = \frac{1}{2}(0.06 - 0.03) = 0.015$ m in this case, we have...
So far, we have considered heat transfer in simple geometries such as large plane walls, long cylinders, and spheres. This is because heat transfer in such geometries can be approximated as one-dimensional, and simple analytical solutions can be obtained easily. But many problems encountered in practice are two- or three-dimensional and involve rather complicated geometries for which no simple solutions are available.

\[
\frac{r_2 + \frac{1}{2}t}{r_1} = \frac{(0.03 + \frac{1}{2} \times 0.002) \text{ m}}{0.015 \text{ m}} = 2.07
\]

\[
(L + \frac{1}{2}t) \sqrt{\frac{h}{kL}} = (0.015 + \frac{1}{2} \times 0.002) \text{ m} \times \sqrt{\frac{60 \text{ W/m}^2 \cdot \ ^\circ\text{C}}{(180 \text{ W/m} \cdot ^\circ\text{C})(0.002 \text{ m})}} = 0.207
\]

\[
A_{\text{fin}} = 2\pi(r_2^2 - r_1^2) + 2\pi r_t t
= 2\pi[(0.03 \text{ m})^2 - (0.015 \text{ m})^2] + 2\pi(0.03 \text{ m})(0.002 \text{ m}) = 0.00462 \text{ m}^2
\]

\[
\dot{Q}_{\text{fin}} = \eta_{\text{fin}} \dot{Q}_{\text{fin, max}} = \eta_{\text{fin}} hA_{\text{fin}}(T_b - T_w)
= 0.95(60 \text{ W/m}^2 \cdot ^\circ\text{C})(0.00462 \text{ m}^2)(120 - 25)^\circ\text{C}
= 25.0 \text{ W}
\]

Heat transfer from the unfinned portion of the tube is

\[
A_{\text{unfin}} = \pi D_t S = \pi(0.03 \text{ m})(0.003 \text{ m}) = 0.000283 \text{ m}^2
\]

\[
\dot{Q}_{\text{unfin}} = hA_{\text{unfin}}(T_b - T_w)
= (60 \text{ W/m}^2 \cdot ^\circ\text{C})(0.000283 \text{ m}^2)(120 - 25)^\circ\text{C}
= 1.60 \text{ W}
\]

Noting that there are 200 fins and thus 200 interfin spacings per meter length of the tube, the total heat transfer from the finned tube becomes

\[
\dot{Q}_{\text{total, fin}} = n(\dot{Q}_{\text{fin}} + \dot{Q}_{\text{unfin}}) = 200(25.0 + 1.6) \text{ W} = 5320 \text{ W}
\]

Therefore, the increase in heat transfer from the tube per meter of its length as a result of the addition of fins is

\[
\dot{Q}_{\text{increase}} = \dot{Q}_{\text{total, fin}} - \dot{Q}_{\text{no fin}} = 5320 - 537 = 4783 \text{ W} \quad \text{(per m tube length)}
\]

**Discussion** The overall effectiveness of the finned tube is

\[
\eta_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{5320 \text{ W}}{537 \text{ W}} = 9.9
\]

That is, the rate of heat transfer from the steam tube increases by a factor of almost 10 as a result of adding fins. This explains the widespread use of finned surfaces.

3-7 *HEAT TRANSFER IN COMMON CONFIGURATIONS*

So far, we have considered heat transfer in simple geometries such as large plane walls, long cylinders, and spheres. This is because heat transfer in such geometries can be approximated as one-dimensional, and simple analytical solutions can be obtained easily. But many problems encountered in practice are two- or three-dimensional and involve rather complicated geometries for which no simple solutions are available.
An important class of heat transfer problems for which simple solutions are obtained encompasses those involving two surfaces maintained at constant temperatures \( T_1 \) and \( T_2 \). The steady rate of heat transfer between these two surfaces is expressed as

\[
Q = Sk(T_1 - T_2)
\]

where \( S \) is the conduction shape factor, which has the dimension of length, and \( k \) is the thermal conductivity of the medium between the surfaces. The conduction shape factor depends on the geometry of the system only.

Conduction shape factors have been determined for a number of configurations encountered in practice and are given in Table 3-5 for some common cases. More comprehensive tables are available in the literature. Once the value of the shape factor is known for a specific geometry, the total steady heat transfer rate can be determined from the equation above using the specified two constant temperatures of the two surfaces and the thermal conductivity of the medium between them. Note that conduction shape factors are applicable only when heat transfer between the two surfaces is by conduction. Therefore, they cannot be used when the medium between the surfaces is a liquid or gas, which involves natural or forced convection currents.

A comparison of Equations 3-4 and 3-79 reveals that the conduction shape factor \( S \) is related to the thermal resistance \( R \) by \( R = 1/kS \) or \( S = 1/kR \). Thus, these two quantities are the inverse of each other when the thermal conductivity of the medium is unity. The use of the conduction shape factors is illustrated with examples 3–13 and 3–14.

### EXAMPLE 3–13 Heat Loss from Buried Steam Pipes

A 30-m-long, 10-cm-diameter hot water pipe of a district heating system is buried in the soil 50 cm below the ground surface, as shown in Figure 3–49. The outer surface temperature of the pipe is 80°C. Taking the surface temperature of the earth to be 10°C and the thermal conductivity of the soil at that location to be 0.9 W/m · °C, determine the rate of heat loss from the pipe.

**SOLUTION**

The hot water pipe of a district heating system is buried in the soil. The rate of heat loss from the pipe is to be determined.

**Assumptions**

1. Steady operating conditions exist.
2. Heat transfer is two-dimensional (no change in the axial direction).
3. Thermal conductivity of the soil is constant.

**Properties**

The thermal conductivity of the soil is given to be \( k = 0.9 \) W/m · °C.

**Analysis**

The shape factor for this configuration is given in Table 3-5 to be

\[
S = \frac{2\pi L}{\ln(4z/D)}
\]

since \( z > 1.5D \), where \( z \) is the distance of the pipe from the ground surface, and \( D \) is the diameter of the pipe. Substituting,

\[
S = \frac{2\pi \times (30 \text{ m})}{\ln(4 \times 0.5/0.1)} = 62.9 \text{ m}
\]
TABLE 3-5

Conduction shape factors $S$ for several configurations for use in $\dot{Q} = kS(T_1 - T_2)$ to determine the steady rate of heat transfer through a medium of thermal conductivity $k$ between the surfaces at temperatures $T_1$ and $T_2$.

<table>
<thead>
<tr>
<th>Configuration</th>
<th>Formula</th>
<th>Diagram</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Isothermal cylinder of length $L$ buried in a semi-infinite medium ($L \gg D$ and $z &gt; 1.5D$)</td>
<td>$S = \frac{2\pi L}{\ln(4z/D)}$</td>
<td><img src="https://via.placeholder.com/150" alt="Diagram" /></td>
</tr>
<tr>
<td>(2) Vertical isothermal cylinder of length $L$ buried in a semi-infinite medium ($L \gg D$)</td>
<td>$S = \frac{2\pi L}{\ln(4L/D)}$</td>
<td><img src="https://via.placeholder.com/150" alt="Diagram" /></td>
</tr>
<tr>
<td>(3) Two parallel isothermal cylinders placed in an infinite medium ($L \gg D_1, D_2, z$)</td>
<td>$S = \frac{2\pi L}{\cosh^{-1}\left(\frac{4z^2-D_1^2-D_2^2}{2D_1D_2}\right)}$</td>
<td><img src="https://via.placeholder.com/150" alt="Diagram" /></td>
</tr>
<tr>
<td>(4) A row of equally spaced parallel isothermal cylinders buried in a semi-infinite medium ($L \gg D, z$ and $w &gt; 1.5D$)</td>
<td>$S = \frac{2\pi L}{\ln\left(\frac{8w}{D}\sinh\frac{2\pi z}{w}\right)}$ (per cylinder)</td>
<td><img src="https://via.placeholder.com/150" alt="Diagram" /></td>
</tr>
<tr>
<td>(5) Circular isothermal cylinder of length $L$ in the midplane of an infinite wall ($z &gt; 0.5D$)</td>
<td>$S = \frac{2\pi L}{\ln(8z/\pi D)}$</td>
<td><img src="https://via.placeholder.com/150" alt="Diagram" /></td>
</tr>
<tr>
<td>(6) Circular isothermal cylinder of length $L$ at the center of a square solid bar of the same length</td>
<td>$S = \frac{2\pi L}{\ln(1.08w/D)}$</td>
<td><img src="https://via.placeholder.com/150" alt="Diagram" /></td>
</tr>
<tr>
<td>(7) Eccentric circular isothermal cylinder of length $L$ in a cylinder of the same length ($L &gt; D_2$)</td>
<td>$S = \frac{2\pi L}{\cosh^{-1}\left(\frac{4z^2-D_1^2-D_2^2}{2D_1D_2}\right)}$</td>
<td><img src="https://via.placeholder.com/150" alt="Diagram" /></td>
</tr>
<tr>
<td>(8) Large plane wall</td>
<td>$S = \frac{A}{L}$</td>
<td><img src="https://via.placeholder.com/150" alt="Diagram" /></td>
</tr>
</tbody>
</table>

(continued)
### TABLE 3-5 (CONCLUDED)

<table>
<thead>
<tr>
<th>No.</th>
<th>Configuration</th>
<th>Heat Transfer Calculation</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>A long cylindrical layer</td>
<td>$S = \frac{2\pi L}{\ln \left(\frac{D_2}{D_1}\right)}$</td>
</tr>
<tr>
<td>10</td>
<td>A square flow passage</td>
<td>(a) For $a/b &gt; 1.4$, $S = \frac{2\pi L}{0.93 \ln (0.948a/b)}$&lt;br&gt;(b) For $a/b &lt; 1.41$, $S = \frac{2\pi L}{0.785 \ln (a/b)}$</td>
</tr>
<tr>
<td>11</td>
<td>A spherical layer</td>
<td>$S = \frac{2\pi D_1 D_2}{D_2 - D_1}$</td>
</tr>
<tr>
<td>12</td>
<td>Disk buried parallel to the surface in a semi-infinite medium ($z &gt;&gt; D$)</td>
<td>$S = 4D$&lt;br&gt;($S = 2D$ when $z = 0$)</td>
</tr>
<tr>
<td>13</td>
<td>The edge of two adjoining walls of equal thickness</td>
<td>$S = 0.54w$</td>
</tr>
<tr>
<td>14</td>
<td>Corner of three walls of equal thickness</td>
<td>$S = 0.15L$</td>
</tr>
<tr>
<td>15</td>
<td>Isothermal sphere buried in a semi-infinite medium</td>
<td>$S = \frac{2\pi D}{1 - 0.25D/z}$</td>
</tr>
<tr>
<td>16</td>
<td>Isothermal sphere buried in a semi-infinite medium at $T_2$ whose surface is insulated</td>
<td>$S = \frac{2\pi D}{1 + 0.25D/z}$</td>
</tr>
</tbody>
</table>
It is well known that insulation reduces heat transfer and saves energy and money. Decisions on the right amount of insulation are based on a heat transfer analysis, followed by an economic analysis to determine the "monetary value" of energy loss. This is illustrated with Example 3–15.

Then the steady rate of heat transfer from the pipe becomes

\[
\dot{Q} = Sk(T_1 - T_2) = (62.9 \text{ m})(0.9 \text{ W/m} \cdot \text{°C})(80 - 10)\text{°C} = 3963 \text{ W}
\]

**Discussion**  
Note that this heat is conducted from the pipe surface to the surface of the earth through the soil and then transferred to the atmosphere by convection and radiation.

**EXAMPLE 3–14**  
**Heat Transfer between Hot and Cold Water Pipes**

A 5-m-long section of hot and cold water pipes run parallel to each other in a thick concrete layer, as shown in Figure 3–50. The diameters of both pipes are 5 cm, and the distance between the centerline of the pipes is 30 cm. The surface temperatures of the hot and cold pipes are 70°C and 15°C, respectively. Taking the thermal conductivity of the concrete to be \( k = 0.75 \text{ W/m} \cdot \text{°C} \), determine the rate of heat transfer between the pipes.

**SOLUTION**  
Hot and cold water pipes run parallel to each other in a thick concrete layer. The rate of heat transfer between the pipes is to be determined.

**Assumptions**  
1. Steady operating conditions exist.  
2. Heat transfer is two-dimensional (no change in the axial direction).  
3. Thermal conductivity of the concrete is constant.

**Properties**  
The thermal conductivity of concrete is given to be \( k = 0.75 \text{ W/m} \cdot \text{°C} \).

**Analysis**  
The shape factor for this configuration is given in Table 3–5 to be

\[
S = \frac{2\pi L}{\cosh^{-1} \left( \frac{4z^2 - D_1^2 - D_2^2}{2D_1D_2} \right)}
\]

where \( z \) is the distance between the centerlines of the pipes and \( L \) is their length. Substituting,

\[
S = \frac{2\pi \times (5 \text{ m})}{\cosh^{-1} \left( \frac{4 \times 0.3^2 - 0.05^2 - 0.05^2}{2 \times 0.05 \times 0.05} \right)} = 6.34 \text{ m}
\]

Then the steady rate of heat transfer between the pipes becomes

\[
\dot{Q} = Sk(T_1 - T_2) = (6.34 \text{ m})(0.75 \text{ W/m} \cdot \text{°C})(70 - 15)\text{°C} = 262 \text{ W}
\]

**Discussion**  
We can reduce this heat loss by placing the hot and cold water pipes further away from each other.
EXAMPLE 3–15 Cost of Heat Loss through Walls in Winter

Consider an electrically heated house whose walls are 9 ft high and have an R-value of insulation of 13 (i.e., a thickness-to-thermal conductivity ratio of \( L/k = 13 \text{ h} \cdot \text{ft}^2 / \text{°F/Btu} \)). Two of the walls of the house are 40 ft long and the others are 30 ft long. The house is maintained at 75°F at all times, while the temperature of the outdoors varies. Determine the amount of heat lost through the walls of the house on a certain day during which the average temperature of the outdoors is 45°F. Also, determine the cost of this heat loss to the homeowner if the unit cost of electricity is $0.075/kWh. For combined convection and radiation heat transfer coefficients, use the ASHRAE (American Society of Heating, Refrigeration, and Air Conditioning Engineers) recommended values of \( h_i = 1.46 \text{ Btu/h} \cdot \text{ft}^2 \cdot \text{°F} \) for the inner surface of the walls and \( h_o = 4.0 \text{ Btu/h} \cdot \text{ft}^2 \cdot \text{°F} \) for the outer surface of the walls under 15 mph wind conditions in winter.

SOLUTION An electrically heated house with R-13 insulation is considered. The amount of heat lost through the walls and its cost are to be determined.

Assumptions  1 The indoor and outdoor air temperatures have remained at the given values for the entire day so that heat transfer through the walls is steady.  
2 Heat transfer through the walls is one-dimensional since any significant temperature gradients in this case will exist in the direction from the indoors to the outdoors.  
3 The radiation effects are accounted for in the heat transfer coefficients.

Analysis This problem involves conduction through the wall and convection at its surfaces and can best be handled by making use of the thermal resistance concept and drawing the thermal resistance network, as shown in Fig. 3–51. The heat transfer area of the walls is

\[
A = \text{Circumference} \times \text{Height} = (2 \times 30 \text{ ft} + 2 \times 40 \text{ ft})(9 \text{ ft}) = 1260 \text{ ft}^2
\]

Then the individual resistances are evaluated from their definitions to be

\[
R_i = R_{\text{conv, } i} = \frac{1}{h_i A} = \frac{1}{(1.46 \text{ Btu/h} \cdot \text{ft}^2 \cdot \text{°F})(1260 \text{ ft}^2)} = 0.00054 \text{ h} \cdot \text{°F/Btu}
\]

\[
R_{\text{wall}} = \frac{L}{k A} = \frac{\text{R-value}}{A} = \frac{13 \text{ h} \cdot \text{ft}^2 \cdot \text{°F/Btu}}{1260 \text{ ft}^2} = 0.01032 \text{ h} \cdot \text{°F/Btu}
\]

\[
R_o = R_{\text{conv, } o} = \frac{1}{h_o A} = \frac{1}{(4.0 \text{ Btu/h} \cdot \text{ft}^2 \cdot \text{°F})(1260 \text{ ft}^2)} = 0.00020 \text{ h} \cdot \text{°F/Btu}
\]

Noting that all three resistances are in series, the total resistance is

\[
R_{\text{total}} = R_i + R_{\text{wall}} + R_o = 0.00054 + 0.01032 + 0.00020 = 0.01106 \text{ h} \cdot \text{°F/Btu}
\]

Then the steady rate of heat transfer through the walls of the house becomes

\[
\dot{Q} = \frac{T_{w1} - T_{w2}}{R_{\text{total}}} = \frac{(75 - 45)\text{°F}}{0.01106 \text{ h} \cdot \text{°F/Btu}} = 2712 \text{ Btu/h}
\]

Finally, the total amount of heat lost through the walls during a 24-h period and its cost to the homeowner are

\[
Q = \dot{Q} \Delta t = (2712 \text{ Btu/h})(24\text{-h/day}) = 65,099 \text{ Btu/day} = 19.1 \text{ kWh/day}
\]
since 1 kWh = 3412 Btu, and

Heating cost = (Energy lost)\((\text{Cost of energy}) = (19.1 \text{kWh/day})(0.075/\text{kWh}) = $1.43/\text{day}

**Discussion** The heat losses through the walls of the house that day will cost the homeowner $1.43 worth of electricity.

**TOPIC OF SPECIAL INTEREST**

*Heat Transfer Through Walls and Roofs*

Under steady conditions, the rate of heat transfer through any section of a building wall or roof can be determined from

\[
\dot{Q} = UA(T_i - T_o) = \frac{A(T_i - T_o)}{R} \tag{3-80}
\]

where \(T_i\) and \(T_o\) are the indoor and outdoor air temperatures, \(A\) is the heat transfer area, \(U\) is the overall heat transfer coefficient (the \(U\)-factor), and \(R = 1/U\) is the overall unit thermal resistance (the \(R\)-value). Walls and roofs of buildings consist of various layers of materials, and the structure and operating conditions of the walls and the roofs may differ significantly from one building to another. Therefore, it is not practical to list the \(R\)-values (or \(U\)-factors) of different kinds of walls or roofs under different conditions. Instead, the overall \(R\)-value is determined from the thermal resistances of the individual components using the thermal resistance network. The overall thermal resistance of a structure can be determined most accurately in a lab by actually assembling the unit and testing it as a whole, but this approach is usually very time consuming and expensive. The analytical approach described here is fast and straightforward, and the results are usually in good agreement with the experimental values.

The unit thermal resistance of a plane layer of thickness \(L\) and thermal conductivity \(k\) can be determined from \(R = L/k\). The thermal conductivity and other properties of common building materials are given in the appendix. The unit thermal resistances of various components used in building structures are listed in Table 3–6 for convenience.

Heat transfer through a wall or roof section is also affected by the convection and radiation heat transfer coefficients at the exposed surfaces. The effects of convection and radiation on the inner and outer surfaces of walls and roofs are usually combined into the *combined convection and radiation heat transfer coefficients* (also called *surface conductances*) \(h\) and \(h_r\), respectively, whose values are given in Table 3–7 for ordinary surfaces (\(e = 0.9\)) and reflective surfaces (\(e = 0.2\) or 0.05). Note that surfaces having a low emittance also have a low surface conductance due to the reduction in radiation heat transfer. The values in the table are based on a surface

*This section can be skipped without a loss of continuity.*

**TABLE 3–7**

Combined convection and radiation heat transfer coefficients at window, wall, or roof surfaces (from ASHRAE Handbook of Fundamentals, Ref. 1, Chap. 22, Table 1).

<table>
<thead>
<tr>
<th>Direction of Flow</th>
<th>Surface Heat Conductance, (h), W/m²·°C</th>
<th>Surface Emittance, (e)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Still air (both indoors and outdoors)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Horiz. Up ↑</td>
<td>9.26</td>
<td>5.17</td>
</tr>
<tr>
<td>Horiz. Down ↓</td>
<td>6.13</td>
<td>2.10</td>
</tr>
<tr>
<td>45° slope Up ↑</td>
<td>9.09</td>
<td>5.00</td>
</tr>
<tr>
<td>45° slope Down ↓</td>
<td>7.50</td>
<td>3.41</td>
</tr>
<tr>
<td>Vertical Horiz.</td>
<td>8.29</td>
<td>4.20</td>
</tr>
<tr>
<td>Moving air (any position, any direction)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Winter condition (winds at 15 mph or 24 km/h)</td>
<td>34.0</td>
<td>—</td>
</tr>
<tr>
<td>Summer condition (winds at 7.5 mph or 12 km/h)</td>
<td>22.7</td>
<td>—</td>
</tr>
</tbody>
</table>

*Multiply by 0.176 to convert to Btu/h · ft² · °F. Surface resistance can be obtained from \(R = 1/h\).*
The temperature of 21°C (72°F) and a surface–air temperature difference of 5.5°C (10°F). Also, the equivalent surface temperature of the environment is assumed to be equal to the ambient air temperature. Despite the convenience it offers, this assumption is not quite accurate because of the additional radiation heat loss from the surface to the clear sky. The effect of sky radiation can be accounted for approximately by taking the outside temperature to be the average of the outdoor air and sky temperatures.

The inner surface heat transfer coefficient \( h_i \) remains fairly constant throughout the year, but the value of \( h_o \) varies considerably because of its dependence on the orientation and wind speed, which can vary from less than 1 km/h in calm weather to over 40 km/h during storms. The commonly used values of \( h_i \) and \( h_o \) for peak load calculations are

\[
    h_i = 8.29 \text{ W/m}^2 \cdot ^\circ\text{C} = 1.46 \text{ Btu/h} \cdot \text{ft}^2 \cdot ^\circ\text{F} \quad \text{(winter and summer)}
\]

\[
    h_o = \begin{cases} 
    34.0 \text{ W/m}^2 \cdot ^\circ\text{C} = 6.0 \text{ Btu/h} \cdot \text{ft}^2 \cdot ^\circ\text{F} & \text{(winter)} \\
    22.7 \text{ W/m}^2 \cdot ^\circ\text{C} = 4.0 \text{ Btu/h} \cdot \text{ft}^2 \cdot ^\circ\text{F} & \text{(summer)}
\end{cases}
\]

which correspond to design wind conditions of 24 km/h (15 mph) for winter and 12 km/h (7.5 mph) for summer. The corresponding surface thermal resistances \( R \)-values are determined from \( R = \frac{1}{h} \). The surface conductance values under still air conditions can be used for interior surfaces as well as exterior surfaces in calm weather.

### Table 3-6

**Unit thermal resistance (the \( R \)-value) of common components used in buildings**

<table>
<thead>
<tr>
<th>Component</th>
<th>( R )-value</th>
<th>( R )-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Outside surface (winter)</td>
<td>0.030 m²·°C/W</td>
<td>0.17 ft²·h·°F/Btu</td>
</tr>
<tr>
<td>Outside surface (summer)</td>
<td>0.044</td>
<td>0.25</td>
</tr>
<tr>
<td>Inside surface, still air</td>
<td>0.12</td>
<td>0.68</td>
</tr>
<tr>
<td>Plane air space, vertical, ordinary surfaces (( \varepsilon_{\text{eff}} = 0.82 ))</td>
<td>13 mm (½ in.)</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>20 mm (¾ in.)</td>
<td>0.17</td>
</tr>
<tr>
<td></td>
<td>40 mm (1.5 in.)</td>
<td>0.16</td>
</tr>
<tr>
<td></td>
<td>90 mm (3.5 in.)</td>
<td>0.16</td>
</tr>
<tr>
<td>Insulation, 25 mm (1 in.)</td>
<td>Glass fiber</td>
<td>0.70</td>
</tr>
<tr>
<td></td>
<td>Mineral fiber batt</td>
<td>0.66</td>
</tr>
<tr>
<td></td>
<td>Urethane rigid foam</td>
<td>0.98</td>
</tr>
<tr>
<td></td>
<td>Stucco, 25 mm (1 in.)</td>
<td>0.037</td>
</tr>
<tr>
<td></td>
<td>Face brick, 100 mm (4 in.)</td>
<td>0.075</td>
</tr>
<tr>
<td></td>
<td>Common brick, 100 mm (4 in.)</td>
<td>0.12</td>
</tr>
<tr>
<td></td>
<td>Steel siding</td>
<td>0.00</td>
</tr>
<tr>
<td></td>
<td>Slag, 13 mm (½ in.)</td>
<td>0.067</td>
</tr>
<tr>
<td></td>
<td>Wood, 25 mm (1 in.)</td>
<td>0.22</td>
</tr>
<tr>
<td></td>
<td>Wood stud, nominal 2 in. × 4 in. (3.5 in. or 90 mm wide)</td>
<td>0.63</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Component</th>
<th>( R )-value</th>
<th>( R )-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wood stud, nominal 2 in. × 6 in. (5.5 in. or 140 mm wide)</td>
<td>0.98</td>
<td>5.56</td>
</tr>
<tr>
<td>Clay tile, 100 mm (4 in.)</td>
<td>0.18</td>
<td>1.01</td>
</tr>
<tr>
<td>Acoustic tile</td>
<td>0.32</td>
<td>1.79</td>
</tr>
<tr>
<td>Asphalt shingle roofing</td>
<td>0.077</td>
<td>0.44</td>
</tr>
<tr>
<td>Building paper</td>
<td>0.011</td>
<td>0.06</td>
</tr>
<tr>
<td>Concrete block, 100 mm (4 in.):</td>
<td>Lightweight</td>
<td>0.27</td>
</tr>
<tr>
<td></td>
<td>Heavyweight</td>
<td>0.13</td>
</tr>
<tr>
<td>Plaster or gypsum board, 13 mm (½ in.)</td>
<td>0.079</td>
<td>0.45</td>
</tr>
<tr>
<td>Wood fiberboard, 13 mm (½ in.)</td>
<td>0.23</td>
<td>1.31</td>
</tr>
<tr>
<td>Plywood, 13 mm (½ in.)</td>
<td>0.11</td>
<td>0.62</td>
</tr>
<tr>
<td>Concrete, 200 mm (8 in.):</td>
<td>Lightweight</td>
<td>1.17</td>
</tr>
<tr>
<td></td>
<td>Heavyweight</td>
<td>0.12</td>
</tr>
<tr>
<td>Cement mortar, 13 mm (1/2 in.)</td>
<td>0.018</td>
<td>0.10</td>
</tr>
<tr>
<td>Wood bevel lapped siding, 13 mm × 200 mm (1/2 in. × 8 in.)</td>
<td>0.14</td>
<td>0.81</td>
</tr>
</tbody>
</table>
Building components often involve trapped air spaces between various layers. Thermal resistances of such air spaces depend on the thickness of the layer, the temperature difference across the layer, the mean air temperature, the emissivity of each surface, the orientation of the air layer, and the direction of heat transfer. The emissivities of surfaces commonly encountered in buildings are given in Table 3–8. The effective emissivity of a plane-parallel air space is given by

$$\frac{1}{\varepsilon_{\text{effective}}} = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1 \quad (3-81)$$

where $\varepsilon_1$ and $\varepsilon_2$ are the emissivities of the surfaces of the air space. Table 3–8 also lists the effective emissivities of air spaces for the cases where (1) the emissivity of one surface of the air space is $\varepsilon$ while the emissivity of the other surface is 0.9 (a building material) and (2) the emissivity of both surfaces is $\varepsilon$. Note that the effective emissivity of an air space between building materials is $0.82/0.03 = 27$ times that of an air space between surfaces covered with aluminum foil. For specified surface temperatures, radiation heat transfer through an air space is proportional to effective emissivity, and thus the rate of radiation heat transfer in the ordinary surface case is 27 times that of the reflective surface case.

Table 3–9 lists the thermal resistances of 20-mm-, 40-mm-, and 90-mm-(0.75-in., 1.5-in., and 3.5-in.) thick air spaces under various conditions. The thermal resistance values in the table are applicable to air spaces of uniform thickness bounded by plane, smooth, parallel surfaces with no air leakage. Thermal resistances for other temperatures, emissivities, and air spaces can be obtained by interpolation and moderate extrapolation. Note that the presence of a low-emissivity surface reduces radiation heat transfer across an air space and thus significantly increases the thermal resistance. The thermal effectiveness of a low-emissivity surface will decline, however, if the condition of the surface changes as a result of some effects such as condensation, surface oxidation, and dust accumulation.

The $R$-value of a wall or roof structure that involves layers of uniform thickness is determined easily by simply adding up the unit thermal resistances of the layers that are in series. But when a structure involves components such as wood studs and metal connectors, then the thermal resistance network involves parallel connections and possible two-dimensional effects. The overall $R$-value in this case can be determined by assuming (1) parallel heat flow paths through areas of different construction or (2) isothermal planes normal to the direction of heat transfer. The first approach usually overpredicts the overall thermal resistance, whereas the second approach usually underpredicts it. The parallel heat flow path approach is more suitable for wood frame walls and roofs, whereas the isothermal planes approach is more suitable for masonry or metal frame walls.

The thermal contact resistance between different components of building structures ranges between 0.01 and 0.1 m$^2$·°C/W, which is negligible in most cases. However, it may be significant for metal building components such as steel framing members.
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TABLE 3–9
Unit thermal resistances (R-values) of well-sealed plane air spaces (from ASHRAE Handbook of Fundamentals, Ref. 1,
Chap. 22, Table 2)
(a) SI units (in m2 · °C/W)

Position
of Air
Space

Direction
of Heat
Flow

Mean
Temp.,
°C

Temp.
Diff.,
°C

32.2
5.6
10.0
16.7
Horizontal Up ↑
10.0
5.6
17.8
11.1
32.2
5.6
10.0
16.7
45° slope Up ↑
10.0
5.6
17.8
11.1
32.2
5.6
10.0
16.7
Vertical Horizontal → 10.0
5.6
17.8
11.1
32.2
5.6
10.0
16.7
45° slope Down ↓
10.0
5.6
17.8
11.1
32.2
5.6
10.0
16.7
Horizontal Down ↓
10.0
5.6
17.8
11.1
(b) English units (in h · ft2 · °F/Btu)

Position
of Air
Space

Direction
of Heat
Flow

Horizontal Up ↑

45° slope Up ↑

Vertical

Horizontal →

45° slope Down ↓

Horizontal Down ↓

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Mean Temp.
Temp., Diff.,
°F
°F
90
50
50
0
90
50
50
0
90
50
50
0
90
50
50
0
90
50
50
0

10
30
10
20
10
30
10
20
10
30
10
20
10
30
10
20
10
30
10
20

20-mm Air Space

40-mm Air Space

90-mm Air Space

Effective
Emissivity, eff

Effective
Emissivity, eff

Effective
Emissivity, eff

0.03 0.05
0.41
0.30
0.40
0.32
0.52
0.35
0.51
0.37
0.62
0.51
0.65
0.55
0.62
0.60
0.67
0.66
0.62
0.66
0.68
0.74

0.39
0.29
0.39
0.32
0.49
0.34
0.48
0.36
0.57
0.49
0.61
0.53
0.58
0.57
0.63
0.63
0.58
0.62
0.63
0.70

0.5
0.18
0.17
0.20
0.20
0.20
0.19
0.23
0.23
0.21
0.23
0.25
0.28
0.21
0.24
0.26
0.30
0.21
0.25
0.26
0.32

0.82 0.03 0.05
0.13
0.14
0.15
0.16
0.14
0.14
0.17
0.18
0.15
0.17
0.18
0.21
0.15
0.17
0.18
0.22
0.15
0.18
0.18
0.23

0.45
0.33
0.44
0.35
0.51
0.38
0.51
0.40
0.70
0.45
0.67
0.49
0.89
0.63
0.90
0.68
1.07
1.10
1.16
1.24

0.42
0.32
0.42
0.34
0.48
0.36
0.48
0.39
0.64
0.43
0.62
0.47
0.80
0.59
0.82
0.64
0.94
0.99
1.04
1.13

0.5
0.19
0.18
0.21
0.22
0.20
0.20
0.23
0.24
0.22
0.22
0.26
0.26
0.24
0.25
0.28
0.31
0.25
0.30
0.30
0.39

0.82 0.03 0.05
0.14
0.14
0.16
0.17
0.14
0.15
0.17
0.18
0.15
0.16
0.18
0.20
0.16
0.18
0.19
0.22
0.17
0.20
0.20
0.26

0.50
0.27
0.49
0.40
0.56
0.40
0.55
0.43
0.65
0.47
0.64
0.51
0.85
0.62
0.83
0.67
1.77
1.69
1.96
1.92

0.47
0.35
0.47
0.38
0.52
0.38
0.52
0.41
0.60
0.45
0.60
0.49
0.76
0.58
0.77
0.64
1.44
1.44
1.63
1.68

0.5

0.82

0.20
0.19
0.23
0.23
0.21
0.20
0.24
0.24
0.22
0.22
0.25
0.27
0.24
0.25
0.28
0.31
0.28
0.33
0.34
0.43

0.14
0.15
0.16
0.18
0.14
0.15
0.17
0.19
0.15
0.16
0.18
0.20
0.16
0.18
0.19
0.22
0.18
0.21
0.22
0.29

0.75-in. Air Space

1.5-in. Air Space

3.5-in. Air Space

Effective
Emissivity, eff

Effective
Emissivity, eff

Effective
Emissivity, eff

0.03 0.05
2.34
1.71
2.30
1.83
2.96
1.99
2.90
2.13
3.50
2.91
3.70
3.14
3.53
3.43
3.81
3.75
3.55
3.77
3.84
4.18

2.22
1.66
2.21
1.79
2.78
1.92
2.75
2.07
3.24
2.77
3.46
3.02
3.27
3.23
3.57
3.57
3.29
3.52
3.59
3.96

0.5
1.04
0.99
1.16
1.16
1.15
1.08
1.29
1.28
1.22
1.30
1.43
1.58
1.22
1.39
1.45
1.72
1.22
1.44
1.45
1.81

0.82 0.03 0.05
0.75
0.77
0.87
0.93
0.81
0.82
0.94
1.00
0.84
0.94
1.01
1.18
0.84
0.99
1.02
1.26
0.85
1.02
1.02
1.30

2.55
1.87
2.50
2.01
2.92
2.14
2.88
2.30
3.99
2.58
3.79
2.76
5.07
3.58
5.10
3.85
6.09
6.27
6.61
7.03

2.41
1.81
2.40
1.95
2.73
2.06
2.74
2.23
3.66
2.46
3.55
2.66
4.55
3.36
4.66
3.66
5.35
5.63
5.90
6.43

0.5
1.08
1.04
1.21
1.23
1.14
1.12
1.29
1.34
1.27
1.23
1.45
1.48
1.36
1.42
1.60
1.74
1.43
1.70
1.73
2.19

0.82 0.03 0.05
0.77
0.80
0.89
0.97
0.80
0.84
0.94
1.04
0.87
0.90
1.02
1.12
0.91
1.00
1.09
1.27
0.94
1.14
1.15
1.49

2.84
2.09
2.80
2.25
3.18
2.26
3.12
2.42
3.69
2.67
3.63
2.88
4.81
3.51
4.74
3.81
10.07
9.60
11.15
10.90

2.66
2.01
2.66
2.18
2.96
2.17
2.95
2.35
3.40
2.55
3.40
2.78
4.33
3.30
4.36
3.63
8.19
8.17
9.27
9.52

0.5

0.82

1.13
1.10
1.28
1.32
1.18
1.15
1.34
1.38
1.24
1.25
1.42
1.51
1.34
1.40
1.57
1.74
1.57
1.88
1.93
2.47

0.80
0.84
0.93
1.03
0.82
0.86
0.96
1.06
0.85
0.91
1.01
1.14
0.90
1.00
1.08
1.27
1.00
1.22
1.24
1.62


EXAMPLE 3–16  The R-Value of a Wood Frame Wall

Determine the overall unit thermal resistance (the \(R\)-value) and the overall heat transfer coefficient (the \(U\)-factor) of a wood frame wall that is built around 38-mm × 90-mm (2 × 4 nominal) wood studs with a center-to-center distance of 400 mm. The 90-mm-wide cavity between the studs is filled with glass fiber insulation. The inside is finished with 13-mm gypsum wallboard and the outside with 13-mm wood fiberboard and 13-mm × 200-mm wood bevel lapped siding. The insulated cavity constitutes 75 percent of the heat transmission area while the studs, plates, and sills constitute 21 percent. The headers constitute 4 percent of the area, and they can be treated as studs.

Also, determine the rate of heat loss through the walls of a house whose perimeter is 50 m and wall height is 2.5 m in Las Vegas, Nevada, whose winter design temperature is \(-2^\circ\text{C}\). Take the indoor design temperature to be \(22^\circ\text{C}\) and assume 20 percent of the wall area is occupied by glazing.

SOLUTION  The \(R\)-value and the \(U\)-factor of a wood frame wall as well as the rate of heat loss through such a wall in Las Vegas are to be determined.

Assumptions  1 Steady operating conditions exist.  2 Heat transfer through the wall is one-dimensional.  3 Thermal properties of the wall and the heat transfer coefficients are constant.

Properties  The \(R\)-values of different materials are given in Table 3–6.

Analysis  The schematic of the wall as well as the different elements used in its construction are shown here. Heat transfer through the insulation and through the studs will meet different resistances, and thus we need to analyze the thermal resistance for each path separately. Once the unit thermal resistances and the \(U\)-factors for the insulation and stud sections are available, the overall average thermal resistance for the entire wall can be determined from

\[
R_{\text{overall}} = \frac{1}{U_{\text{overall}}}
\]

where

\[
U_{\text{overall}} = (U \times f_{\text{area}})_{\text{insulation}} + (U \times f_{\text{area}})_{\text{stud}}
\]

and the value of the area fraction \(f_{\text{area}}\) is 0.75 for the insulation section and 0.25 for the stud section since the headers that constitute a small part of the wall are to be treated as studs. Using the available \(R\)-values from Table 3–6 and calculating others, the total \(R\)-values for each section can be determined in a systematic manner in the table in this sample.

We conclude that the overall unit thermal resistance of the wall is 2.23 \(\text{m}^2 \cdot \degree\text{C}/\text{W}\), and this value accounts for the effects of the studs and headers. It corresponds to an \(R\)-value of 2.23 × 5.68 = 12.7 (or nearly \(R-13\)) in English units. Note that if there were no wood studs and headers in the wall, the overall thermal resistance would be 3.05 \(\text{m}^2 \cdot \degree\text{C}/\text{W}\), which is 37 percent greater than 2.23 \(\text{m}^2 \cdot \degree\text{C}/\text{W}\). Therefore, the wood studs and headers in this case serve as thermal bridges in wood frame walls, and their effect must be considered in the thermal analysis of buildings.
HEAT TRANSFER

**Schematic**

<table>
<thead>
<tr>
<th>Construction</th>
<th>Between Studs</th>
<th>At Studs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Outside surface, 24 km/h wind</td>
<td>0.030</td>
<td>0.030</td>
</tr>
<tr>
<td>2. Wood bevel lapped siding</td>
<td>0.14</td>
<td>0.14</td>
</tr>
<tr>
<td>3. Wood fiberboard sheeting, 13 mm</td>
<td>0.23</td>
<td>0.23</td>
</tr>
<tr>
<td>4a. Glass fiber insulation, 90 mm</td>
<td>2.45</td>
<td>—</td>
</tr>
<tr>
<td>4b. Wood stud, 38 mm × 90 mm</td>
<td>—</td>
<td>0.63</td>
</tr>
<tr>
<td>5. Gypsum wallboard, 13 mm</td>
<td>0.079</td>
<td>0.079</td>
</tr>
<tr>
<td>6. Inside surface, still air</td>
<td>0.12</td>
<td>0.12</td>
</tr>
</tbody>
</table>

**Total unit thermal resistance of each section, R (in m² · °C/W)**

3.05 1.23

**The U-factor of each section, U, in W/m² · °C**

0.328 0.813

**Area fraction of each section, f_area**

0.75 0.25

**Overall U-factor: U = ∑U_i * f_area**

0.449 W/m² · °C

**Overall unit thermal resistance:**

\[ R = 1/U = 2.23 \text{ m}^2 \cdot \text{°C/W} \]

The perimeter of the building is 50 m and the height of the walls is 2.5 m. Noting that glazing constitutes 20 percent of the walls, the total wall area is

\[ A_{wall} = 0.80(\text{Perimeter})(\text{Height}) = 0.80(50 \text{ m})(2.5 \text{ m}) = 100 \text{ m}^2 \]

Then the rate of heat loss through the walls under design conditions becomes

\[ Q_{wall} = (UA)_{wall} (T_i - T_o) \]

\[ = (0.449 \text{ W/m}^2 \cdot \text{°C})(100 \text{ m}^2)[22 - (-2)\text{°C}] \]

\[ = 1078 \text{ W} \]

**Discussion**

Note that a 1-kW resistance heater in this house will make up almost all the heat lost through the walls, except through the doors and windows, when the outdoor air temperature drops to −2°C.

---

**EXAMPLE 3–17 The R-Value of a Wall with Rigid Foam**

The 13-mm-thick wood fiberboard sheathing of the wood stud wall discussed in the previous example is replaced by a 25-mm-thick rigid foam insulation. Determine the percent increase in the R-value of the wall as a result.
SOLUTION  The overall $R$-value of the existing wall was determined in Example 3–16 to be $2.23 \text{ m}^2 \cdot \text{°C}/\text{W}$. Noting that the $R$-values of the fiberboard and the foam insulation are $0.23 \text{ m}^2 \cdot \text{°C}/\text{W}$ and $0.98 \text{ m}^2 \cdot \text{°C}/\text{W}$, respectively, and the added and removed thermal resistances are in series, the overall $R$-value of the wall after modification becomes

$$R_{\text{new}} = R_{\text{old}} - R_{\text{removed}} + R_{\text{added}}$$

$$= 2.23 - 0.23 + 0.98$$

$$= 2.98 \text{ m}^2 \cdot \text{°C}/\text{W}$$

This represents an increase of $(2.98 - 2.23)/2.23 = 0.34$ or 34 percent in the $R$-value of the wall. This example demonstrated how to evaluate the new $R$-value of a structure when some structural members are added or removed.

EXAMPLE 3–18  The $R$-Value of a Masonry Wall

Determine the overall unit thermal resistance (the $R$-value) and the overall heat transfer coefficient (the $U$-factor) of a masonry cavity wall that is built around 6-in.-thick concrete blocks made of lightweight aggregate with 3 cores filled with perlite ($R = 4.2 \text{ h} \cdot \text{ft}^2 \cdot \text{°F}/\text{Btu}$). The outside is finished with 4-in. face brick with $\frac{1}{4}$-in. cement mortar between the bricks and concrete blocks. The inside finish consists of $\frac{1}{4}$ in. gypsum wallboard separated from the concrete block by $\frac{3}{4}$-in.-thick (1-in. × 3-in. nominal) vertical furring ($R = 4.2 \text{ h} \cdot \text{ft}^2 \cdot \text{°F}/\text{Btu}$) whose center-to-center distance is 16 in. Both sides of the $\frac{3}{4}$-in.-thick air space between the concrete block and the gypsum board are coated with reflective aluminum foil ($e = 0.05$) so that the effective emissivity of the air space is 0.03. For a mean temperature of 50°F and a temperature difference of 30°F, the $R$-value of the air space is $2.91 \text{ h} \cdot \text{ft}^2 \cdot \text{°F}/\text{Btu}$. The reflective air space constitutes 80 percent of the heat transmission area, while the vertical furring constitutes 20 percent.

SOLUTION  The $R$-value and the $U$-factor of a masonry cavity wall are to be determined.

Assumptions  1 Steady operating conditions exist. 2 Heat transfer through the wall is one-dimensional. 3 Thermal properties of the wall and the heat transfer coefficients are constant.

Properties  The $R$-values of different materials are given in Table 3–6.

Analysis  The schematic of the wall as well as the different elements used in its construction are shown below. Following the approach described here and using the available $R$-values from Table 3–6, the overall $R$-value of the wall is determined in this table.
### EXAMPLE 3–19  The \( R \)-Value of a Pitched Roof

Determine the overall unit thermal resistance (the \( R \)-value) and the overall heat transfer coefficient (the \( U \)-factor) of a 45° pitched roof built around nominal 2-in. × 4-in. wood studs with a center-to-center distance of 16 in. The 3.5-in.-wide air space between the studs does not have any reflective surface and thus its effective emissivity is 0.84. For a mean temperature of 90°F and a temperature difference of 30°F, the \( R \)-value of the air space is 0.86 \( h \cdot \text{ft}^2 \cdot ^\circ \text{F}/\text{Btu} \). The lower part of the roof is finished with 1/8-in. gypsum wallboard and the upper part with 1/16-in. plywood, building paper, and asphalt shingle roofing. The air space constitutes 75 percent of the heat transmission area, while the studs and headers constitute 25 percent.

**SOLUTION** The \( R \)-value and the \( U \)-factor of a 45° pitched roof are to be determined.

**Assumptions** 1 Steady operating conditions exist. 2 Heat transfer through the roof is one-dimensional. 3 Thermal properties of the roof and the heat transfer coefficients are constant.

**Properties** The \( R \)-values of different materials are given in Table 3–6.

---

### Table 3–6: R-values

<table>
<thead>
<tr>
<th>Construction</th>
<th>Between Furring</th>
<th>At Furring</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Outside surface, 15 mph wind</td>
<td>0.17</td>
<td>0.17</td>
</tr>
<tr>
<td>2. Face brick, 4 in.</td>
<td>0.43</td>
<td>0.43</td>
</tr>
<tr>
<td>3. Cement mortar, 0.5 in.</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>4. Concrete block, 6 in.</td>
<td>4.20</td>
<td>4.20</td>
</tr>
<tr>
<td>5a. Reflective air space, ( \frac{1}{4} ) in.</td>
<td>2.91</td>
<td>—</td>
</tr>
<tr>
<td>5b. Nominal 1 × 3 vertical furring</td>
<td>—</td>
<td>0.94</td>
</tr>
<tr>
<td>6. Gypsum wallboard, 0.5 in.</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>7. Inside surface, still air</td>
<td>0.68</td>
<td>0.68</td>
</tr>
</tbody>
</table>

Total unit thermal resistance of each section, \( R \):

\[
\text{Overall} \quad R = 8.94 \quad \text{and} \quad 6.97
\]

The \( U \)-factor of each section, \( U = 1/R \), in \( \text{Btu/h} \cdot \text{ft}^2 \cdot ^\circ \text{F} \):

\[
\text{Overall} \quad U = 0.112 \quad \text{and} \quad 0.143
\]

Area fraction of each section, \( f_{\text{area}} \):

\[
\text{Overall} \quad f_{\text{area}} = 0.80 \quad \text{and} \quad 0.20
\]

Overall \( U \)-factor: \( U = \sum f_{\text{area}} \cdot U_i = 0.80 \times 0.112 + 0.20 \times 0.143 = 0.118 \text{ Btu/h} \cdot \text{ft}^2 \cdot ^\circ \text{F} \)

Overall unit thermal resistance:

\[
R = \frac{1}{U} = 8.46 \quad h \cdot \text{ft}^2 \cdot ^\circ \text{F}/\text{Btu}
\]

Therefore, the overall unit thermal resistance of the wall is 8.46 \( h \cdot \text{ft}^2 \cdot ^\circ \text{F}/\text{Btu} \) and the overall \( U \)-factor is 0.118 Btu/h · ft² · °F. These values account for the effects of the vertical furring.
The construction of wood frame flat ceilings typically involve 2-in. × 6-in. joists on 400-mm (16-in.) or 600-mm (24-in.) centers. The fraction of framing is usually taken to be 0.10 for joists on 400-mm centers and 0.07 for joists on 600-mm centers.

Most buildings have a combination of a ceiling and a roof with an attic space in between, and the determination of the $R$-value of the roof–attic–ceiling combination depends on whether the attic is vented or not. For adequately ventilated attics, the attic air temperature is practically the same as the outdoor air temperature, and thus heat transfer through the roof is governed by the $R$-value of the ceiling only. However, heat is also transferred between the roof and the ceiling by radiation, and it needs to be considered (Fig. 3–52). The major function of the roof in this case is to serve as a radiation shield by blocking off solar radiation. Effectively ventilating the attic in summer should not lead one to believe that heat gain to the building through the attic is greatly reduced. This is because most of the heat transfer through the attic is by radiation.

### Analysis

The schematic of the pitched roof as well as the different elements used in its construction are shown below. Following the approach described above and using the available $R$-values from Table 3–6, the overall $R$-value of the roof can be determined in the table here.

<table>
<thead>
<tr>
<th>Construction</th>
<th>Between Studs</th>
<th>At Studs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Outside surface, 15 mph wind</td>
<td>0.17</td>
<td>0.17</td>
</tr>
<tr>
<td>2. Asphalt shingle roofing</td>
<td>0.44</td>
<td>0.44</td>
</tr>
<tr>
<td>3. Building paper</td>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>4. Plywood deck, ⅜ in.</td>
<td>0.78</td>
<td>0.78</td>
</tr>
<tr>
<td>5a. Nonreflective air space, 3.5 in.</td>
<td>0.86</td>
<td>—</td>
</tr>
<tr>
<td>5b. Wood stud, 2 in. by 4 in.</td>
<td>—</td>
<td>3.58</td>
</tr>
<tr>
<td>6. Gypsum wallboard, 0.5 in.</td>
<td>0.45</td>
<td>0.45</td>
</tr>
<tr>
<td>7. Inside surface, 45° slope, still air</td>
<td>0.63</td>
<td>0.63</td>
</tr>
</tbody>
</table>

Total unit thermal resistance of each section, $R = 3.43 + 6.15 = 9.58$ h · ft² · °F/Btu

The $U$-factor of each section, $U = 1/R$, in Btu/h · ft² · °F

Area fraction of each section, $f_{area} = 0.75 + 0.25 = 1.00$

Overall $U$-factor: $U = \sum f_{area} \cdot U_i = 0.75 \times 0.292 + 0.25 \times 0.163 = 0.260$ Btu/h · ft² · °F

Overall unit thermal resistance: $R = 1/U = 3.85$ h · ft² · °F/Btu

Therefore, the overall unit thermal resistance of this pitched roof is 3.85 h · ft² · °F/Btu and the overall $U$-factor is 0.260 Btu/h · ft² · °F. Note that the wood studs offer much larger thermal resistance to heat flow than the air space between the studs.
Radiation heat transfer between the ceiling and the roof can be minimized by covering at least one side of the attic (the roof or the ceiling side) by a reflective material, called radiant barrier, such as aluminum foil or aluminum-coated paper. Tests on houses with R-19 attic floor insulation have shown that radiant barriers can reduce summer ceiling heat gains by 16 to 42 percent compared to an attic with the same insulation level and no radiant barrier. Considering that the ceiling heat gain represents about 15 to 25 percent of the total cooling load of a house, radiant barriers will reduce the air conditioning costs by 2 to 10 percent. Radiant barriers also reduce the heat loss in winter through the ceiling, but tests have shown that the percentage reduction in heat losses is less. As a result, the percentage reduction in heating costs will be less than the reduction in the air-conditioning costs. Also, the values given are for new and undusted radiant barrier installations, and percentages will be lower for aged or dusty radiant barriers.

Some possible locations for attic radiant barriers are given in Figure 3–53. In whole house tests on houses with R-19 attic floor insulation, radiant barriers have reduced the ceiling heat gain by an average of 35 percent when the radiant barrier is installed on the attic floor, and by 24 percent when it is attached to the bottom of roof rafters. Test cell tests also demonstrated that the best location for radiant barriers is the attic floor, provided that the attic is not used as a storage area and is kept clean.

For unvented attics, any heat transfer must occur through (1) the ceiling, (2) the attic space, and (3) the roof (Fig. 3–54). Therefore, the overall R-value of the roof–ceiling combination with an unvented attic depends on the combined effects of the R-value of the ceiling and the R-value of the roof as well as the thermal resistance of the attic space. The attic space can be treated as an air layer in the analysis. But a more practical way of accounting for its effect is to consider surface resistances on the roof and ceiling surfaces facing each other. In this case, the R-values of the ceiling and the roof are first determined separately (by using convection resistances for the still-air case for the attic surfaces). Then it can be shown that the overall R-value of the ceiling–roof combination per unit area of the ceiling can be expressed as
where $A_{\text{ceiling}}$ and $A_{\text{roof}}$ are the ceiling and roof areas, respectively. The area ratio is equal to 1 for flat roofs and is less than 1 for pitched roofs. For a 45° pitched roof, the area ratio is $A_{\text{ceiling}}/A_{\text{roof}} = 1/\sqrt{2} = 0.707$. Note that the pitched roof has a greater area for heat transfer than the flat ceiling, and the area ratio accounts for the reduction in the unit $R$-value of the roof when expressed per unit area of the ceiling. Also, the direction of heat flow is up in winter (heat loss through the roof) and down in summer (heat gain through the roof).

The $R$-value of a structure determined by analysis assumes that the materials used and the quality of workmanship meet the standards. Poor workmanship and substandard materials used during construction may result in $R$-values that deviate from predicted values. Therefore, some engineers use a safety factor in their designs based on experience in critical applications.

**SUMMARY**

One-dimensional heat transfer through a simple or composite body exposed to convection from both sides to mediums at temperatures $T_{s1}$ and $T_{s2}$ can be expressed as

$$Q = \frac{T_{s1} - T_{s2}}{R_{\text{total}}},$$

where $R_{\text{total}}$ is the total thermal resistance between the two mediums. For a plane wall exposed to convection on both sides, the total resistance is expressed as

$$R_{\text{total}} = R_{\text{conv},1} + R_{\text{wall}} + R_{\text{conv},2} = \frac{1}{h_1 A} + \frac{L}{k A} + \frac{1}{h_2 A}$$

This relation can be extended to plane walls that consist of two or more layers by adding an additional resistance for each additional layer. The elementary thermal resistance relations can be expressed as follows:

**Conduction resistance (plane wall):**

$$R_{\text{wall}} = \frac{L}{k A}$$

**Conduction resistance (cylinder):**

$$R_{\text{cyl}} = \frac{\ln(r_2/r_1)}{2\pi L k}$$

**Conduction resistance (sphere):**

$$R_{\text{sph}} = \frac{r_2 - r_1}{4\pi r_1 r_2 k}$$

**Convection resistance:**

$$R_{\text{conv}} = \frac{1}{h A}$$

**Interface resistance:**

$$R_{\text{interface}} = \frac{1}{h A} = \frac{R_s}{A}$$

**Radiation resistance:**

$$R_{\text{rad}} = \frac{1}{h A}$$

where $h$ is the thermal contact conductance, $R_s$ is the thermal contact resistance, and the radiation heat transfer coefficient is defined as

$$h_{\text{rad}} = e\sigma(T_s^4 + T_{\text{surr}}^4)(T_s + T_{\text{surr}})$$

Once the rate of heat transfer is available, the temperature drop across any layer can be determined from

$$\Delta T = \frac{Q}{R}$$

The thermal resistance concept can also be used to solve steady heat transfer problems involving parallel layers or combined series-parallel arrangements.

Adding insulation to a cylindrical pipe or a spherical shell will increase the rate of heat transfer if the outer radius of the insulation is less than the critical radius of insulation, defined as

$$r_{\text{cr, cylinder}} = \frac{k_{\text{ins}}}{h}$$

$$r_{\text{cr, sphere}} = \frac{2k_{\text{ins}}}{h}$$

The effectiveness of an insulation is often given in terms of its $R$-value, the thermal resistance of the material per unit surface area, expressed as

$$R\text{-value} = \frac{L}{k}$$

(flatt insulation)

where $L$ is the thickness and $k$ is the thermal conductivity of the material.
Finned surfaces are commonly used in practice to enhance heat transfer. Fins enhance heat transfer from a surface by exposing a larger surface area to convection. The temperature distribution along the fin for very long fins and for fins with negligible heat transfer at the fin are given by

\[
\text{Very long fin: } \quad \frac{T(x) - T_\infty}{T_b - T_\infty} = e^{-x \sqrt{\frac{h_p}{k_A}}},
\]

\[
\text{Adiabatic fin tip: } \quad \frac{T(x) - T_\infty}{T_b - T_\infty} = \cosh \left( \frac{a(L - x)}{L} \right)
\]

where \( a = \sqrt{\frac{h_p}{k_A}} \), \( p \) is the perimeter, and \( A \) is the cross sectional area of the fin. The rates of heat transfer for both cases are given to be

\[
\text{Very long fin: } \quad \dot{Q}_{\text{long fin}} = -kA_b \frac{dT}{dx} \bigg|_{x = 0} = \sqrt{h_p k_A} (T_b - T_\infty)
\]

\[
\text{Adiabatic fin tip: } \quad \dot{Q}_{\text{adiabatic tip}} = -kA_b \frac{dT}{dx} \bigg|_{x = 0} = \sqrt{h_p k_A} (T_b - T_\infty) \tanh aL
\]

Fins exposed to convection at their tips can be treated as fins with insulated tips by using the corrected length \( L_c = L + A/p \) instead of the actual fin length.

The temperature of a fin drops along the fin, and thus the heat transfer from the fin will be less because of the decreasing temperature difference toward the fin tip. To account for the effect of this decrease in temperature on heat transfer, we define fin efficiency as

\[
\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{fin, max}}} = \frac{\text{Actual heat transfer rate from the fin}}{\text{Ideal heat transfer rate from the fin if the entire fin were at base temperature}}
\]

When the fin efficiency is available, the rate of heat transfer from a fin can be determined from

\[
\dot{Q}_{\text{fin}} = \eta_{\text{fin}} \dot{Q}_{\text{fin, max}} = \eta_{\text{fin}} hA_b (T_b - T_\infty)
\]

The performance of the fins is judged on the basis of the enhancement in heat transfer relative to the no-fin case and is expressed in terms of the fin effectiveness \( \varepsilon_{\text{fin}} \) defined as

\[
\varepsilon_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{no fin}}} \quad \text{Heat transfer rate from the fin of base area } A_b
\]

\[
\varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} \quad \text{Heat transfer rate from the surface of area } A_b
\]

Here, \( A_b \) is the cross-sectional area of the fin at the base and \( \dot{Q}_{\text{no fin}} \) represents the rate of heat transfer from this area if no fins are attached to the surface. The overall effectiveness for a finned surface is defined as the ratio of the total heat transfer from the finned surface to the heat transfer from the same surface if there were no fins,

\[
\varepsilon_{\text{fin, overall}} = \frac{\dot{Q}_{\text{total, fin}}}{\dot{Q}_{\text{total, no fin}}} = \frac{h(A_{\text{fin}} + \eta_{\text{fin}} A_b)(T_b - T_\infty)}{hA_{\text{no fin}} (T_b - T_\infty)}
\]

Fin efficiency and fin effectiveness are related to each other by

\[
\varepsilon_{\text{fin}} = \frac{A_{\text{fin}}}{A_b} \eta_{\text{fin}}
\]

Certain multidimensional heat transfer problems involve two surfaces maintained at constant temperatures \( T_1 \) and \( T_2 \). The steady rate of heat transfer between these two surfaces is expressed as

\[
\dot{Q} = Sk(T_1 - T_2)
\]

where \( S \) is the conduction shape factor that has the dimension of length and \( k \) is the thermal conductivity of the medium between the surfaces.

References and Suggested Reading


**PROBLEMS**

### Steady Heat Conduction in Plane Walls

3–1C Consider one-dimensional heat conduction through a cylindrical rod of diameter $D$ and length $L$. What is the heat transfer area of the rod if (a) the lateral surfaces of the rod are insulated and (b) the top and bottom surfaces of the rod are insulated?

3–2C Consider heat conduction through a plane wall. Does the energy content of the wall change during steady heat conduction? How about during transient conduction? Explain.

3–3C Consider steady one-dimensional heat transfer through a wall of thickness $L$ and area $A$. Under what conditions will the temperature distributions in the wall be a straight line?

3–4C What does the thermal resistance of a medium represent?

3–5C How is the combined heat transfer coefficient defined? What convenience does it offer in heat transfer calculations?

3–6C Can we define the convection resistance per unit surface area as the inverse of the convection heat transfer coefficient?

3–7C Why are the convection and the radiation resistances at a surface in parallel instead of being in series?

3–8C Consider a surface of area $A$ at which the convection and radiation heat transfer coefficients are $h_{con}$ and $h_{rad}$, respectively. Explain how you would determine (a) the single equivalent heat transfer coefficient, and (b) the equivalent thermal resistance. Assume the medium and the surrounding surfaces are at the same temperature.

3–9C How does the thermal resistance network associated with a single-layer plane wall differ from the one associated with a five-layer composite wall?

3–10C Consider steady one-dimensional heat transfer through a multilayer medium. If the rate of heat transfer $\dot{Q}$ is known, explain how you would determine the temperature drop across each layer.

3–11C Consider steady one-dimensional heat transfer through a plane wall exposed to convection from both sides to environments at known temperatures $T_{a1}$ and $T_{a2}$ with known heat transfer coefficients $h_1$ and $h_2$. Once the rate of heat transfer $\dot{Q}$ has been evaluated, explain how you would determine the temperature of each surface.

3–12C Someone comments that a microwave oven can be viewed as a conventional oven with zero convection resistance at the surface of the food. Is this an accurate statement?

3–13C Consider a window glass consisting of two 4-mm-thick glass sheets pressed tightly against each other. Compare the heat transfer rate through this window with that of one consisting of a single 8-mm-thick glass sheet under identical conditions.

3–14C Consider steady heat transfer through the wall of a room in winter. The convection heat transfer coefficient at the outer surface of the wall is three times that of the inner surface as a result of the winds. On which surface of the wall do you think the temperature will be closer to the surrounding air temperature? Explain.

3–15C The bottom of a pan is made of a 4-mm-thick aluminum layer. In order to increase the rate of heat transfer through the bottom of the pan, someone proposes a design for the bottom that consists of a 3-mm-thick copper layer sandwiched between two 2-mm-thick aluminum layers. Will the new design conduct heat better? Explain. Assume perfect contact between the layers.

3–16C Consider two cold canned drinks, one wrapped in a blanket and the other placed on a table in the same room. Which drink will warm up faster?

3–17 Consider a 4-m-high, 6-m-wide, and 0.3-m-thick brick wall whose thermal conductivity is $k = 0.8 \text{ W/m} \cdot \text{°C}$. On a certain day, the temperatures of the inner and the outer surfaces of the wall are measured to be $14^\circ\text{C}$ and $6^\circ\text{C}$, respectively. Determine the rate of heat loss through the wall on that day.

---

*Problems designated by a “C” are concept questions, and students are encouraged to answer them all. Problems designated by an “E” are in English units, and the SI users can ignore them. Problems with an EES-CD icon are solved using EES, and complete solutions together with parametric studies are included on the enclosed CD. Problems with a computer-EES icon are comprehensive in nature, and are intended to be solved with a computer, preferably using the EES software that accompanies this text.*
3–18 Consider a 1.2-m-high and 2-m-wide glass window whose thickness is 6 mm and thermal conductivity is \( k = 0.78 \text{ W/m} \cdot ^\circ \text{C} \). Determine the steady rate of heat transfer through this glass window and the temperature of its inner surface for a day during which the room is maintained at 24\(^\circ\)C while the temperature of the outdoors is \(-5\)^\circC. Take the convection heat transfer coefficients on the inner and outer surfaces of the window to be \( h_1 = 10 \text{ W/m}^2 \cdot ^\circ\text{C} \) and \( h_2 = 25 \text{ W/m}^2 \cdot ^\circ\text{C} \), and disregard any heat transfer by radiation.

**Answers:** 114 W, 19.2\(^\circ\)C

3–19 Consider a 1.2-m-high and 2-m-wide double-pane window consisting of two 3-mm-thick layers of glass \( (k = 0.78 \text{ W/m} \cdot ^\circ \text{C}) \) separated by a 12-mm-wide stagnant air space \( (k = 0.026 \text{ W/m} \cdot ^\circ \text{C}) \). Determine the steady rate of heat transfer through this double-pane window and the temperature of its inner surface for a day during which the room is maintained at 24\(^\circ\)C while the temperature of the outdoors is \(-5\)^\circC. Take the convection heat transfer coefficients on the inner and outer surfaces of the window to be \( h_1 = 10 \text{ W/m}^2 \cdot ^\circ\text{C} \) and \( h_2 = 25 \text{ W/m}^2 \cdot ^\circ\text{C} \), and disregard any heat transfer by radiation.

3–20 Repeat Problem 3–19, assuming the space between the two glass layers is evacuated.

3–21 Reconsider Problem 3–19. Using EES (or other) software, plot the rate of heat transfer through the window as a function of the width of air space in the range of 2 mm to 20 mm, assuming pure conduction through the air. Discuss the results.

3–22E Consider an electrically heated brick house \( (k = 0.40 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ\text{F}) \) whose walls are 9 ft high and 1 ft thick. Two of the walls of the house are 40 ft long and the others are 30 ft long. The house is maintained at 70°F at all times while the temperature of the outdoors varies. On a certain day, the temperature of the inner surface of the walls is measured to be at 55°F while the average temperature of the outer surface is observed to remain at 45°F during the day for 10 h and at 35°F at night for 14 h. Determine the amount of heat lost from the house that day. Also determine the cost of that heat loss to the homeowner for an electricity price of $0.09/kWh.

3–23 A cylindrical resistor element on a circuit board dissipates 0.15 W of power in an environment at 40°C. The resistor is 1.2 cm long, and has a diameter of 0.3 cm. Assuming heat to be transferred uniformly from all surfaces, determine \( a \) the amount of heat this resistor dissipates during a 24-h period, \( b \) the heat flux on the surface of the resistor, in W/m², and \( c \) the surface temperature of the resistor for a combined convection and radiation heat transfer coefficient of 9 W/m² \cdot ^\circ\text{C}.

3–24 Consider a power transistor that dissipates 0.2 W of power in an environment at 30°C. The transistor is 0.4 cm long and has a diameter of 0.5 cm. Assuming heat to be transferred uniformly from all surfaces, determine \( a \) the amount of heat this resistor dissipates during a 24-h period, in kWh; \( b \) the heat flux on the surface of the transistor, in W/m²; and \( c \) the surface temperature of the resistor for a combined convection and radiation heat transfer coefficient of 18 W/m² \cdot ^\circ\text{C}.

3–25 A 12-cm \times 18-cm circuit board houses on its surface 100 closely spaced logic chips, each dissipating 0.07 W in an environment at 40°C. The heat transfer from the back surface of the board is negligible. If the heat transfer coefficient on the
surface of the board is 10 W/m²·°C, determine (a) the heat flux on the surface of the circuit board, in W/m²; (b) the surface temperature of the chips; and (c) the thermal resistance between the surface of the circuit board and the cooling medium, in °C/W.

3–26 Consider a person standing in a room at 20°C with an exposed surface area of 1.7 m². The deep body temperature of the human body is 37°C, and the thermal conductivity of the human tissue near the skin is about 0.3 W/m·°C. The body is losing heat at a rate of 150 W by natural convection and radiation to the surroundings. Taking the body temperature 0.5 cm beneath the skin to be 37°C, determine the skin temperature of the person. Answer: 35.5°C

3–27 Water is boiling in a 25-cm-diameter aluminum pan (k = 237 W/m·°C) at 95°C. Heat is transferred steadily to the boiling water in the pan through its 0.5-cm-thick flat bottom at a rate of 800 W. If the inner surface temperature of the bottom of the pan is 108°C, determine (a) the boiling heat transfer coefficient on the inner surface of the pan, and (b) the outer surface temperature of the bottom of the pan.

3–28E A wall is constructed of two layers of 0.5-in-thick sheetrock (k = 0.10 Btu/h·ft·°F), which is a plasterboard made of two layers of heavy paper separated by a layer of gypsum, placed 5 in. apart. The space between the sheetrocks is filled with fiberglass insulation (k = 0.020 Btu/h·ft·°F). Determine (a) the thermal resistance of the wall, and (b) its R-value of insulation in English units.

3–29 The roof of a house consists of a 3-cm-thick concrete slab (k = 2 W/m·°C) that is 15 m wide and 20 m long. The convection heat transfer coefficients on the inner and outer surfaces of the roof are 5 and 12 W/m²·°C, respectively. On a clear winter night, the ambient air is reported to be at 10°C, while the night sky temperature is 100 K. The house and the interior surfaces of the wall are maintained at a constant temperature of 20°C. The emissivity of both surfaces of the concrete roof is 0.9. Considering both radiation and convection heat transfers, determine the rate of heat transfer through the roof, and the inner surface temperature of the roof.

If the house is heated by a furnace burning natural gas with an efficiency of 80 percent, and the price of natural gas is $0.60/therm (1 therm = 105,500 kJ of energy content), determine the money lost through the roof that night during a 14-h period.

3–30 A 2-m × 1.5-m section of wall of an industrial furnace burning natural gas is not insulated, and the temperature at the outer surface of this section is measured to be 80°C. The temperature of the furnace room is 30°C, and the combined convection and radiation heat transfer coefficient at the surface of the outer furnace is 10 W/m²·°C. It is proposed to insulate this section of the furnace wall with glass wool insulation (k = 0.038 W/m·°C) in order to reduce the heat loss by 90 percent. Assuming the outer surface temperature of the metal section still remains at about 80°C, determine the thickness of the insulation that needs to be used.

The furnace operates continuously and has an efficiency of 78 percent. The price of the natural gas is $0.55/therm (1 therm = 105,500 kJ of energy content). If the installation of the insulation will cost $250 for materials and labor, determine how long it will take for the insulation to pay for itself from the energy it saves.

3–31 Repeat Problem. 3–30 for expanded perlite insulation assuming conductivity is k = 0.052 W/m·°C.
3–32 Reconsider Problem 3–30. Using EES (or other) software, investigate the effect of thermal conductivity on the required insulation thickness. Plot the thickness of insulation as a function of the thermal conductivity of the insulation in the range of 0.02 W/m · °C to 0.08 W/m · °C, and discuss the results.

3–33E Consider a house whose walls are 12 ft high and 40 ft long. Two of the walls of the house have no windows, while each of the other two walls has four windows made of 0.25-in.-thick glass (\(k = 0.45\) Btu/h · ft · °F), 3 ft × 5 ft in size. The walls are certified to have an R-value of 19 (i.e., an \(L/k\) value of 19 h · ft² · °F/Btu). Disregarding any direct radiation gain or loss through the windows and taking the heat transfer coefficients at the inner and outer surfaces of the house to be 2 and 4 Btu/h · ft² · °F, respectively, determine the ratio of the heat transfer through the walls with and without windows.

3–34 Consider a house that has a 10-m × 20-m base and a 4-m-high wall. All four walls of the house have an R-value of 2.31 m² · °C/W. The two 10-m × 4-m walls have no windows. The third wall has five windows made of 0.5-cm-thick glass (\(k = 0.78\) W/m · °C), 1.2 m × 1.8 m in size. The fourth wall has the same size and number of windows, but they are double-paneled with a 1.5-cm-thick stagnant air space (\(k = 0.026\) W/m · °C) enclosed between two 0.5-cm-thick glass layers. The thermostat in the house is set at 22°C and the average temperature outside at that location is 8°C during the seven-month-long heating season. Disregarding any direct radiation gain or loss through the windows and taking the heat transfer coefficients at the inner and outer surfaces of the house to be 7 and 15 W/m² · °C, respectively, determine the average rate of heat transfer through each wall.

If the house is electrically heated and the price of electricity is $0.08/kWh, determine the amount of money this household will save per heating season by converting the single-pane windows to double-pane windows.

3–35 The wall of a refrigerator is constructed of fiberglass insulation (\(k = 0.035\) W/m · °C) sandwiched between two layers of 1-mm-thick sheet metal (\(k = 15.1\) W/m · °C). The refrigerated space is maintained at 3°C, and the average heat transfer coefficients at the inner and outer surfaces of the wall are 4 W/m² · °C and 9 W/m² · °C, respectively. The kitchen temperature averages 25°C. It is observed that condensation occurs on the outer surfaces of the refrigerator when the temperature of the outer surface drops to 20°C. Determine the minimum thickness of fiberglass insulation that needs to be used in the wall in order to avoid condensation on the outer surfaces.

3–36 Reconsider Problem 3–35. Using EES (or other) software, investigate the effects of the thermal conductivities of the insulation material and the sheet metal on the thickness of the insulation. Let the thermal conductivity vary from 0.02 W/m · °C to 0.08 W/m · °C for insulation and 10 W/m · °C to 400 W/m · °C for sheet metal. Plot the thickness of the insulation as the functions of the thermal conductivities of the insulation and the sheet metal, and discuss the results.

3–37 Heat is to be conducted along a circuit board that has a copper layer on one side. The circuit board is 15 cm long and 15 cm wide, and the thicknesses of the copper and epoxy layers are 0.1 mm and 1.2 mm, respectively. Disregarding heat transfer from side surfaces, determine the percentages of heat conduction along the copper (\(k = 386\) W/m · °C) and epoxy (\(k = 0.26\) W/m · °C) layers. Also determine the effective thermal conductivity of the board.

Answers: 0.8 percent, 99.2 percent, and 29.9 W/m · °C

3–38E A 0.03-in-thick copper plate (\(k = 223\) Btu/h · ft · °F) is sandwiched between two 0.1-in-thick epoxy boards (\(k = 0.15\) Btu/h · ft · °F) that are 7 in. × 9 in. in size. Determine the effective thermal conductivity of the board along its 9-in.-long side. What fraction of the heat conducted along that side is conducted through copper?

Thermal Contact Resistance

3–39C What is thermal contact resistance? How is it related to thermal contact conductance?

3–40C Will the thermal contact resistance be greater for smooth or rough plain surfaces?
3–41C A wall consists of two layers of insulation pressed against each other. Do we need to be concerned about the thermal contact resistance at the interface in a heat transfer analysis or can we just ignore it?

3–42C A plate consists of two thin metal layers pressed against each other. Do we need to be concerned about the thermal contact resistance at the interface in a heat transfer analysis or can we just ignore it?

3–43C Consider two surfaces pressed against each other. Now the air at the interface is evacuated. Will the thermal contact resistance at the interface increase or decrease as a result?

3–44C Explain how the thermal contact resistance can be minimized.

3–45 The thermal contact conductance at the interface of two 1-cm-thick copper plates is measured to be 18,000 W/m² °C. Determine the thickness of the copper plate whose thermal resistance is equal to the thermal resistance of the interface between the plates.

3–46 Six identical power transistors with aluminum casing are attached on one side of a 1.2-cm-thick 20-cm × 30-cm copper plate (k = 386 W/m · °C) by screws that exert an average pressure of 10 MPa. The base area of each transistor is 9 cm², and each transistor is placed at the center of a 10-cm × 10-cm section of the plate. The interface roughness is estimated to be about 1.4 μm. All transistors are covered by a thick Plexiglas layer, which is a poor conductor of heat, and thus all the heat generated at the junction of the transistor must be dissipated to the ambient at 15°C through the back surface of the copper plate. The combined convection/radiation heat transfer coefficient at the back surface can be taken to be 30 W/m² °C. If the case temperature of the transistor is not to exceed 85°C, determine the maximum power each transistor can dissipate safely, and the temperature jump at the case-plate interface.

3–47 Two 5-cm-diameter, 15–cm-long aluminum bars (k = 176 W/m · °C) with ground surfaces are pressed against each other with a pressure of 20 atm. The bars are enclosed in an insulation sleeve and, thus, heat transfer from the lateral surfaces is negligible. If the top and bottom surfaces of the two-bar system are maintained at temperatures of 150°C and 20°C, respectively, determine (a) the rate of heat transfer along the cylinders under steady conditions and (b) the temperature drop at the interface.

Answers: (a) 142.4 W, (b) 6.4°C

3–48 A 1-mm-thick copper plate (k = 386 W/m · °C) is sandwiched between two 5-mm-thick epoxy boards (k = 0.26 W/m · °C) that are 15 cm × 20 cm in size. If the thermal conductance on both sides of the copper plate is estimated to be 6000 W/m · °C, determine the error involved in the total thermal resistance of the plate if the thermal contact conductances are ignored.
Generalized Thermal Resistance Networks

3–49C When plotting the thermal resistance network associated with a heat transfer problem, explain when two resistances are in series and when they are in parallel.

3–50C The thermal resistance networks can also be used approximately for multidimensional problems. For what kind of multidimensional problems will the thermal resistance approach give adequate results?

3–51C What are the two approaches used in the development of the thermal resistance network for two-dimensional problems?

3–52 A 4-m-high and 6-m-wide wall consists of a long 18-cm × 30-cm cross section of horizontal bricks (k = 0.72 W/m · °C) separated by 3-cm-thick plaster layers (k = 0.22 W/m · °C). There are also 2-cm-thick plaster layers on each side of the wall, and a 2-cm-thick rigid foam (k = 0.026 W/m · °C) on the inner side of the wall. The indoor and the outdoor temperatures are 22°C and −4°C, and the convection heat transfer coefficients on the inner and the outer sides are \( h_1 = 10 \text{ W/m}^2 \text{ · °C} \) and \( h_2 = 20 \text{ W/m}^2 \text{ · °C} \), respectively. Assuming one-dimensional heat transfer and disregarding radiation, determine the rate of heat transfer through the wall.

3–53 Using EES (or other) software, plot the rate of heat transfer through the wall as a function of the thickness of the rigid foam in the range of 1 cm to 10 cm. Discuss the results.

3–54 A 10-cm-thick wall is to be constructed with 2.5-m-long wood studs (k = 0.11 W/m · °C) that have a cross section of 10 cm × 10 cm. At some point the builder ran out of those studs and started using pairs of 2.5-m-long wood studs that have a cross section of 5 cm × 10 cm nailed to each other instead. The manganese steel nails (k = 50 W/m · °C) are 10 cm long and have a diameter of 0.4 cm. A total of 50 nails are used to connect the two studs, which are mounted to the wall such that the nails cross the wall. The temperature difference between the inner and outer surfaces of the wall is 8°C. Assuming the thermal contact resistance between the two layers to be negligible, determine the rate of heat transfer (a) through a solid stud and (b) through a stud pair of equal length and width nailed to each other. (c) Also determine the effective conductivity of the nailed stud pair.

3–55 A 12-m-long and 5-m-high wall is constructed of two layers of 1-cm-thick sheetrock (k = 0.17 W/m · °C) spaced 12 cm by wood studs (k = 0.11 W/m · °C) whose cross section is 12 cm × 5 cm. The studs are placed vertically 60 cm apart, and the space between them is filled with fiberglass insulation (k = 0.034 W/m · °C). The house is maintained at 20°C and the ambient temperature outside is −5°C. Taking the heat transfer coefficients at the inner and outer surfaces of the house to be 8.3 and 34 W/m² · °C, respectively, determine (a) the thermal resistance of the wall considering a representative section of it and (b) the rate of heat transfer through the wall.

3–56E A 10-in.-thick, 30-ft-long, and 10-ft-high wall is to be constructed using 9-in.-long solid bricks (k = 0.40 Btu/h · ft · °F) of cross section 7 in. × 7 in., or identical size bricks with nine square air holes (k = 0.015 Btu/h · ft · °F) that are 9 in. long and have a cross section of 1.5 in. × 1.5 in. There is a 0.5-in.-thick plaster layer (k = 0.10 Btu/h · ft · °F) between two adjacent bricks on all four sides and on both sides of the wall. The house is maintained at 80°F and the ambient temperature outside is 30°F. Taking the heat transfer coefficients at the inner and outer surfaces of the wall to be 1.5 and 4 Btu/h · ft² · °F, respectively, determine the rate of heat transfer through the wall constructed of (a) solid bricks and (b) bricks with air holes.
3–57 Consider a 5-m-high, 8-m-long, and 0.22-m-thick wall whose representative cross section is as given in the figure. The thermal conductivities of various materials used, in W/m · °C, are $k_A = k_F = 2$, $k_B = 8$, $k_C = 20$, $k_D = 15$, and $k_E = 35$. The left and right surfaces of the wall are maintained at uniform temperatures of 300°C and 100°C, respectively. Assuming heat transfer through the wall to be one-dimensional, determine (a) the rate of heat transfer through the wall; (b) the temperature at the point where the sections B, D, and E meet; and (c) the temperature drop across the section F. Disregard any contact resistances at the interfaces.

![Figure P3–57](image)

3–58 Repeat Problem 3–57 assuming that the thermal contact resistance at the interfaces D-F and E-F is 0.00012 m² · °C/W.

3–59 Clothing made of several thin layers of fabric with trapped air in between, often called ski clothing, is commonly used in cold climates because it is light, fashionable, and a very effective thermal insulator. So it is no surprise that such clothing has largely replaced thick and heavy old-fashioned coats.

Consider a jacket made of five layers of 0.1-mm-thick synthetic fabric $(k = 0.13$ W/m · °C) with 1.5-mm-thick air space $(k = 0.026$ W/m · °C) between the layers. Assuming the inner surface temperature of the jacket to be 28°C and the surface area to be 1.1 m², determine the rate of heat loss through the jacket when the temperature of the outdoors is −5°C and the heat transfer coefficient at the outer surface is 25 W/m² · °C.

What would your response be if the jacket is made of a single layer of 0.5-mm-thick synthetic fabric? What should be the thickness of a wool fabric $(k = 0.035$ W/m · °C) if the person is to achieve the same level of thermal comfort wearing a thick wool coat instead of a five-layer ski jacket?

3–60 Repeat Problem 3–59 assuming the layers of the jacket are made of cotton fabric $(k = 0.06$ W/m · °C).

3–61 A 5-m-wide, 4-m-high, and 40-m-long kiln used to cure concrete pipes is made of 20-cm-thick concrete walls and ceiling $(k = 0.9$ W/m · °C). The kiln is maintained at 40°C by injecting hot steam into it. The two ends of the kiln, 4 m × 5 m in size, are made of a 3-mm-thick sheet metal covered with 2-cm-thick Styrofoam $(k = 0.033$ W/m · °C). The convection heat transfer coefficients on the inner and the outer surfaces of the kiln are 3000 W/m² · °C and 25 W/m² · °C, respectively. Disregarding any heat loss through the floor, determine the rate of heat loss from the kiln when the ambient air is at −4°C.

![Figure P3–61](image)

3–62 Reconsider Problem 3–61. Using EES (or other) software, investigate the effects of the thickness of the walls and the convection heat transfer coefficient on the outer surface of the rate of heat loss from the kiln. Let the thickness vary from 10 cm to 30 cm and the convection heat transfer coefficient from 5 W/m² · °C to 50 W/m² · °C. Plot the rate of heat transfer as functions of wall thickness and the convection heat transfer coefficient, and discuss the results.

3–63E Consider a 6-in. × 8-in. epoxy glass laminate $(k = 0.10$ Btuꞏh⁻¹ꞏft⁻¹ꞏ°F) whose thickness is 0.05 in. In order to reduce the thermal resistance across its thickness, cylindrical copper fillings $(k = 223$ Btuꞏh⁻¹ꞏft⁻¹ꞏ°F) of 0.02 in. diameter are to be planted throughout the board, with a center-to-center distance of 0.06 in. Determine the new value of the thermal resistance of the epoxy board for heat conduction across its thickness as a result of this modification.

Answer: 0.00064 h · °F/Btu
Heat Conduction in Cylinders and Spheres

3–64C What is an infinitely long cylinder? When is it proper to treat an actual cylinder as being infinitely long, and when is it not?

3–65C Consider a short cylinder whose top and bottom sur-
faces are insulated. The cylinder is initially at a uniform tem-
perature \( T_i \) and is subjected to convection from its side surface to a medium at temperature \( T_H \) with a heat transfer coefficient of \( h \).

Is the heat transfer in this short cylinder one- or two-
dimensional? Explain.

3–66C Can the thermal resistance concept be used for a solid cylinder or sphere in steady operation? Explain.

3–67 A 5-m-internal-diameter spherical tank made of 1.5-cm-thick stainless steel \( (k = 15 \text{ W/m} \cdot \text{°C}) \) is used to store iced water at 0°C. The tank is located in a room whose temperature is 30°C. The walls of the room are also at 30°C. The outer surface of the tank is black (emissivity \( \varepsilon = 1 \)), and heat transfer between the outer surface of the tank and the surroundings is by natural convection and radiation. The convection heat transfer coefficients at the inner and the outer surfaces of the tank are 80 W/m² · °C and 10 W/m² · °C, respectively. Determine \( (a) \) the rate of heat transfer to the iced water in the tank and \( (b) \) the amount of ice at 0°C that melts during a 24-h period. The heat of fusion of water at atmospheric pressure is \( h_f = 333.7 \text{ kJ/kg} \).

3–68 Steam at 320°C flows in a stainless steel pipe \( (k = 15 \text{ W/m} \cdot \text{°C}) \) whose inner and outer diameters are 5 cm and 5.5 cm, respectively. The pipe is covered with 3-cm-thick glass wool insulation \( (k = 0.038 \text{ W/m} \cdot \text{°C}) \). Heat is lost to the sur-
roundings at 15°C by natural convection and radiation, with a combined natural convection and radiation heat transfer co-
efficient of 15 W/m² · °C. Taking the heat transfer coefficient inside the pipe to be 80 W/m² · °C, determine the rate of heat loss from the steam per unit length of the pipe. Also determine the temperature drops across the pipe shell and the insulation.

3–69 Reconsider Problem 3–68. Using EES (or other) software, investigate the effect of the thickness of the insulation on the rate of heat loss from the steam and the temperature drop across the insulation layer. Let the insulation thickness vary from 1 cm to 10 cm. Plot the rate of heat loss and the temperature drop as a function of insulation thickness, and discuss the results.

3–70 A 50-m-long section of a steam pipe whose outer diameter is 10 cm passes through an open space at 15°C. The average temperature of the outer surface of the pipe is measured to be 150°C. If the combined heat transfer co-
efficient on the outer surface of the pipe is 20 W/m² · °C, de-
termine \( (a) \) the rate of heat loss from the steam pipe, \( (b) \) the annual cost of this energy lost if steam is generated in a natural gas furnace that has an efficiency of 75 percent and the price of natural gas is $0.52/therm \( (1 \text{ therm} = 105,500 \text{ kJ}) \), and \( (c) \) the thickness of fiberglass insulation \( (k = 0.035 \text{ W/m} \cdot \text{°C}) \) needed in order to save 90 percent of the heat lost. Assume the pipe temperature to remain constant at 150°C.
27°C, and the heat transfer coefficients on the inner and outer surfaces of the heater are 50 and 12 W/m\(^2\)\(\cdot\)°C, respectively. The tank is placed in another 46-cm-diameter sheet metal tank of negligible thickness, and the space between the two tanks is filled with foam insulation \(k = 0.03\) W/m\(\cdot\)°C. The thermal resistances of the water tank and the outer thin sheet metal shell are very small and can be neglected. The price of electricity is $0.08/kWh, and the home owner pays $280 a year for water heating. Determine the fraction of the hot water energy cost of this household that is due to the heat loss from the tank.

Hot water tank insulation kits consisting of 3-cm-thick fiberglass insulation \(k = 0.035\) W/m\(\cdot\)°C) large enough to wrap the entire tank are available in the market for about $30. If such an insulation is installed on this water tank by the home owner himself, how long will it take for this additional insulation to pay for itself? **Answers:** 17.5 percent, 1.5 years

3–72 **Reconsider** Problem 3–71. Using EES (or other) software, plot the fraction of energy cost of hot water due to the heat loss from the tank as a function of the hot water temperature in the range of 40°C to 90°C. Discuss the results.

3–73 Consider a cold aluminum canned drink that is initially at a uniform temperature of 3°C. The can is 12.5 cm high and has a diameter of 6 cm. If the combined convection/radiation heat transfer coefficient between the can and the surrounding air at 25°C is 10 W/m\(^2\)\(\cdot\)°C, determine how long it will take for the average temperature of the drink to rise to 10°C.

In an effort to slow down the warming of the cold drink, a person puts the can in a perfectly fitting 1-cm-thick cylindrical rubber insulation \(k = 0.13\) W/m\(\cdot\)°C). Now how long will it take for the average temperature of the drink to rise to 10°C? Assume the top of the can is not covered.

3–74 Repeat Problem 3–73, assuming a thermal contact resistance of 0.00008 m\(^2\)\(\cdot\)°C/W between the can and the insulation.

3–75E Steam at 450°F is flowing through a steel pipe \((k = 8.7\) Btu/h \(\cdot\) ft \(\cdot\)°F) whose inner and outer diameters are 3.5 in. and 4.0 in., respectively, in an environment at 55°F. The pipe is insulated with 2-in.-thick fiberglass insulation \((k = 0.020\) Btu/h \(\cdot\) ft \(\cdot\)°F). If the heat transfer coefficients on the inside and the outside of the pipe are 30 and 5 Btu/h \(\cdot\) ft\(^2\) \(\cdot\)°F, respectively, determine the rate of heat loss from the steam per foot length of the pipe. What is the error involved in neglecting the thermal resistance of the steel pipe in calculations?

3–76 Hot water at an average temperature of 90°C is flowing through a 15-m section of a cast iron pipe \((k = 52\) W/m \(\cdot\)°C) whose inner and outer diameters are 4 cm and 4.6 cm, respectively. The outer surface of the pipe, whose emissivity is 0.7, is exposed to the cold air at 10°C in the basement, with a heat transfer coefficient of 15 W/m\(^2\)\(\cdot\)°C. The heat transfer coefficient at the inner surface of the pipe is 120 W/m\(^2\)\(\cdot\)°C. Taking the walls of the basement to be at 10°C also, determine the rate of heat loss from the hot water. Also, determine the average
velocity of the water in the pipe if the temperature of the water drops by 3°C as it passes through the basement.

3–77 Repeat Problem 3–76 for a pipe made of copper \((k = 386 \text{ W/m} \cdot ^\circ\text{C})\) instead of cast iron.

3–78E Steam exiting the turbine of a steam power plant at 100°F is to be condensed in a large condenser by cooling water flowing through copper pipes \((k = 223 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ\text{F})\) of inner diameter 0.4 in. and outer diameter 0.6 in. at an average temperature of 70°F. The heat of vaporization of water at 100°F is 1037 Btu/lbm. The heat transfer coefficients are 1500 Btu/h \cdot \text{ft}^2 \cdot ^\circ\text{F} on the steam side and 35 Btu/h \cdot \text{ft}^2 \cdot ^\circ\text{F} on the water side. Determine the length of the tube required to condense steam at a rate of 120 lbm/h. \textit{Answer: 1148 ft}

3–79E Repeat Problem 3–78E, assuming that a 0.01-in.-thick layer of mineral deposit \((k = 0.5 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ\text{F})\) has formed on the inner surface of the pipe.

3–80 Reconsider Problem 3–78E. Using EES (or other) software, investigate the effects of the thermal conductivity of the pipe material and the outer diameter of the pipe on the length of the tube required. Let the thermal conductivity vary from 10 Btu/h \cdot \text{ft} \cdot ^\circ\text{F} to 400 Btu/h \cdot \text{ft} \cdot ^\circ\text{F} and the outer diameter from 0.5 in. to 1.0 in. Plot the length of the tube as functions of pipe conductivity and the outer pipe diameter, and discuss the results.

3–81 The boiling temperature of nitrogen at atmospheric pressure at sea level (1 atm pressure) is \(-196^\circ\text{C}\). Therefore, nitrogen is commonly used in low-temperature scientific studies since the temperature of liquid nitrogen in a tank open to the atmosphere will remain constant at \(-196^\circ\text{C}\) until it is depleted. Any heat transfer to the tank will result in the evaporation of some liquid nitrogen, which has a heat of vaporization of 198 kJ/kg and a density of 810 kg/m³ at 1 atm.

Consider a 3-m-diameter spherical tank that is initially filled with liquid nitrogen at 1 atm and \(-196^\circ\text{C}\). The tank is exposed to ambient air at 15°C, with a combined convection and radiation heat transfer coefficient of 35 W/m² \cdot ^\circ\text{C}. The temperature of the thin-shelled spherical tank is observed to be almost the same as the temperature of the nitrogen inside. Determine the rate of evaporation of the liquid nitrogen in the tank as a result of the heat transfer from the ambient air if the tank is (a) not insulated, (b) insulated with 5-cm-thick fiberglass insulation \((k = 0.035 \text{ W/m} \cdot ^\circ\text{C})\), and (c) insulated with 2-cm-thick superinsulation which has an effective thermal conductivity of 0.00005 W/m \cdot ^\circ\text{C}.

3–82 Repeat Problem 3–81 for liquid oxygen, which has a boiling temperature of \(-183^\circ\text{C}\), a heat of vaporization of 213 kJ/kg, and a density of 1140 kg/m³ at 1 atm pressure.

**Critical Radius of Insulation**

3–83C What is the critical radius of insulation? How is it defined for a cylindrical layer?

3–84C A pipe is insulated such that the outer radius of the insulation is less than the critical radius. Now the insulation is taken off. Will the rate of heat transfer from the pipe increase or decrease for the same pipe surface temperature?

3–85C A pipe is insulated to reduce the heat loss from it. However, measurements indicate that the rate of heat loss has increased instead of decreasing. Can the measurements be right?

3–86C Consider a pipe at a constant temperature whose radius is greater than the critical radius of insulation. Someone claims that the rate of heat loss from the pipe has increased when some insulation is added to the pipe. Is this claim valid?

3–87C Consider an insulated pipe exposed to the atmosphere. Will the critical radius of insulation be greater on calm days or on windy days? Why?
3–88 A 2-mm-diameter and 10-m-long electric wire is tightly wrapped with a 1-mm-thick plastic cover whose thermal conductivity is $k = 0.15 \text{ W/m} \cdot ^\circ \text{C}$. Electrical measurements indicate that a current of 10 A passes through the wire and there is a voltage drop of 8 V along the wire. If the insulated wire is exposed to a medium at $T_m = 30^\circ \text{C}$ with a heat transfer coefficient of $h = 24 \text{ W/m}^2 \cdot ^\circ \text{C}$, determine the temperature at the interface of the wire and the plastic cover in steady operation. Also determine if doubling the thickness of the plastic cover will increase or decrease this interface temperature.

3–89 A 0.083-in.-diameter electrical wire at 115°F is covered by 0.02-in.-thick plastic insulation ($k = 0.075 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ \text{F}$). The wire is exposed to a medium at 50°F, with a combined convection and radiation heat transfer coefficient of 2.5 $\text{Btu/h} \cdot \text{ft}^2 \cdot ^\circ \text{F}$. Determine if the plastic insulation on the wire will increase or decrease heat transfer from the wire.

**Answer:** It helps

3–90 Repeat Problem 3–89E, assuming a thermal contact resistance of 0.001 $\text{h} \cdot \text{ft}^2 \cdot ^\circ \text{F/Btu}$ at the interface of the wire and the insulation.

3–91 A 5-mm-diameter spherical ball at 50°C is covered by a 1-mm-thick plastic insulation ($k = 0.13 \text{ W/m} \cdot ^\circ \text{C}$). The ball is exposed to a medium at 15°C, with a combined convection and radiation heat transfer coefficient of 20 $\text{W/m}^2 \cdot ^\circ \text{C}$. Determine if the plastic insulation on the ball will help or hurt heat transfer from the ball.

**FIGURE P3–88**

3–92 Reconsider Problem 3–91. Using EES (or other) software, plot the rate of heat transfer from the ball as a function of the plastic insulation thickness in the range of 0.5 mm to 20 mm. Discuss the results.

3–93C What is the reason for the widespread use of fins on surfaces?

**FIGURE P3–91**

3–94C What is the difference between the fin effectiveness and the fin efficiency?

3–95C The fins attached to a surface are determined to have an effectiveness of 0.9. Do you think the rate of heat transfer from the surface has increased or decreased as a result of the addition of these fins?

3–96C Explain how the fins enhance heat transfer from a surface. Also, explain how the addition of fins may actually decrease heat transfer from a surface.

3–97C How does the overall effectiveness of a finned surface differ from the effectiveness of a single fin?

3–98C Hot water is to be cooled as it flows through the tubes exposed to atmospheric air. Fins are to be attached in order to enhance heat transfer. Would you recommend attaching the fins inside or outside the tubes? Why?

3–99C Hot air is to be cooled as it is forced to flow through the tubes exposed to atmospheric air. Fins are to be added in order to enhance heat transfer. Would you recommend attaching the fins inside or outside the tubes? Why? When would you recommend attaching fins both inside and outside the tubes?

3–100C Consider two finned surfaces that are identical except that the fins on the first surface are formed by casting or extrusion, whereas they are attached to the second surface afterwards by welding or tight fitting. For which case do you think the fins will provide greater enhancement in heat transfer? Explain.

3–101C The heat transfer surface area of a fin is equal to the sum of all surfaces of the fin exposed to the surrounding medium, including the surface area of the fin tip. Under what conditions can we neglect heat transfer from the fin tip?

3–102C Does the (a) efficiency and (b) effectiveness of a fin increase or decrease as the fin length is increased?

3–103C Two pin fins are identical, except that the diameter of one of them is twice the diameter of the other. For which fin will the (a) fin effectiveness and (b) fin efficiency be higher? Explain.

3–104C Two plate fins of constant rectangular cross section are identical, except that the thickness of one of them is twice the thickness of the other. For which fin will the (a) fin effectiveness and (b) fin efficiency be higher? Explain.

3–105C Two finned surfaces are identical, except that the convection heat transfer coefficient of one of them is twice that of the other. For which finned surface will the (a) fin effectiveness and (b) fin efficiency be higher? Explain.

3–106 Obtain a relation for the fin efficiency for a fin of constant cross-sectional area $A_f$, perimeter $p$, length $L$, and thermal conductivity $k$ exposed to convection to a medium at $T_m$ with a heat transfer coefficient $h$. Assume the fins are sufficiently long so that the temperature of the fin at the tip is nearly $T_m$. Take the temperature of the fin at the base to be $T_b$ and neglect heat...
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transfer from the fin tips. Simplify the relation for (a) a circular fin of diameter \( D \) and (b) rectangular fins of thickness \( t \).

3–107 The case-to-ambient thermal resistance of a power transistor that has a maximum power rating of 15 W is given to be 25°C/W. If the case temperature of the transistor is not to exceed 80°C, determine the power at which this transistor can be operated safely in an environment at 40°C.

3–108 A 40-W power transistor is to be cooled by attaching it to one of the commercially available heat sinks shown in Table 3–4. Select a heat sink that will allow the case temperature of the transistor not to exceed 90°C in the ambient air at 20°C.

3–109 A 30-W power transistor is to be cooled by attaching it to one of the commercially available heat sinks shown in Table 3–4. Select a heat sink that will allow the case temperature of the transistor not to exceed 80°C in the ambient air at 35°C.

3–110 Steam in a heating system flows through tubes whose outer diameter is 5 cm and whose walls are maintained at a temperature of 180°C. Circular aluminum alloy 2024-T6 fins \( (k = 186 \text{ W/m} \cdot \text{°C}) \) of outer diameter 6 cm and constant thickness 1 mm are attached to the tube. The space between the fins is 3 mm, and thus there are 250 fins per meter length of the tube. Heat is transferred to the surrounding air at \( T_a = 25°C \), with a heat transfer coefficient of 40 W/m²·°C. Determine the increase in heat transfer from the tube per meter of its length as a result of adding fins. Answer: 2639 W

3–111E Repeat Problem 3–111 for a silver spoon \((k = 247 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F})\).

3–112E Consider a stainless steel spoon \((k = 8.7 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F})\) partially immersed in boiling water at 200°F in a kitchen at 75°F. The handle of the spoon has a cross section of 0.08 in. × 0.5 in., and extends 7 in. in the air from the free surface of the water. If the heat transfer coefficient at the exposed surfaces of the spoon handle is 3 Btu/h · ft² · °F, determine the temperature difference across the exposed surface of the spoon handle. State your assumptions. Answer: 124.6°F

3–113E Reconsider Problem 3–111E. Using EES (or other) software, investigate the effects of the thermal conductivity of the spoon material and the length of its extension in the air on the temperature difference across the exposed surface of the spoon handle. Let the thermal conductivity vary from 5 Btu/h · ft · °F to 225 Btu/h · ft · °F and the length from 5 in. to 12 in. Plot the temperature difference as the functions of thermal conductivity and length, and discuss the results.

3–114 A 0.3-cm-thick, 12-cm-high, and 18-cm-long circuit board houses 80 closely spaced logic chips on one side, each dissipating 0.04 W. The board is impregnated with copper fillings and has an effective thermal conductivity of 20 W/m · °C. All the heat generated in the chips is conducted across the circuit board and is dissipated from the back side of the board to a medium at 40°C, with a heat transfer coefficient of 50 W/m² · °C. (a) Determine the temperatures on the two sides of the circuit board. (b) Now a 0.2-cm-thick, 12-cm-high, and
18-cm-long aluminum plate \((k = 237 \text{ W/m} \cdot \text{°C})\) with 864
2-cm-long aluminum pin fins of diameter 0.25 cm is attached
to the back side of the circuit board with a 0.02-cm-thick epoxy
adhesive \((k = 1.8 \text{ W/m} \cdot \text{°C})\). Determine the new temperatures
on the two sides of the circuit board.

3–115 Repeat Problem 3–114 using a copper plate with cop-
per fins \((k = 386 \text{ W/m} \cdot \text{°C})\) instead of aluminum ones.

3–116 A hot surface at 100°C is to be cooled by attach-
ing 3-cm-long, 0.25-cm-diameter aluminum pin fins \((k =
237 \text{ W/m} \cdot \text{°C})\) to it, with a center-to-center distance of 0.6 cm.
The temperature of the surrounding medium is 30°C, and the
heat transfer coefficient on the surfaces is 35 \text{ W/m}^2 \cdot \text{°C}.
Determine the rate of heat transfer from the surface for a
1-m \times 1-m section of the plate. Also determine the overall
effectiveness of the fins.

3–117 Repeat Problem 3–116 using copper fins \((k = 386
\text{ W/m} \cdot \text{°C})\) instead of aluminum ones.

3–118 Reconsider Problem 3–116. Using EES (or
other) software, investigate the effect of the cen-
ter-to-center distance of the fins on the rate of heat transfer
from the surface and the overall effectiveness of the fins. Let
the center-to-center distance vary from 0.4 cm to 2.0 cm. Plot
the rate of heat transfer and the overall effectiveness as a func-
tion of the center-to-center distance, and discuss the results.

3–119 Two 3-m-long and 0.4-cm-thick cast iron \((k = 52
\text{ W/m} \cdot \text{°C})\) steam pipes of outer diameter 10 cm are connected
each other through two 1-cm-thick flanges of outer diameter
20 cm. The steam flows inside the pipe at an average tempera-
ture of 200°C with a heat transfer coefficient of 180 \text{ W/m}^2 \cdot \text{°C}.
The outer surface of the pipe is exposed to an ambient at 12°C,
with a heat transfer coefficient of 25 \text{ W/m}^2 \cdot \text{°C}. \((a)\) Disregard-
ing the flanges, determine the average outer surface tempera-
ture of the pipe. \((b)\) Using this temperature for the base of
the flange and treating the flanges as the fins, determine the fin ef-
ciciency and the rate of heat transfer from the flanges. \((c)\) What
length of pipe is the flange section equivalent to for heat trans-
fer purposes?

3–120 \What is a conduction shape factor? How is it related
to the thermal resistance?

3–121 \What is the value of conduction shape factors in
engineering?

3–122 A 20-m-long and 8-cm-diameter hot water pipe of a
district heating system is buried in the soil 80 cm below the
ground surface. The outer surface temperature of the pipe is
60°C. Taking the surface temperature of the earth to be 5°C
and the thermal conductivity of the soil at that location to be
0.9 \text{ W/m} \cdot \text{°C}, determine the rate of heat loss from the pipe.

3–123 Reconsider Problem 3–122. Using EES (or
other) software, plot the rate of heat loss from
the pipe as a function of the burial depth in the range of 20 cm
to 2.0 m. Discuss the results.

3–124 Hot and cold water pipes 8 m long run parallel to each
other in a thick concrete layer. The diameters of both pipes are
5 cm, and the distance between the centerlines of the pipes is
40 cm. The surface temperatures of the hot and cold pipes are
60°C and 15°C, respectively. Taking the thermal conductivity
of the concrete to be \(k = 0.75 \text{ W/m} \cdot \text{°C}\), determine the rate of
heat transfer between the pipes. \(\text{Answer: 306 W}\)
3–125 Reconsider Problem 3–124. Using EES (or other) software, plot the rate of heat transfer between the pipes as a function of the distance between the centerlines of the pipes in the range of 10 cm to 1.0 m. Discuss the results.

3–126E A row of 3-ft-long and 1-in.-diameter used uranium fuel rods that are still radioactive are buried in the ground parallel to each other with a center-to-center distance of 8 in. at a depth 15 ft from the ground surface at a location where the thermal conductivity of the soil is 0.6 Btu/h·ft·°F. If the surface temperature of the rods and the ground are 350°F and 60°F, respectively, determine the rate of heat transfer from the fuel rods to the atmosphere through the soil.

3–127 Hot water at an average temperature of 60°C and an average velocity of 0.6 m/s is flowing through a 5-m section of a thin-walled hot water pipe that has an outer diameter of 2.5 cm. The pipe passes through the center of a 14-cm-thick wall filled with fiberglass insulation (k = 0.035 W/m·°C). If the surfaces of the wall are at 18°C, determine (a) the rate of heat transfer from the pipe to the air in the rooms and (b) the temperature drop of the hot water as it flows through this 5-m-long section of the wall. Answers: 23.5 W, 0.02°C

3–128 Hot water at an average temperature of 80°C and an average velocity of 1.5 m/s is flowing through a 25-m section of a pipe that has an outer diameter of 5 cm. The pipe extends 2 m in the ambient air above the ground, dips into the ground (k = 1.5 W/m·°C) vertically for 3 m, and continues horizontally at this depth for 20 m more before it enters the next building. The first section of the pipe is exposed to the ambient air at 8°C, with a heat transfer coefficient of 22 W/m²·°C. If the surface of the ground is covered with snow at 0°C, determine (a) the total rate of heat loss from the hot water and (b) the temperature drop of the hot water as it flows through this 25-m-long section of the pipe.

3–129 Consider a house with a flat roof whose outer dimensions are 12 m × 12 m. The outer walls of the house are 6 m high. The walls and the roof of the house are made of 20-cm-thick concrete (k = 0.75 W/m·°C). The temperatures of the inner and outer surfaces of the house are 15°C and 3°C, respectively. Accounting for the effects of the edges of adjoining surfaces, determine the rate of heat loss from the house through its walls and the roof. What is the error involved in ignoring the effects of the edges and corners and treating the roof as a 12 m × 12 m surface and the walls as 6 m × 12 m surfaces for simplicity?

3–130 Consider a 10-m-long thick-walled concrete duct (k = 0.75 W/m·°C) of square cross-section. The outer dimensions of the duct are 20 cm × 20 cm, and the thickness of the duct wall is 2 cm. If the inner and outer surfaces of the duct are at 100°C and 15°C, respectively, determine the rate of heat transfer through the walls of the duct. Answer: 22.9 kW

3–131 A 3-m-diameter spherical tank containing some radioactive material is buried in the ground (k = 1.4 W/m·°C). The distance between the top surface of the tank and the ground surface is 4 m. If the surface temperatures of the tank and the
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ground are 140°C and 15°C, respectively, determine the rate of heat transfer from the tank.

3-132 Reconsider Problem 3–131. Using EES (or other) software, plot the rate of heat transfer from the tank as a function of the tank diameter in the range of 0.5 m to 5.0 m. Discuss the results.

3-133 Hot water at an average temperature of 85°C passes through a row of eight parallel pipes that are 4 m long and have an outside diameter of 3 cm, located vertically in the middle of a concrete wall (k = 0.75 W/m · °C) that is 4 m high, 8 m long, and 15 cm thick. If the surfaces of the concrete walls are exposed to a medium at 32°C, with a heat transfer coefficient of 12 W/m² · °C, determine the rate of heat loss from the hot water and the surface temperature of the wall.

Special Topics:
Heat Transfer through the Walls and Roofs

3–134C What is the R-value of a wall? How does it differ from the unit thermal resistance of the wall? How is it related to the U-factor?

3–135C What is effective emissivity for a plane-parallel air space? How is it determined? How is radiation heat transfer through the air space determined when the effective emissivity is known?

3–136C The unit thermal resistances (R-values) of both 40-mm and 90-mm vertical air spaces are given in Table 3–9 to be 0.22 m² · °C/W, which implies that more than doubling the thickness of air space in a wall has no effect on heat transfer through the wall. Do you think this is a typing error? Explain.

3–137C What is a radiant barrier? What kind of materials are suitable for use as radiant barriers? Is it worthwhile to use radiant barriers in the attics of homes?

3–138C Consider a house whose attic space is ventilated effectively so that the air temperature in the attic is the same as the ambient air temperature at all times. Will the roof still have any effect on heat transfer through the ceiling? Explain.

3–139 Determine the summer R-value and the U-factor of a wood frame wall that is built around 38-mm × 140-mm wood studs with a center-to-center distance of 400 mm. The 140-mm-wide cavity between the studs is filled with mineral fiber batt insulation. The inside is finished with 13-mm gypsum wallboard and the outside with 13-mm plywood, a layer of felt (R = 0.166 m² · °C/W), a 13-mm plywood, a layer of felt (R = 0.011 m² · °C/W), and linoleum (R = 0.009 m² · °C/W). Both

Answers: 3.213 m² · °C/W, 0.311 W/m² · °C

3–140 The 13-mm-thick wood fiberboard sheathing of the wood stud wall in Problem 3–139 is replaced by a 25-mm-thick rigid foam insulation. Determine the percent increase in the R-value of the wall as a result.

3–141E Determine the winter R-value and the U-factor of a masonry cavity wall that is built around 4-in.-thick concrete blocks made of lightweight aggregate. The outside is finished with 4-in. face brick with ½-in. cement mortar between the bricks and concrete blocks. The inside finish consists of ½-in. gypsum wallboard separated from the concrete block by ½-in.-thick (1-in. by 3-in. nominal) vertical furring whose center-to-center distance is 16 in. Neither side of the ½-in.-thick air space between the concrete block and the gypsum board is coated with any reflective film. When determining the R-value of the air space, the temperature difference across it can be taken to be 30°F with a mean air temperature of 50°F. The air space constitutes 80 percent of the heat transmission area, while the vertical furring and similar structures constitute 20 percent.

3–142 Consider a flat ceiling that is built around 38-mm × 90-mm wood studs with a center-to-center distance of 400 mm. The lower part of the ceiling is finished with 13-mm gypsum wallboard, while the upper part consists of a wood subfloor (R = 0.166 m² · °C/W), a 13-mm plywood, a layer of felt (R = 0.011 m² · °C/W), and linoleum (R = 0.009 m² · °C/W). Both
sides of the ceiling are exposed to still air. The air space constitutes 82 percent of the heat transmission area, while the studs and headers constitute 18 percent. Determine the winter $R$-value and the $U$-factor of the ceiling assuming the 90-mm-wide air space between the studs (a) does not have any reflective surface, (b) has a reflective surface with $\varepsilon = 0.05$ on one side, and (c) has reflective surfaces with $\varepsilon = 0.05$ on both sides. Assume a mean temperature of 10°C and a temperature difference of 5.6°C for the air space.

3–143 Determine the winter $R$-value and the $U$-factor of a masonry cavity wall that consists of 100-mm common bricks, a 90-mm air space, 100-mm concrete blocks made of lightweight aggregate, 20-mm air space, and 13-mm gypsum wallboard separated from the concrete block by 20-mm-thick (1-in. × 3-in. nominal) vertical furring whose center-to-center distance is 400 mm. Neither side of the two air spaces is coated with any reflective films. When determining the $R$-value of the air spaces, the temperature difference across them can be taken to be 16.7°C with a mean air temperature of 10°C. The air space constitutes 84 percent of the heat transmission area, while the vertical furring and similar structures constitute 16 percent.  

Answers: 1.02 m² · °C/W, 0.978 W/m² · °C

3–144 Repeat Problem 3–143 assuming one side of both air spaces is coated with a reflective film of $\varepsilon = 0.05$.

3–145 Determine the winter $R$-value and the $U$-factor of a masonry wall that consists of the following layers: 100-mm face bricks, 100-mm common bricks, 25-mm urethane rigid foam insulation, and 13-mm gypsum wallboard.

Answers: 1.404 m² · °C/W, 0.712 W/m² · °C

3–146 The overall heat transfer coefficient (the $U$-value) of a wall under winter design conditions is $U = 1.55$ W/m² · °C. Determine the $U$-value of the wall under summer design conditions.

3–147 The overall heat transfer coefficient (the $U$-value) of a wall under winter design conditions is $U = 2.25$ W/m² · °C. Now a layer of 100-mm face brick is added to the outside, leaving a 20-mm air space between the wall and the bricks. Determine the new $U$-value of the wall. Also, determine the rate of heat transfer through a 3-m-high, 7-m-long section of the wall after modification when the indoor and outdoor temperatures are 22°C and −5°C, respectively.

3–148 Determine the summer and winter $R$-values, in m² · °C/W, of a masonry wall that consists of 100-mm face bricks, 13-mm of cement mortar, 100-mm lightweight concrete block, 40-mm air space, and 20-mm plasterboard.

Answers: 0.809 and 0.795 m² · °C/W

3–149E The overall heat transfer coefficient of a wall is determined to be $U = 0.09$ Btu/h · ft² · °F under the conditions of still air inside and winds of 7.5 mph outside. What will the $U$-factor be when the wind velocity outside is doubled?

Answer: 0.0907 Btu/h · ft² · °F

3–150 Two homes are identical, except that the walls of one house consist of 200-mm lightweight concrete blocks, 20-mm air space, and 20-mm plasterboard, while the walls of the other house involve the standard $R$-2.4 m² · °C/W frame wall construction. Which house do you think is more energy efficient?
Review Problems

3–151 Determine the R-value of a ceiling that consists of a layer of 19-mm acoustical tiles whose top surface is covered with a highly reflective aluminum foil for winter conditions. Assume still air below and above the tiles.

![Diagram of a ceiling with acoustical tiles and highly reflective foil](image)

**FIGURE P3–151**

**Review Problems**

3–152E Steam is produced in the copper tubes \((k = 223 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ \text{F})\) of a heat exchanger at a temperature of 250°F by another fluid condensing on the outside surfaces of the tubes at 350°F. The inner and outer diameters of the tube are 1 in. and 1.3 in., respectively. When the heat exchanger was new, the rate of heat transfer per foot length of the tube was \(2 \times 10^4 \text{ Btu/h}\). Determine the rate of heat transfer per foot length of the tube when a 0.01-in.-thick layer of limestone \((k = 1.7 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ \text{F})\) has formed on the inner surface of the tube after extended use.

3–153E Repeat Problem 3–152E, assuming that a 0.01-in.-thick limestone layer has formed on both the inner and outer surfaces of the tube.

3–154 A 1.2-m-diameter and 6-m-long cylindrical propane tank is initially filled with liquid propane whose density is 581 \(\text{kg/m}^3\). The tank is exposed to the ambient temperature at 30°C, with a heat transfer coefficient of 25 \(\text{W/m}^2 \cdot ^\circ \text{C}\). Now a crack develops at the top of the tank and the pressure inside drops to 1 atm while the temperature drops to \(-42^\circ \text{C}\), which is the boiling temperature of propane at 1 atm. The heat of vaporization of propane at 1 atm is 425 \(\text{kJ/kg}\). The propane is slowly vaporized as a result of the heat transfer from the ambient air into the tank, and the propane vapor escapes the tank at \(-42^\circ \text{C}\) through the crack. Assuming the propane tank to be at about the same temperature as the propane inside at all times, determine how long it will take for the propane tank to empty if the tank is \((a)\) not insulated and \((b)\) insulated with 7.5-cm-thick glass wool insulation \((k = 0.038 \text{ W/m} \cdot ^\circ \text{C})\).

3–155 Hot water is flowing at an average velocity of 1.5 \(\text{m/s}\) through a cast iron pipe \((k = 52 \text{ W/m} \cdot ^\circ \text{C})\) whose inner and outer diameters are 3 cm and 3.5 cm, respectively. The pipe passes through a 15-m-long section of a basement whose temperature is 15°C. If the temperature of the water drops from 70°C to 67°C as it passes through the basement and the heat transfer coefficient on the inner surface of the pipe is 400 \(\text{W/m}^2 \cdot ^\circ \text{C}\), determine the combined convection and radiation heat transfer coefficient at the outer surface of the pipe.

**Answer:** \(272.5 \text{ W/m}^2 \cdot ^\circ \text{C}\)

3–156 Newly formed concrete pipes are usually cured first overnight by steam in a curing kiln maintained at a temperature of 45°C before the pipes are cured for several days outside. The heat and moisture to the kiln is provided by steam flowing in a pipe whose outer diameter is 12 cm. During a plant inspection, it was noticed that the pipe passes through a 10-m section that is completely exposed to the ambient air before it reaches the kiln. The temperature measurements indicate that the average temperature of the outer surface of the steam pipe is 82°C when the ambient temperature is 8°C. The combined convection and radiation heat transfer coefficient at the outer surface of the pipe is estimated to be 25 \(\text{W/m}^2 \cdot ^\circ \text{C}\). Determine the amount of heat lost from the steam during a 10-h curing process that night.

Steam is supplied by a gas-fired steam generator that has an efficiency of 80 percent, and the plant pays $0.60/therm of natural gas (1 therm = 105,500 \(\text{kJ}\)). If the pipe is insulated and 90 percent of the heat loss is saved as a result, determine the amount of money this facility will save a year as a result of insulating the steam pipes. Assume that the concrete pipes are cured 110 nights a year. State your assumptions.

![Diagram of a steam pipe passing through a basement](image)

**FIGURE P3–156**

3–157 Consider an 18-cm \(\times\) 18-cm multilayer circuit board dissipating 27 \(\text{W}\) of heat. The board consists of four layers of 0.2-mm-thick copper \((k = 386 \text{ W/m} \cdot ^\circ \text{C})\) and three layers of
1.5-mm-thick epoxy glass \((k = 0.26 \text{ W/m} \cdot \text{°C})\) sandwiched together, as shown in the figure. The circuit board is attached to a heat sink from both ends, and the temperature of the board at those ends is 35°C. Heat is considered to be uniformly generated in the epoxy layers of the board at a rate of 0.5 W per 1-cm \(\times\) 18-cm epoxy laminate strip (or 1.5 W per 1-cm \(\times\) 18-cm strip of the board). Considering only a portion of the board because of symmetry, determine the magnitude and location of the maximum temperature that occurs in the board. Assume heat transfer from the top and bottom faces of the board to be negligible.

3–158 The plumbing system of a house involves a 0.5-m section of a plastic pipe \((k = 0.16 \text{ W/m} \cdot \text{°C})\) of inner diameter 2 cm and outer diameter 2.4 cm exposed to the ambient air. During a cold and windy night, the ambient air temperature remains at about −5°C for a period of 14 h. The combined convection and radiation heat transfer coefficient on the outer surface of the pipe is estimated to be 40 W/m² \(\cdot\) °C, and the heat of fusion of water is 333.7 kJ/kg. Assuming the pipe to contain stationary water initially at 0°C, determine if the water in that section of the pipe will completely freeze that night.

3–159 Repeat Problem 3–158 for the case of a heat transfer coefficient of 10 W/m² \(\cdot\) °C on the outer surface as a result of putting a fence around the pipe that blocks the wind.

3–160E The surface temperature of a 3-in.-diameter baked potato is observed to drop from 300°F to 200°F in 5 minutes in an environment at 70°F. Determine the average heat transfer coefficient between the potato and its surroundings. Using this heat transfer coefficient and the same surface temperature, determine how long it will take for the potato to experience the same temperature drop if it is wrapped completely in a 0.12-in.-thick towel \((k = 0.035 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F})\). You may use the properties of water for potato.

3–161E Repeat Problem 3–160E assuming there is a 0.02-in.-thick air space \((k = 0.015 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F})\) between the potato and the towel.

3–162 An ice chest whose outer dimensions are 30 cm \(\times\) 40 cm \(\times\) 50 cm is made of 3-cm-thick Styrofoam \((k = 0.033 \text{ W/m} \cdot \text{°C})\). Initially, the chest is filled with 45 kg of ice at 0°C, and the inner surface temperature of the ice chest can be taken to be 0°C at all times. The heat of fusion of ice at 0°C is 333.7 kJ/kg, and the heat transfer coefficient between the outer surface of the ice chest and surrounding air at 35°C is 18 W/m² \(\cdot\) °C. Disregarding any heat transfer from the 40-cm \(\times\) 50-cm base of the ice chest, determine how long it will take for the ice in the chest to melt completely.
remaining space between the steel plates is filled with fiberglass insulation \( (k = 0.035 \text{ W/m} \cdot \text{°C}) \). If the temperature difference between the inner and the outer surfaces of the walls is 22°C, determine the rate of heat transfer through the wall. Can we ignore the steel bars between the plates in heat transfer analysis since they occupy only 1 percent of the heat transfer surface area?

**3–164** A 0.2-cm-thick, 10-cm-high, and 15-cm-long circuit board houses electronic components on one side that dissipate a total of 15 W of heat uniformly. The board is impregnated with conducting metal fillings and has an effective thermal conductivity of 12 W/m · °C. All the heat generated in the components is conducted across the circuit board and is dissipated from the back side of the board to a medium at 37°C, with a heat transfer coefficient of 45 W/m² · °C. (a) Determine the surface temperatures on the two sides of the circuit board. (b) Now a 0.1-cm-thick, 10-cm-high, and 15-cm-long aluminum plate \( (k = 237 \text{ W/m} \cdot \text{°C}) \) with 20 0.2-cm-thick, 2-cm-long, and 15-cm-wide aluminum fins of rectangular profile are attached to the back side of the circuit board with a 0.03-cm-thick epoxy adhesive \( (k = 1.8 \text{ W/m} \cdot \text{°C}) \). Determine the new temperatures on the two sides of the circuit board.

**3–165** Repeat Problem 3–164 using a copper plate with copper fins \( (k = 386 \text{ W/m} \cdot \text{°C}) \) instead of aluminum ones.

**3–166** A row of 10 parallel pipes that are 5 m long and have an outer diameter of 6 cm are used to transport steam at 150°C through the concrete floor \( (k = 0.75 \text{ W/m} \cdot \text{°C}) \) of a 10-m × 5-m room that is maintained at 25°C. The combined convection and radiation heat transfer coefficient at the floor is 12 W/m² · °C. If the surface temperature of the concrete floor is not to exceed 40°C, determine how deep the steam pipes should be buried below the surface of the concrete floor.

**3–167** Consider two identical people each generating 60 W of metabolic heat steadily while doing sedentary work, and dissipating it by convection and perspiration. The first person is wearing clothes made of 1-mm-thick leather \( (k = 0.159 \text{ W/m} \cdot \text{°C}) \) that covers half of the body while the second one is wearing clothes made of 1-mm-thick synthetic fabric \( (k = 0.13 \text{ W/m} \cdot \text{°C}) \) that covers the body completely. The ambient air is at 30°C, the heat transfer coefficient at the outer surface is 15 W/m² · °C, and the inner surface temperature of the clothes can be taken to be 32°C. Treating the body of each person as a 25-cm-diameter 1.7-m-long cylinder, determine the fractions of heat lost from each person by perspiration.

**3–168** A 6-m-wide 2.8-m-high wall is constructed of one layer of common brick \( (k = 0.72 \text{ W/m} \cdot \text{°C}) \) of thickness 20 cm, one inside layer of light-weight plaster \( (k = 0.36 \text{ W/m} \cdot \text{°C}) \) of thickness 1 cm, and one outside layer of cement based covering \( (k = 1.40 \text{ W/m} \cdot \text{°C}) \) of thickness 2 cm. The inner surface of the wall is maintained at 23°C while the outer surface is exposed to outdoors at 8°C with a combined convection and radiation heat transfer coefficient of 17 W/m² · °C. Determine the rate of heat transfer through the wall and temperature drops across the plaster, brick, covering, and surface-ambient air.

**3–169** Reconsider Problem 3–168. It is desired to insulate the wall in order to decrease the heat loss by 85 percent. For the same inner surface temperature, determine the thickness of insulation and the outer surface temperature if the wall is insulated with (a) polyurethane foam \( (k = 0.025 \text{ W/m} \cdot \text{°C}) \) and (b) glass fiber \( (k = 0.036 \text{ W/m} \cdot \text{°C}) \).

**3–170** Cold conditioned air at 12°C is flowing inside a 1.5-cm-thick square aluminum \( (k = 237 \text{ W/m} \cdot \text{°C}) \) duct of inner cross section 22 cm × 22 cm at a mass flow rate of 0.8 kg/s. The duct is exposed to air at 33°C with a combined convection-radiation heat transfer coefficient of 8 W/m² · °C. The convection heat transfer coefficient at the inner surface is 75 W/m² · °C. If the air temperature in the duct should not increase by more than 1°C determine the maximum length of the duct.

**3–171** When analyzing heat transfer through windows, it is important to consider the frame as well as the glass area. Consider a 2-m-wide 1.5-m-high wood-framed window with...
85 percent of the area covered by 3-mm-thick single-pane glass ($k = 0.7 \text{ W/m} \cdot \text{°C}$). The frame is 5 cm thick, and is made of pine wood ($k = 0.12 \text{ W/m} \cdot \text{°C}$). The heat transfer coefficient is 7 W/m² · °C inside and 13 W/m² · °C outside. The room is maintained at 24°C, and the temperature outdoors is 40°C. Determine the percent error involved in heat transfer when the window is assumed to consist of glass only.

3–172 Steam at 235°C is flowing inside a steel pipe ($k = 61 \text{ W/m} \cdot \text{°C}$) whose inner and outer diameters are 10 cm and 12 cm, respectively, in an environment at 20°C. The heat transfer coefficients inside and outside the pipe are 105 W/m² · °C and 14 W/m² · °C, respectively. Determine (a) the thickness of the insulation ($k = 0.038 \text{ W/m} \cdot \text{°C}$) needed to reduce the heat loss by 95 percent and (b) the thickness of the insulation needed to reduce the exposed surface temperature of insulated pipe to 40°C for safety reasons.

3–173 When the transportation of natural gas in a pipeline is not feasible for economic or other reasons, it is first liquefied at about −160°C, and then transported in specially insulated tanks placed in marine ships. Consider a 6-m-diameter spherical tank that is filled with liquefied natural gas (LNG) at −160°C. The tank is exposed to ambient air at 18°C with a heat transfer coefficient of 22 W/m² · °C. The tank is thin-shelled and its temperature can be taken to be the same as the LNG temperature. The tank is insulated with 5-cm-thick super insulation that has an effective thermal conductivity of 0.00008 W/m · °C. Taking the density and the specific heat of LNG to be 425 kg/m³ and 3.475 kJ/kg · °C, respectively, estimate how long it will take for the LNG temperature to rise to −150°C.

3–174 A 15-cm × 20-cm hot surface at 85°C is to be cooled by attaching 4-cm-long aluminum ($k = 237 \text{ W/m} \cdot \text{°C}$) fins of 2-mm × 2-mm square cross section. The temperature of surrounding medium is 25°C and the heat transfer coefficient on the surfaces can be taken to be 20 W/m² · °C. If it is desired to triple the rate of heat transfer from the bare hot surface, determine the number of fins that needs to be attached.

3–175 Reconsider Problem 3–174. Using EES (or other) software, plot the number of fins as a function of the increase in the heat loss by fins relative to no fin case (i.e., overall effectiveness of the fins) in the range of 1.5 to 5. Discuss the results. Is it realistic to assume the heat transfer coefficient to remain constant?

3–176 A 1.4-m-diameter spherical steel tank filled with iced water at 0°C is buried underground at a location where the thermal conductivity of the soil is $k = 0.55 \text{ W/m} \cdot \text{°C}$. The distance between the tank center and the ground surface is 2.4 m. For ground surface temperature of 18°C, determine the rate of heat transfer to the iced water in the tank. What would your answer be if the soil temperature were 18°C and the ground surface were insulated?

3–177 A 0.6-m-diameter 1.9-m-long cylindrical tank containing liquefied natural gas (LNG) at −160°C is placed at the center of a 1.9-m-long 1.4-m × 1.4-m square solid bar made of an insulating material with $k = 0.0006 \text{ W/m} \cdot \text{°C}$. If the outer surface temperature of the bar is 20°C, determine the rate of heat transfer to the tank. Also, determine the LNG temperature after one month. Take the density and the specific heat of LNG to be 425 kg/m³ and 3.475 kJ/kg · °C, respectively.

### Design and Essay Problems

3–178 The temperature in deep space is close to absolute zero, which presents thermal challenges for the astronauts who do space walks. Propose a design for the clothing of the astronauts that will be most suitable for the thermal environment in space. Defend the selections in your design.

3–179 In the design of electronic components, it is very desirable to attach the electronic circuitry to a substrate material that is a very good thermal conductor but also a very effective electrical insulator. If the high cost is not a major concern, what material would you propose for the substrate?

3–180 Using cylindrical samples of the same material, devise an experiment to determine the thermal contact resistance. Cylindrical samples are available at any length, and the thermal conductivity of the material is known.

3–181 Find out about the wall construction of the cabins of large commercial airplanes, the range of ambient conditions under which they operate, typical heat transfer coefficients on the inner and outer surfaces of the wall, and the heat generation rates inside. Determine the size of the heating and air-conditioning system that will be able to maintain the cabin at 20°C at all times for an airplane capable of carrying 400 people.

3–182 Repeat Problem 3–181 for a submarine with a crew of 60 people.

3–183 A house with 200-m² floor space is to be heated with geothermal water flowing through pipes laid in the ground under the floor. The walls of the house are 4 m high, and there are 10 single-paned windows in the house that are 1.2 m wide and 1.8 m high. The house has R-19 (in h · ft² · °F/Btu) insulation in the walls and R-30 on the ceiling. The floor temperature is not to exceed 40°C. Hot geothermal water is available at 90°C, and the inner and outer diameter of the pipes to be used are 2.4 cm and 3.0 cm. Design such a heating system for this house in your area.

3–184 Using a timer (or watch) and a thermometer, conduct this experiment to determine the rate of heat gain of your refrigerator. First, make sure that the door of the refrigerator is not opened for at least a few hours to make sure that steady operating conditions are established. Start the timer when the refrigerator stops running and measure the time $\Delta t$, it stays off...
before it kicks in. Then measure the time $\Delta t_2$ it stays on. Noting that the heat removed during $\Delta t_2$ is equal to the heat gain of the refrigerator during $\Delta t_1 + \Delta t_2$ and using the power consumed by the refrigerator when it is running, determine the average rate of heat gain for your refrigerator, in watts. Take the COP (coefficient of performance) of your refrigerator to be 1.3 if it is not available.

Now, clean the condenser coils of the refrigerator and remove any obstacles on the way of airflow through the coils. By replacing these measurements, determine the improvement in the COP of the refrigerator.
The temperature of a body, in general, varies with time as well as position. In rectangular coordinates, this variation is expressed as $T(x, y, z, t)$, where $(x, y, z)$ indicates variation in the $x$, $y$, and $z$ directions, respectively, and $t$ indicates variation with time. In the preceding chapter, we considered heat conduction under steady conditions, for which the temperature of a body at any point does not change with time. This certainly simplified the analysis, especially when the temperature varied in one direction only, and we were able to obtain analytical solutions. In this chapter, we consider the variation of temperature with time as well as position in one- and multidimensional systems.

We start this chapter with the analysis of lumped systems in which the temperature of a solid varies with time but remains uniform throughout the solid at any time. Then we consider the variation of temperature with time as well as position for one-dimensional heat conduction problems such as those associated with a large plane wall, a long cylinder, a sphere, and a semi-infinite medium using transient temperature charts and analytical solutions. Finally, we consider transient heat conduction in multidimensional systems by utilizing the product solution.
4–1 LUMPED SYSTEM ANALYSIS

In heat transfer analysis, some bodies are observed to behave like a “lump” whose interior temperature remains essentially uniform at all times during a heat transfer process. The temperature of such bodies can be taken to be a function of time only, \( T(t) \). Heat transfer analysis that utilizes this idealization is known as lumped system analysis, which provides great simplification in certain classes of heat transfer problems without much sacrifice from accuracy.

Consider a small hot copper ball coming out of an oven (Fig. 4–1). Measurements indicate that the temperature of the copper ball changes with time, but it does not change much with position at any given time. Thus the temperature of the ball remains uniform at all times, and we can talk about the temperature of the ball with no reference to a specific location.

Now let us go to the other extreme and consider a large roast in an oven. If you have done any roasting, you must have noticed that the temperature distribution within the roast is not even close to being uniform. You can easily verify this by taking the roast out before it is completely done and cutting it in half. You will see that the outer parts of the roast are well done while the center part is barely warm. Thus, lumped system analysis is not applicable in this case. Before presenting a criterion about applicability of lumped system analysis, we develop the formulation associated with it.

Consider a body of arbitrary shape of mass \( m \), volume \( V \), surface area \( A_s \), density \( \rho \), and specific heat \( C_p \) initially at a uniform temperature \( T_i \) (Fig. 4–2).

At time \( t = 0 \), the body is placed into a medium at temperature \( T_m \), and heat transfer takes place between the body and its environment, with a heat transfer coefficient \( h \). For the sake of discussion, we will assume that \( T_m > T_i \), but the analysis is equally valid for the opposite case. We assume lumped system analysis to be applicable, so that the temperature remains uniform within the body at all times and changes with time only, \( T = T(t) \).

During a differential time interval \( dt \), the temperature of the body rises by a differential amount \( dT \). An energy balance of the solid for the time interval \( dt \) can be expressed as

\[
\left( \text{Heat transfer into the body during } dt \right) = \left( \text{The increase in the energy of the body during } dt \right)
\]

or

\[
hA_s(T_m - T) \, dt = mC_p \, dT
\]  

(4-1)

Noting that \( m = \rho V \) and \( dT = d(T - T_m) \) since \( T_m \) = constant, Eq. 4–1 can be rearranged as

\[
\frac{d(T - T_m)}{T - T_m} = - \frac{hA_s}{\rho VC_p} \, dt
\]  

(4-2)

Integrating from \( t = 0 \), at which \( T = T_m \), to any time \( t \), at which \( T = T(t) \), gives

\[
\ln \left( \frac{T(t) - T_m}{T_i - T_m} \right) = - \frac{hA_s}{\rho VC_p} \, t
\]  

(4-3)
Taking the exponential of both sides and rearranging, we obtain

\[ \frac{T(t) - T_a}{T_f - T_a} = e^{-bt} \] (4-4)

where

\[ b = \frac{hA_s}{\rho V C_p} \quad (1/s) \] (4-5)

is a positive quantity whose dimension is \((\text{time})^{-1}\). The reciprocal of \(b\) has time unit (usually s), and is called the **time constant**. Equation 4–4 is plotted in Fig. 4–3 for different values of \(b\). There are two observations that can be made from this figure and the relation above:

1. Equation 4–4 enables us to determine the temperature \(T(t)\) of a body at time \(t\), or alternatively, the time \(t\) required for the temperature to reach a specified value \(T_i\).
2. The temperature of a body approaches the ambient temperature \(T_a\) exponentially. The temperature of the body changes rapidly at the beginning, but rather slowly later on. A large value of \(b\) indicates that the body will approach the environment temperature in a short time. The larger the value of the exponent \(b\), the higher the rate of decay in temperature. Note that \(b\) is proportional to the surface area, but inversely proportional to the mass and the specific heat of the body. This is not surprising since it takes longer to heat or cool a larger mass, especially when it has a large specific heat.

Once the temperature \(T(t)\) at time \(t\) is available from Eq. 4–4, the rate of convection heat transfer between the body and its environment at that time can be determined from Newton’s law of cooling as

\[ \dot{Q}(t) = hA_s[T(t) - T_a] \quad (\text{W}) \] (4-6)

The **total amount** of heat transfer between the body and the surrounding medium over the time interval \(t = 0\) to \(t\) is simply the change in the energy content of the body:

\[ Q = mC_p[T(t) - T_i] \quad (\text{kJ}) \] (4-7)

The amount of heat transfer reaches its **upper limit** when the body reaches the surrounding temperature \(T_a\). Therefore, the **maximum** heat transfer between the body and its surroundings is (Fig. 4–4)

\[ Q_{\text{max}} = mC_p(T_a - T_i) \quad (\text{kJ}) \] (4-8)

We could also obtain this equation by substituting the \(T(t)\) relation from Eq. 4–4 into the \(\dot{Q}(t)\) relation in Eq. 4–6 and integrating it from \(t = 0\) to \(t \to \infty\).

**Criteria for Lumped System Analysis**

The lumped system analysis certainly provides great convenience in heat transfer analysis, and naturally we would like to know when it is appropriate
to use it. The first step in establishing a criterion for the applicability of the lumped system analysis is to define a **characteristic length** as

$$L_c = \frac{V}{A_i}$$

and a **Biot number** $Bi$ as

$$Bi = \frac{hL_c}{k}$$ (4.9)

It can also be expressed as (Fig. 4–5)

$$Bi = \frac{h}{k/L_c} \frac{\Delta T}{\Delta T} = \frac{\text{Convection at the surface of the body}}{\text{Conduction within the body}}$$

or

$$Bi = \frac{L_c/k}{1/h} = \frac{\text{Conduction resistance within the body}}{\text{Convection resistance at the surface of the body}}$$

When a solid body is being heated by the hotter fluid surrounding it (such as a potato being baked in an oven), heat is first **convected** to the body and subsequently **conducted** within the body. The Biot number is the **ratio** of the internal resistance of a body to heat conduction to its external resistance to heat convection. Therefore, a small Biot number represents small resistance to heat conduction, and thus small temperature gradients within the body.

**Lumped system analysis** assumes a **uniform** temperature distribution throughout the body, which will be the case only when the thermal resistance of the body to heat conduction (the **conduction resistance**) is zero. Thus, lumped system analysis is **exact** when $Bi = 0$ and **approximate** when $Bi > 0$. Of course, the smaller the Biot number, the more accurate the lumped system analysis. Then the question we must answer is, How much accuracy are we willing to sacrifice for the convenience of the lumped system analysis?

Before answering this question, we should mention that a 20 percent uncertainty in the convection heat transfer coefficient $h$ in most cases is considered “normal” and “expected.” Assuming $h$ to be **constant** and **uniform** is also an approximation of questionable validity, especially for irregular geometries. Therefore, in the absence of sufficient experimental data for the specific geometry under consideration, we cannot claim our results to be better than ±20 percent, even when $Bi = 0$. This being the case, introducing another source of uncertainty in the problem will hardly have any effect on the overall uncertainty, provided that it is minor. It is generally accepted that lumped system analysis is **applicable** if

$$Bi \leq 0.1$$

When this criterion is satisfied, the temperatures within the body relative to the surroundings (i.e., $T - T_s$) remain within 5 percent of each other even for well-rounded geometries such as a spherical ball. Thus, when $Bi < 0.1$, the variation of temperature with location within the body will be slight and can reasonably be approximated as being uniform.
The first step in the application of lumped system analysis is the calculation of the Biot number, and the assessment of the applicability of this approach. One may still wish to use lumped system analysis even when the criterion $\text{Bi} < 0.1$ is not satisfied, if high accuracy is not a major concern.

Note that the Biot number is the ratio of the convection at the surface to conduction within the body, and this number should be as small as possible for lumped system analysis to be applicable. Therefore, small bodies with high thermal conductivity are good candidates for lumped system analysis, especially when they are in a medium that is a poor conductor of heat (such as air or another gas) and motionless. Thus, the hot small copper ball placed in quiescent air, discussed earlier, is most likely to satisfy the criterion for lumped system analysis (Fig. 4-6).

Some Remarks on Heat Transfer in Lumped Systems

To understand the heat transfer mechanism during the heating or cooling of a solid by the fluid surrounding it, and the criterion for lumped system analysis, consider this analogy (Fig. 4-7). People from the mainland are to go by boat to an island whose entire shore is a harbor, and from the harbor to their destinations on the island by bus. The overcrowding of people at the harbor depends on the boat traffic to the island and the ground transportation system on the island. If there is an excellent ground transportation system with plenty of buses, there will be no overcrowding at the harbor, especially when the boat traffic is light. But when the opposite is true, there will be a huge overcrowding at the harbor, creating a large difference between the populations at the harbor and inland. The chance of overcrowding is much lower in a small island with plenty of fast buses.

In heat transfer, a poor ground transportation system corresponds to poor heat conduction in a body, and overcrowding at the harbor to the accumulation of heat and the subsequent rise in temperature near the surface of the body relative to its inner parts. Lumped system analysis is obviously not applicable when there is overcrowding at the surface. Of course, we have disregarded radiation in this analogy and thus the air traffic to the island. Like passengers at the harbor, heat changes vehicles at the surface from convection to conduction. Noting that a surface has zero thickness and thus cannot store any energy, heat reaching the surface of a body by convection must continue its journey within the body by conduction.

Consider heat transfer from a hot body to its cooler surroundings. Heat will be transferred from the body to the surrounding fluid as a result of a temperature difference. But this energy will come from the region near the surface, and thus the temperature of the body near the surface will drop. This creates a temperature gradient between the inner and outer regions of the body and initiates heat flow by conduction from the interior of the body toward the outer surface.

When the convection heat transfer coefficient $h$ and thus convection heat transfer from the body are high, the temperature of the body near the surface will drop quickly (Fig. 4-8). This will create a larger temperature difference between the inner and outer regions unless the body is able to transfer heat from the inner to the outer regions just as fast. Thus, the magnitude of the maximum temperature difference within the body depends strongly on the ability of a body to conduct heat toward its surface relative to the ability of

![Figure 4-6](image-url)

Small bodies with high thermal conductivities and low convection coefficients are most likely to satisfy the criterion for lumped system analysis.

![Figure 4-7](image-url)

Analogy between heat transfer to a solid and passenger traffic to an island.

![Figure 4-8](image-url)

When the convection coefficient $h$ is high and $k$ is low, large temperature differences occur between the inner and outer regions of a large solid.
the surrounding medium to convect this heat away from the surface. The Biot number is a measure of the relative magnitudes of these two competing effects.

Recall that heat conduction in a specified direction \( n \) per unit surface area is expressed as \( \dot{q} = -k \frac{\partial T}{\partial n} \), where \( \frac{\partial T}{\partial n} \) is the temperature gradient and \( k \) is the thermal conductivity of the solid. Thus, the temperature distribution in the body will be uniform only when its thermal conductivity is infinite, and no such material is known to exist. Therefore, temperature gradients and thus temperature differences must exist within the body, no matter how small, in order for heat conduction to take place. Of course, the temperature gradient and the thermal conductivity are inversely proportional for a given heat flux. Therefore, the larger the thermal conductivity, the smaller the temperature gradient.

### EXAMPLE 4–1 Temperature Measurement by Thermocouples

The temperature of a gas stream is to be measured by a thermocouple whose junction can be approximated as a 1-mm-diameter sphere, as shown in Fig. 4–9. The properties of the junction are \( k = 35 \text{ W/m} \cdot \text{°C} \), \( \rho = 8500 \text{ kg/m}^3 \), and \( C_p = 320 \text{ J/kg} \cdot \text{°C} \), and the convection heat transfer coefficient between the junction and the gas is \( h = 210 \text{ W/m}^2 \cdot \text{°C} \). Determine how long it will take for the thermocouple to read 99 percent of the initial temperature difference.

**SOLUTION**  The temperature of a gas stream is to be measured by a thermocouple. The time it takes to register 99 percent of the initial \( \Delta T \) is to be determined.

**Assumptions**
1. The junction is spherical in shape with a diameter of \( D = 0.001 \text{ m} \).
2. The thermal properties of the junction and the heat transfer coefficient are constant.
3. Radiation effects are negligible.

**Properties**  The properties of the junction are given in the problem statement.

**Analysis**  The characteristic length of the junction is

\[
L_c = \frac{V}{A} = \frac{\frac{4}{3} \pi D^3}{\pi D^2} = \frac{1}{6} D = \frac{1}{6} (0.001 \text{ m}) = 1.67 \times 10^{-4} \text{ m}
\]

Then the Biot number becomes

\[
\text{Bi} = \frac{hL_c}{k} = \frac{(210 \text{ W/m}^2 \cdot \text{°C})(1.67 \times 10^{-4} \text{ m})}{35 \text{ W/m} \cdot \text{°C}} = 0.001 < 0.1
\]

Therefore, lumped system analysis is applicable, and the error involved in this approximation is negligible.

In order to read 99 percent of the initial temperature difference \( T_j - T_w \) between the junction and the gas, we must have

\[
\frac{T(t) - T_w}{T_j - T_w} = 0.01
\]

For example, when \( T_j = 0 \text{°C} \) and \( T_w = 100 \text{°C} \), a thermocouple is considered to have read 99 percent of this applied temperature difference when its reading indicates \( T(t) = 99 \text{°C} \).
The value of the exponent $b$ is

$$b = \frac{hA_e}{\rho c_p V} = \frac{h}{\rho c_p L} = \frac{210 \text{ W/m}^2 \cdot \text{C}}{(8500 \text{ kg/m}^3)(320 \text{ J/kg} \cdot \text{C})(1.67 \times 10^{-4} \text{ m})} = 0.462 \text{ s}^{-1}$$

We now substitute these values into Eq. 4–4 and obtain

$$\frac{T(t) - T_a}{T_i - T_a} = e^{-bt} \rightarrow 0.01 = e^{-(0.462 \text{ s}^{-1})t}$$

which yields

$$t = 10 \text{ s}$$

Therefore, we must wait at least 10 s for the temperature of the thermocouple junction to approach within 1 percent of the initial junction-gas temperature difference.

**Discussion** Note that conduction through the wires and radiation exchange with the surrounding surfaces will affect the result, and should be considered in a more refined analysis.

---

**EXAMPLE 4–2 Predicting the Time of Death**

A person is found dead at 5 PM in a room whose temperature is 20°C. The temperature of the body is measured to be 25°C when found, and the heat transfer coefficient is estimated to be $h = 8 \text{ W/m}^2 \cdot \text{C}$. Modeling the body as a 30-cm-diameter, 1.70-m-long cylinder, estimate the time of death of that person (Fig. 4–10).

**SOLUTION** A body is found while still warm. The time of death is to be estimated.

**Assumptions** 1 The body can be modeled as a 30-cm-diameter, 1.70-m-long cylinder. 2 The thermal properties of the body and the heat transfer coefficient are constant. 3 The radiation effects are negligible. 4 The person was healthy(!) when he or she died with a body temperature of 37°C.

**Properties** The average human body is 72 percent water by mass, and thus we can assume the body to have the properties of water at the average temperature of $(37 + 25)/2 = 31°C$; $k = 0.617 \text{ W/m} \cdot \text{C}$, $\rho = 996 \text{ kg/m}^3$, and $C_p = 4178 \text{ J/kg} \cdot \text{C}$ (Table A-9).

**Analysis** The characteristic length of the body is

$$L_c = \frac{V}{A_e} = \frac{\pi r_o^2 L}{2\pi r_o L + 2\pi r_o^2} = \frac{\pi(0.15 \text{ m})^2(1.7 \text{ m})}{2\pi(0.15 \text{ m})(1.7 \text{ m}) + 2\pi(0.15 \text{ m})^2} = 0.0689 \text{ m}$$

Then the Biot number becomes

$$Bi = \frac{hL_c}{k} = \frac{(8 \text{ W/m}^2 \cdot \text{C})(0.0689 \text{ m})}{0.617 \text{ W/m} \cdot \text{C}} = 0.89 > 0.1$$

**FIGURE 4–10** Schematic for Example 4–2.
In Section 4–1, we considered bodies in which the variation of temperature within the body was negligible; that is, bodies that remain nearly isothermal during a process. Relatively small bodies of highly conductive materials approximate this behavior. In general, however, the temperature within a body will change from point to point as well as with time. In this section, we consider the variation of temperature with time and position in one-dimensional problems such as those associated with a large plane wall, a long cylinder, and a sphere.

Consider a plane wall of thickness $2L$, a long cylinder of radius $r_o$, and a sphere of radius $r_o$ initially at a uniform temperature $T_i$, as shown in Fig. 4–11. At time $t = 0$, each geometry is placed in a large medium that is at a constant temperature $T_w$ and kept in that medium for $t > 0$. Heat transfer takes place between these bodies and their environments by convection with a uniform and constant heat transfer coefficient $h$. Note that all three cases possess geometric and thermal symmetry: the plane wall is symmetric about its center plane ($x = 0$), the cylinder is symmetric about its centerline ($r = 0$), and the sphere is symmetric about its center point ($r = 0$). We neglect radiation heat transfer between these bodies and their surrounding surfaces, or incorporate the radiation effect into the convection heat transfer coefficient $h$.

The variation of the temperature profile with time in the plane wall is illustrated in Fig. 4–12. When the wall is first exposed to the surrounding medium at $T_w < T_i$ at $t = 0$, the entire wall is at its initial temperature $T_i$. But the wall temperature at and near the surfaces starts to drop as a result of heat transfer from the wall to the surrounding medium. This creates a temperature
gradient in the wall and initiates heat conduction from the inner parts of the wall toward its outer surfaces. Note that the temperature at the center of the wall remains at $T_i$ until $t = t_2$, and that the temperature profile within the wall remains symmetric at all times about the center plane. The temperature profile gets flatter and flatter as time passes as a result of heat transfer, and eventually becomes uniform at $T = T \infty$. That is, the wall reaches thermal equilibrium with its surroundings. At that point, the heat transfer stops since there is no longer a temperature difference. Similar discussions can be given for the long cylinder or sphere.

The formulation of the problems for the determination of the one-dimensional transient temperature distribution $T(x, t)$ in a wall results in a partial differential equation, which can be solved using advanced mathematical techniques. The solution, however, normally involves infinite series, which are inconvenient and time-consuming to evaluate. Therefore, there is clear motivation to present the solution in tabular or graphical form. However, the solution involves the parameters $x, L, t, k, h, T_i$, and $T \infty$, which are too many to make any graphical presentation of the results practical. In order to reduce the number of parameters, we nondimensionalize the problem by defining the following dimensionless quantities:

Dimensionless temperature: $\theta(x, t) = \frac{T(x, t) - T_i}{T_i - T_\infty}$

Dimensionless distance from the center: $X = \frac{x}{L}$

Dimensionless heat transfer coefficient: $Bi = \frac{hL}{k}$ (Biot number)

Dimensionless time: $\tau = \frac{at}{L^2}$ (Fourier number)

The nondimensionalization enables us to present the temperature in terms of three parameters only: $X, Bi,$ and $\tau$. This makes it practical to present the solution in graphical form. The dimensionless quantities defined above for a plane wall can also be used for a cylinder or sphere by replacing the space variable $x$ by $r$ and the half-thickness $L$ by the outer radius $r_o$. Note that the characteristic length in the definition of the Biot number is taken to be the...
half-thickness \( L \) for the plane wall, and the radius \( r_o \) for the long cylinder and sphere instead of \( V/A \) used in lumped system analysis.

The one-dimensional transient heat conduction problem just described can be solved exactly for any of the three geometries, but the solution involves infinite series, which are difficult to deal with. However, the terms in the solutions converge rapidly with increasing time, and for \( \tau > 0.2 \), keeping the first term and neglecting all the remaining terms in the series results in an error under 2 percent. We are usually interested in the solution for times with \( \tau > 0.2 \), and thus it is very convenient to express the solution using this one-term approximation, given as

\[
\text{Plane wall: } \theta(x, t)_{\text{wall}} = \frac{T(x, t) - T_w}{T_i - T_w} = A_1 e^{-\lambda_1^2 \frac{x^2}{L^2}}, \quad \tau > 0.2 \tag{4-10}
\]

\[
\text{Cylinder: } \theta(r, t)_{\text{cyl}} = \frac{T(r, t) - T_w}{T_i - T_w} = A_1 e^{-\lambda_1^2 \frac{r^2}{r_o^2}}, \quad \tau > 0.2 \tag{4-11}
\]

\[
\text{Sphere: } \theta(r, t)_{\text{sph}} = \frac{T(r, t) - T_w}{T_i - T_w} = A_1 e^{-\lambda_1^2 \frac{r^2}{r_o^2}}, \quad \tau > 0.2 \tag{4-12}
\]

where the constants \( A_1 \) and \( \lambda_1 \) are functions of the Bi number only, and their values are listed in Table 4–1 against the Bi number for all three geometries. The function \( J_0 \) is the zeroth-order Bessel function of the first kind, whose value can be determined from Table 4–2. Noting that \( \cos(0) = J_0(0) = 1 \) and the limit of \( \sin(x)/x \) is also 1, these relations simplify to the next ones at the center of a plane wall, cylinder, or sphere:

\[
\text{Center of plane wall } (x = 0): \quad \theta_{0, \text{wall}} = \frac{T_o - T_w}{T_i - T_w} = A_1 e^{-\lambda_1^2 \tau} \tag{4-13}
\]

\[
\text{Center of cylinder } (r = 0): \quad \theta_{0, \text{cyl}} = \frac{T_o - T_w}{T_i - T_w} = A_1 e^{-\lambda_1^2 \tau} \tag{4-14}
\]

\[
\text{Center of sphere } (r = 0): \quad \theta_{0, \text{sph}} = \frac{T_o - T_w}{T_i - T_w} = A_1 e^{-\lambda_1^2 \tau} \tag{4-15}
\]

Once the Bi number is known, the above relations can be used to determine the temperature anywhere in the medium. The determination of the constants \( A_1 \) and \( \lambda_1 \) usually requires interpolation. For those who prefer reading charts to interpolating, the relations above are plotted and the one-term approximation solutions are presented in graphical form, known as the transient temperature charts. Note that the charts are sometimes difficult to read, and they are subject to reading errors. Therefore, the relations above should be preferred to the charts.

The transient temperature charts in Figs. 4–13, 4–14, and 4–15 for a large plane wall, long cylinder, and sphere were presented by M. P. Heisler in 1947 and are called Heisler charts. They were supplemented in 1961 with transient heat transfer charts by H. Gröber. There are three charts associated with each geometry: the first chart is to determine the temperature \( T_o \) at the center of the geometry at a given time \( t \). The second chart is to determine the temperature at other locations at the same time in terms of \( T_o \). The third chart is to determine the total amount of heat transfer up to the time \( t \). These plots are valid for \( \tau > 0.2 \).
Note that the case \( 1/\text{Bi} = k/L = 0 \) corresponds to \( h \rightarrow \infty \), which corresponds to the case of **specified surface temperature** \( T_\infty \). That is, the case in which the surfaces of the body are suddenly brought to the temperature \( T_\infty \) at \( t = 0 \) and kept at \( T_\infty \) at all times can be handled by setting \( h \) to infinity (Fig. 4–16).

The temperature of the body changes from the initial temperature \( T_i \) to the temperature of the surroundings \( T_\infty \) at the end of the transient heat conduction process. Thus, the **maximum** amount of heat that a body can gain (or lose if \( T_i > T_\infty \)) is simply the **change** in the **energy content** of the body. That is,

\[
Q_{\text{max}} = mC_p(T_\infty - T_i) = \rho V C_p(T_\infty - T_i) \quad (\text{kJ}) \tag{4-16}
\]
where $m$ is the mass, $V$ is the volume, $\rho$ is the density, and $C_p$ is the specific heat of the body. Thus, $Q_{\text{max}}$ represents the amount of heat transfer for $t \to \infty$.

The amount of heat transfer $Q$ at a finite time $t$ will obviously be less than this.

---

**FIGURE 4–13**

Transient temperature and heat transfer charts for a plane wall of thickness $2L$ initially at a uniform temperature $T_i$, subjected to convection from both sides to an environment at temperature $T_e$ with a convection coefficient of $h$.

(a) Midplane temperature (from M. P. Heisler)

(b) Temperature distribution (from M. P. Heisler)

(c) Heat transfer (from H. Gröber et al.)
Transient temperature and heat transfer charts for a long cylinder of radius $r_o$ initially at a uniform temperature $T_i$ subjected to convection from all sides to an environment at temperature $T_w$ with a convection coefficient of $h$. 

\[ \theta = \frac{T - T_w}{T_i - T_w} \]

\[ \theta = \frac{1}{\text{Bi}^2} - \frac{k}{hr_o} \]

\[ \frac{Q}{Q_{\text{max}}} \]

\[ \text{Bi}^2 \tau = \frac{h^2 \alpha t}{k^2} \]

\[ \tau = \frac{\alpha t}{r_o^2} \]
maximum. The ratio $Q/Q_{\text{max}}$ is plotted in Figures 4–13c, 4–14c, and 4–15c against the variables $\text{Bi}$ and $h^2\alpha\tau/k^2$ for the large plane wall, long cylinder, and
sphere, respectively. Note that once the fraction of heat transfer \( \frac{Q}{Q_{\text{max}}} \) has been determined from these charts for the given \( t \), the actual amount of heat transfer by that time can be evaluated by multiplying this fraction by \( Q_{\text{max}} \). A negative sign for \( Q_{\text{max}} \) indicates that heat is leaving the body (Fig. 4–17).

The fraction of heat transfer can also be determined from these relations, which are based on the one-term approximations already discussed:

\[
\text{Plane wall: } Q/Q_{\text{max/wall}} = 1 - \frac{\sin \lambda_1}{\lambda_1} \tag{4-17}
\]

\[
\text{Cylinder: } Q/Q_{\text{max/cyl}} = 1 - \frac{\lambda_1}{\lambda_1} J_1(\lambda_1) \tag{4-18}
\]

\[
\text{Sphere: } Q/Q_{\text{max/sph}} = 1 - \frac{\sin \lambda_1 - \lambda_1 \cos \lambda_1}{\lambda_1^2} \tag{4-19}
\]

The use of the Heisler/Gröber charts and the one-term solutions already discussed is limited to the conditions specified at the beginning of this section: the body is initially at a uniform temperature, the temperature of the medium surrounding the body and the convection heat transfer coefficient are constant and uniform, and there is no energy generation in the body.

We discussed the physical significance of the Biot number earlier and indicated that it is a measure of the relative magnitudes of the two heat transfer mechanisms: convection at the surface and conduction through the solid. A small value of Bi indicates that the inner resistance of the body to heat conduction is small relative to the resistance to convection between the surface and the fluid. As a result, the temperature distribution within the solid becomes fairly uniform, and lumped system analysis becomes applicable. Recall that when \( \text{Bi} < 0.1 \), the error in assuming the temperature within the body to be uniform is negligible.

To understand the physical significance of the Fourier number \( \tau \), we express it as (Fig. 4–18)

\[
\tau = \frac{\rho L^2 (1/L) \Delta T}{\Delta T} = \frac{kL^2}{\rho C_p L^3 / \nu} = \frac{\text{The rate at which heat is conducted across } L \text{ of a body of volume } L^3}{\text{The rate at which heat is stored in a body of volume } L^3} \tag{4-20}
\]

Therefore, the Fourier number is a measure of heat conducted through a body relative to heat stored. Thus, a large value of the Fourier number indicates faster propagation of heat through a body.

Perhaps you are wondering about what constitutes an infinitely large plate or an infinitely long cylinder. After all, nothing in this world is infinite. A plate whose thickness is small relative to the other dimensions can be modeled as an infinitely large plate, except very near the outer edges. But the edge effects on large bodies are usually negligible, and thus a large plane wall such as the wall of a house can be modeled as an infinitely large wall for heat transfer purposes. Similarly, a long cylinder whose diameter is small relative to its length can be analyzed as an infinitely long cylinder. The use of the transient temperature charts and the one-term solutions is illustrated in the following examples.
**EXAMPLE 4–3  Boiling Eggs**

An ordinary egg can be approximated as a 5-cm-diameter sphere (Fig. 4–19). The egg is initially at a uniform temperature of 5°C and is dropped into boiling water at 95°C. Taking the convection heat transfer coefficient to be \( h = 1200 \text{ W/m}^2 \cdot ^\circ\text{C} \), determine how long it will take for the center of the egg to reach 70°C.

**SOLUTION**  An egg is cooked in boiling water. The cooking time of the egg is to be determined.

**Assumptions**  1 The egg is spherical in shape with a radius of \( r_0 = 2.5 \text{ cm} \).  
2 Heat conduction in the egg is one-dimensional because of thermal symmetry about the midpoint.  
3 The thermal properties of the egg and the heat transfer coefficient are constant.  
4 The Fourier number is \( \tau > 0.2 \) so that the one-term approximate solutions are applicable.

**Properties**  The water content of eggs is about 74 percent, and thus the thermal conductivity and diffusivity of eggs can be approximated by those of water at the average temperature of \( \left( \frac{5 + 70}{2} \right) = 37.5°C \); \( k = 0.627 \text{ W/m} \cdot ^\circ\text{C} \) and \( \alpha = \frac{k}{\rho C_p} = 0.151 \times 10^{-6} \text{ m}^2/\text{s} \) (Table A-9).

**Analysis**  The temperature within the egg varies with radial distance as well as time, and the temperature at a specified location at a given time can be determined from the Heisler charts or the one-term solutions. Here we will use the latter to demonstrate their use. The Biot number for this problem is

\[
\text{Bi} = \frac{h r_0}{k} = \frac{1200 \text{ W/m}^2 \cdot ^\circ\text{C} (0.025 \text{ m})}{0.627 \text{ W/m} \cdot ^\circ\text{C}} = 47.8
\]

which is much greater than 0.1, and thus the lumped system analysis is not applicable. The coefficients \( \lambda_1 \) and \( A_1 \) for a sphere corresponding to this Bi are, from Table 4–1,

\[
\lambda_1 = 3.0753, \quad A_1 = 1.9958
\]

Substituting these and other values into Eq. 4–15 and solving for \( \tau \) gives

\[
\frac{T_c - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1 \tau} \quad \longrightarrow \quad \frac{70 - 95}{5 - 95} = 1.9958 e^{-\left(3.0753\right)\tau} \quad \longrightarrow \quad \tau = 0.209
\]

which is greater than 0.2, and thus the one-term solution is applicable with an error of less than 2 percent. Then the cooking time is determined from the definition of the Fourier number to be

\[
t = \frac{\tau r_0^2}{\alpha} = \frac{(0.209)(0.025 \text{ m})^2}{0.151 \times 10^{-6} \text{ m}^2/\text{s}} = 865 \text{ s} \approx 14.4 \text{ min}
\]

Therefore, it will take about 15 min for the center of the egg to be heated from 5°C to 70°C.

**Discussion**  Note that the Biot number in lumped system analysis was defined differently as \( \text{Bi} = \frac{h r_0}{k} = \frac{h(r/3)}{k} \). However, either definition can be used in determining the applicability of the lumped system analysis unless \( \text{Bi} \approx 0.1 \).
EXAMPLE 4–4 Heating of Large Brass Plates in an Oven

In a production facility, large brass plates of 4 cm thickness that are initially at a uniform temperature of 20°C are heated by passing them through an oven that is maintained at 500°C (Fig. 4–20). The plates remain in the oven for a period of 7 min. Taking the combined convection and radiation heat transfer coefficient to be \( h = 120 \text{ W/m}^2 \cdot ^\circ \text{C} \), determine the surface temperature of the plates when they come out of the oven.

SOLUTION Large brass plates are heated in an oven. The surface temperature of the plates leaving the oven is to be determined.

Assumptions 1 Heat conduction in the plate is one-dimensional since the plate is large relative to its thickness and there is thermal symmetry about the center plane. 2 The thermal properties of the plate and the heat transfer coefficient are constant. 3 The Fourier number is \( \tau > 0.2 \) so that the one-term approximate solutions are applicable.

Properties The properties of brass at room temperature are \( k = 110 \text{ W/m} \cdot ^\circ \text{C} \), \( \rho = 8530 \text{ kg/m}^3 \), \( C_p = 380 \text{ J/kg} \cdot ^\circ \text{C} \), and \( \alpha = 3.39 \times 10^{-6} \text{ m}^2/\text{s} \) (Table A-3). More accurate results are obtained by using properties at average temperature.

Analysis The temperature at a specified location at a given time can be determined from the Heisler charts or one-term solutions. Here we will use the charts to demonstrate their use. Noting that the half-thickness of the plate is \( L = 0.02 \text{ m} \), from Fig. 4–13 we have

\[
\frac{1}{\text{Bi}} = \frac{k}{hL} = \frac{100 \text{ W/m} \cdot ^\circ \text{C}}{(120 \text{ W/m}^2 \cdot ^\circ \text{C})(0.02 \text{ m})} = 45.8
\]

\[
\frac{T_o - T_s}{T_i - T_s} = 0.46
\]

Also,

\[
\frac{1}{\text{Bi}} = \frac{k}{hL} = 45.8
\]

\[
\frac{\alpha}{L^2} = \frac{33.9 \times 10^{-6} \text{ m}^2/\text{s}}{(0.02 \text{ m})^2} = 0.46
\]

Therefore,

\[
\frac{T - T_s}{T_i - T_s} = \frac{T - T_s}{T_o - T_s} \frac{T_o - T_s}{T_i - T_s} = 0.46 \times 0.99 = 0.455
\]

and

\[
T = T_s + 0.455(T_i - T_s) = 500 + 0.455(20 - 500) = 282^\circ \text{C}
\]

Therefore, the surface temperature of the plates will be 282°C when they leave the oven.

Discussion We notice that the Biot number in this case is \( \text{Bi} = 1/45.8 = 0.022 \), which is much less than 0.1. Therefore, we expect the lumped system analysis to be applicable. This is also evident from \( T(T_i - T_s)/(T_o - T_i) \) = 0.99, which indicates that the temperatures at the center and the surface of the plate relative to the surrounding temperature are within 1 percent of each other.
Noting that the error involved in reading the Heisler charts is typically at least a few percent, the lumped system analysis in this case may yield just as accurate results with less effort.

The heat transfer surface area of the plate is $2A$, where $A$ is the face area of the plate (the plate transfers heat through both of its surfaces), and the volume of the plate is $V = (2L)A$, where $L$ is the half-thickness of the plate. The exponent $b$ used in the lumped system analysis is determined to be

$$b = \frac{hA}{\rho C_p V} = \frac{h(2A)}{\rho C_p (2LA)} = \frac{h}{\rho C_p L} = \frac{120 \text{ W/m}^2 \cdot \text{°C}}{\left(\frac{8530 \text{ kg/m}^3)(380 \text{ J/kg} \cdot \text{°C})(0.02 \text{ m})}{1005\text{ s}}\right) = 0.00185 \text{ s}^{-1}$$

Then the temperature of the plate at $t = 7 \text{ min} = 420 \text{ s}$ is determined from

$$\frac{T(t) - T_w}{T_i - T_w} = e^{-bt} \quad \Rightarrow \quad \frac{T(t) - 500}{20 - 500} = e^{-0.00185 \text{ s}^{-1}(420 \text{ s})}$$

It yields

$$T(t) = 279^\circ \text{C}$$

which is practically identical to the result obtained above using the Heisler charts. Therefore, we can use lumped system analysis with confidence when the Biot number is sufficiently small.

---

**EXAMPLE 4-5  Cooling of a Long Stainless Steel Cylindrical Shaft**

A long 20-cm-diameter cylindrical shaft made of stainless steel 304 comes out of an oven at a uniform temperature of $600^\circ \text{C}$ (Fig. 4-21). The shaft is then allowed to cool slowly in an environment chamber at $200^\circ \text{C}$ with an average heat transfer coefficient of $h = 80 \text{ W/m}^2 \cdot \text{°C}$. Determine the temperature at the center of the shaft 45 min after the start of the cooling process. Also, determine the heat transfer per unit length of the shaft during this time period.

**SOLUTION** A long cylindrical shaft at $600^\circ \text{C}$ is allowed to cool slowly. The center temperature and the heat transfer per unit length are to be determined.

**Assumptions** 1 Heat conduction in the shaft is one-dimensional since it is long and it has thermal symmetry about the centerline. 2 The thermal properties of the shaft and the heat transfer coefficient are constant. 3 The Fourier number is $\tau > 0.2$ so that the one-term approximate solutions are applicable.

**Properties** The properties of stainless steel 304 at room temperature are $k = 14.9 \text{ W/m} \cdot \text{°C}$, $\rho = 7900 \text{ kg/m}^3$, $C_p = 477 \text{ J/kg} \cdot \text{°C}$, and $\alpha = 3.95 \times 10^{-6} \text{ m}^2/\text{s}$ (Table A-3). More accurate results can be obtained by using properties at average temperature.

**Analysis** The temperature within the shaft may vary with the radial distance $r$ as well as time, and the temperature at a specified location at a given time can
be determined from the Heisler charts. Noting that the radius of the shaft is 
\( r_o = 0.1 \text{ m} \), from Fig. 4–14 we have

\[
\frac{1}{\text{Bi}} = \frac{k}{h r_o} = \frac{14.9 \text{ W/m} \cdot \text{°C}}{(80 \text{ W/m}^2 \cdot \text{°C})(0.1 \text{ m})} = 1.86
\]

\[
\tau = \frac{\alpha L}{r_o^2} = \frac{(3.95 \times 10^{-6} \text{ m}^2/\text{s})(45 \times 60 \text{ s})}{(0.1 \text{ m})^2} = 1.07
\]

Therefore,

\[
T_o = T_\infty + 0.4(T_i - T_\infty) = 200 + 0.4(600 - 200) = 360^\circ \text{C}
\]

Therefore, the center temperature of the shaft will drop from 600°C to 360°C in 45 min.

To determine the actual heat transfer, we first need to calculate the maximum heat that can be transferred from the cylinder, which is the sensible energy of the cylinder relative to its environment. Taking \( L = 1 \text{ m} \),

\[
m = \rho V = \rho \pi r_o^2 L = (7900 \text{ kg/m}^3)\pi(0.1 \text{ m})^2(1 \text{ m}) = 248.2 \text{ kg}
\]

\[
Q_{\max} = mC_p(T_\infty - T_i) = (248.2 \text{ kg})(0.477 \text{ kJ/kg} \cdot \text{°C})(600 - 200)^\circ \text{C}
\]

\[
= 47,354 \text{ kJ}
\]

The dimensionless heat transfer ratio is determined from Fig. 4–14c for a long cylinder to be

\[
\text{Bi} = \frac{1}{1/\text{Bi}} = \frac{1}{1.86} = 0.537
\]

\[
\frac{h^2\alpha L}{k^2} = \text{Bi}^2\tau = (0.537)^2(1.07) = 0.309
\]

Therefore,

\[
Q = 0.62Q_{\max} = 0.62 \times 47,354 \text{ kJ} = 29,360 \text{ kJ}
\]

which is the total heat transfer from the shaft during the first 45 min of the cooling.

**ALTERNATIVE SOLUTION** We could also solve this problem using the one-term solution relation instead of the transient charts. First we find the Biot number

\[
\text{Bi} = \frac{h r_o}{k} = \frac{(80 \text{ W/m}^2 \cdot \text{°C})(0.1 \text{ m})}{14.9 \text{ W/m} \cdot \text{°C}} = 0.537
\]

The coefficients \( \lambda_1 \) and \( A_1 \) for a cylinder corresponding to this Bi are determined from Table 4–1 to be

\[
\lambda_1 = 0.970, \quad A_1 = 1.122
\]

Substituting these values into Eq. 4–14 gives

\[
\theta_0 = \frac{T_o - T_\infty}{T_i - T_\infty} = A_1 e^{-\lambda_1 \tau} = 1.122e^{-(0.970)^2(1.07)} = 0.41
\]
4–3 TRANSIENT HEAT CONDUCTION IN SEMI-INFINITE SOLIDS

A semi-infinite solid is an idealized body that has a single plane surface and extends to infinity in all directions, as shown in Fig. 4–22. This idealized body is used to indicate that the temperature change in the part of the body in which we are interested (the region close to the surface) is due to the thermal conditions on a single surface. The earth, for example, can be considered to be a semi-infinite medium in determining the variation of temperature near its surface. Also, a thick wall can be modeled as a semi-infinite medium if all we are interested in is the variation of temperature in the region near one of the surfaces, and the other surface is too far to have any impact on the region of interest during the time of observation.

Consider a semi-infinite solid that is at a uniform temperature \( T_i \). At time \( t = 0 \), the surface of the solid at \( x = 0 \) is exposed to convection by a fluid at a constant temperature \( T_w \), with a heat transfer coefficient \( h \). This problem can be formulated as a partial differential equation, which can be solved analytically for the transient temperature distribution \( T(x, t) \). The solution obtained is presented in Fig. 4–23 graphically for the nondimensionalized temperature defined as

\[
1 - 0(x, t) = 1 - \frac{T(x, t) - T_w}{T_s - T_w} = \frac{T(x, t) - T_i}{T_s - T_i} \quad (4-21)
\]

against the dimensionless variable \( x/(2 \sqrt{\alpha t}) \) for various values of the parameter \( h \sqrt{\alpha t}/k \).

Note that the values on the vertical axis correspond to \( x = 0 \), and thus represent the surface temperature. The curve \( h \sqrt{\alpha t}/k = \infty \) corresponds to \( h \rightarrow \infty \), which corresponds to the case of specified temperature \( T_w \) at the surface at \( x = 0 \). That is, the case in which the surface of the semi-infinite body is suddenly brought to temperature \( T_w \) at \( t = 0 \) and kept at \( T_w \) at all times can be handled by setting \( h \) to infinity. The specified surface temperature case is closely
approximated in practice when condensation or boiling takes place on the surface. For a finite heat transfer coefficient $h$, the surface temperature approaches the fluid temperature $T_i$ as the time $t$ approaches infinity.

The exact solution of the transient one-dimensional heat conduction problem in a semi-infinite medium that is initially at a uniform temperature of $T_i$ and is suddenly subjected to convection at time $t = 0$ has been obtained, and is expressed as

$$\frac{T(x, t) - T_i}{T_e - T_i} = \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right) - \exp \left( \frac{hx}{k} + \frac{h^2\alpha t}{k^2} \right) \left[ \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k} \right) \right]$$

(4-22)

where the quantity $\text{erfc} (\xi)$ is the complementary error function, defined as

$$\text{erfc} (\xi) = 1 - \frac{2}{\sqrt{\pi}} \int_0^\xi e^{-u^2} \, du$$

(4-23)

Despite its simple appearance, the integral that appears in the above relation cannot be performed analytically. Therefore, it is evaluated numerically for different values of $\xi$, and the results are listed in Table 4–3. For the special case of $h \to \infty$, the surface temperature $T_s$ becomes equal to the fluid temperature $T_i$, and Eq. 4–22 reduces to

$$\frac{T(x, t) - T_i}{T_e - T_i} = \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right)$$

(4-24)
This solution corresponds to the case when the temperature of the exposed surface of the medium is suddenly raised (or lowered) to $T_s$ at $t = 0$ and is maintained at that value at all times. Although the graphical solution given in Fig. 4–23 is a plot of the exact analytical solution given by Eq. 4–23, it is subject to reading errors, and thus is of limited accuracy.

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<td>1.34</td>
<td>0.05809</td>
<td>1.72</td>
<td>0.01500</td>
</tr>
<tr>
<td>0.22</td>
<td>0.7557</td>
<td>0.60</td>
<td>0.3961</td>
<td>0.98</td>
<td>0.1658</td>
<td>1.36</td>
<td>0.05444</td>
<td>1.74</td>
<td>0.01387</td>
</tr>
<tr>
<td>0.24</td>
<td>0.7343</td>
<td>0.62</td>
<td>0.3806</td>
<td>1.00</td>
<td>0.1573</td>
<td>1.38</td>
<td>0.05098</td>
<td>1.76</td>
<td>0.01281</td>
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<tr>
<td>0.26</td>
<td>0.7131</td>
<td>0.64</td>
<td>0.3654</td>
<td>1.02</td>
<td>0.1492</td>
<td>1.40</td>
<td>0.04772</td>
<td>1.78</td>
<td>0.01183</td>
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<tr>
<td>0.28</td>
<td>0.6921</td>
<td>0.66</td>
<td>0.3506</td>
<td>1.04</td>
<td>0.1413</td>
<td>1.42</td>
<td>0.04462</td>
<td>1.80</td>
<td>0.01091</td>
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<tr>
<td>0.30</td>
<td>0.6714</td>
<td>0.68</td>
<td>0.3362</td>
<td>1.06</td>
<td>0.1339</td>
<td>1.44</td>
<td>0.04170</td>
<td>1.82</td>
<td>0.01006</td>
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<td>0.32</td>
<td>0.6509</td>
<td>0.70</td>
<td>0.3222</td>
<td>1.08</td>
<td>0.1267</td>
<td>1.46</td>
<td>0.03895</td>
<td>1.84</td>
<td>0.00926</td>
</tr>
<tr>
<td>0.34</td>
<td>0.6306</td>
<td>0.72</td>
<td>0.3086</td>
<td>1.10</td>
<td>0.1198</td>
<td>1.48</td>
<td>0.03635</td>
<td>1.86</td>
<td>0.00853</td>
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<td>0.36</td>
<td>0.6107</td>
<td>0.74</td>
<td>0.2953</td>
<td>1.12</td>
<td>0.1132</td>
<td>1.50</td>
<td>0.03390</td>
<td>1.88</td>
<td>0.00784</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Example 4–6 Minimum Burial Depth of Water Pipes to Avoid Freezing</th>
</tr>
</thead>
</table>

In areas where the air temperature remains below 0°C for prolonged periods of time, the freezing of water in underground pipes is a major concern. Fortunately, the soil remains relatively warm during those periods, and it takes weeks for the subfreezing temperatures to reach the water mains in the ground. Thus, the soil effectively serves as an insulation to protect the water from subfreezing temperatures in winter.

The ground at a particular location is covered with snow pack at $-10^\circ$C for a continuous period of three months, and the average soil properties at that location are $k = 0.4$ W/m · °C and $\alpha = 0.15 \times 10^{-6}$ m$^2$/s (Fig. 4–24). Assuming an initial uniform temperature of $15^\circ$C for the ground, determine the minimum burial depth to prevent the water pipes from freezing.

SOLUTION

The water pipes are buried in the ground to prevent freezing. The minimum burial depth at a particular location is to be determined.

Assumptions

1. The temperature in the soil is affected by the thermal conditions at one surface only, and thus the soil can be considered to be a semi-infinite medium with a specified surface temperature of $-10^\circ$C.
2. The thermal properties of the soil are constant.
The transient temperature charts presented earlier can be used to determine the temperature distribution and heat transfer in one-dimensional heat conduction problems associated with a large plane wall, a long cylinder, a sphere, and a semi-infinite medium. Using a superposition approach called the product solution, these charts can also be used to construct solutions for the two-dimensional transient heat conduction problems encountered in geometries such as a short cylinder, a long rectangular bar, or a semi-infinite cylinder or plate, and even three-dimensional problems associated with geometries such as a rectangular prism or a semi-infinite rectangular bar, provided that all surfaces of the solid are subjected to convection to the same fluid at temperature.

**Properties** The properties of the soil are as given in the problem statement.

**Analysis** The temperature of the soil surrounding the pipes will be 0°C after three months in the case of minimum burial depth. Therefore, from Fig. 4–23, we have

\[
\frac{h \sqrt{\alpha t}}{k} = \infty \quad \text{(since } h \to \infty) \\
1 - \frac{T(x, t) - T_s}{T_i - T_s} = 1 - \frac{0 - (-10)}{15 - (-10)} = 0.6 \\
\xi = \frac{x}{2 \sqrt{\alpha t}} = 0.36
\]

We note that

\[
t = (90 \text{ days})(24 \text{ h/day})(3600 \text{ s/h}) = 7.78 \times 10^6 \text{ s}
\]

and thus

\[
x = 2\xi \sqrt{\alpha t} = 2 \times 0.36 \sqrt{(0.15 \times 10^{-6} \text{ m}^2/\text{s})(7.78 \times 10^6 \text{ s})} = 0.77 \text{ m}
\]

Therefore, the water pipes must be buried to a depth of at least 77 cm to avoid freezing under the specified harsh winter conditions.

**ALTERNATIVE SOLUTION** The solution of this problem could also be determined from Eq. 4–24:

\[
\frac{T(x, t) - T_i}{T_s - T_i} = \text{erfc} \left( \frac{x}{2 \sqrt{\alpha t}} \right) \quad \rightarrow \quad \frac{0 - 15}{-10 - 15} = \text{erfc} \left( \frac{x}{2 \sqrt{\alpha t}} \right) = 0.60
\]

The argument that corresponds to this value of the complementary error function is determined from Table 4–3 to be \(\xi = 0.37\). Therefore,

\[
x = 2\xi \sqrt{\alpha t} = 2 \times 0.37 \sqrt{(0.15 \times 10^{-6} \text{ m}^2/\text{s})(7.78 \times 10^6 \text{ s})} = 0.80 \text{ m}
\]

Again, the slight difference is due to the reading error of the chart.

**4–4 TRANSIENT HEAT CONDUCTION IN MULTIDIMENSIONAL SYSTEMS**

The transient temperature charts presented earlier can be used to determine the temperature distribution and heat transfer in one-dimensional heat conduction problems associated with a large plane wall, a long cylinder, a sphere, and a semi-infinite medium. Using a superposition approach called the product solution, these charts can also be used to construct solutions for the two-dimensional transient heat conduction problems encountered in geometries such as a short cylinder, a long rectangular bar, or a semi-infinite cylinder or plate, and even three-dimensional problems associated with geometries such as a rectangular prism or a semi-infinite rectangular bar, provided that all surfaces of the solid are subjected to convection to the same fluid at temperature.

**FIGURE 4–25** The temperature in a short cylinder exposed to convection from all surfaces varies in both the radial and axial directions, and thus heat is transferred in both directions.
A long solid bar of rectangular profile $a \times b$ is the intersection of two plane walls of thicknesses $a$ and $b$.

**FIGURE 4–27**

A long solid bar of rectangular profile $a \times b$ is the intersection of two plane walls of thicknesses $a$ and $b$.

For example, the solution for a long solid bar whose cross section is an $a \times b$ rectangle is the intersection of the two infinite plane walls of thicknesses $a$ and $b$, as shown in Fig. 4–27, and thus the transient temperature distribution for this rectangular bar can be expressed as

$$
\frac{(T(x, y, t) - T_\infty)}{T_1 - T_\infty} = \theta_{\text{wall}}(x, t)\theta_{\text{wall}}(y, t)
$$

(4-27)

The proper forms of the product solutions for some other geometries are given in Table 4–4. It is important to note that the $x$-coordinate is measured from the surface in a semi-infinite solid, and from the midplane in a plane wall. The radial distance $r$ is always measured from the centerline.

Note that the solution of a two-dimensional problem involves the product of two one-dimensional solutions, whereas the solution of a three-dimensional problem involves the product of three one-dimensional solutions.

A modified form of the product solution can also be used to determine the total transient heat transfer to or from a multidimensional geometry by using the one-dimensional values, as shown by L. S. Langston in 1982. The
<table>
<thead>
<tr>
<th>Table 4-4</th>
<th>Multidimensional solutions expressed as products of one-dimensional solutions for bodies that are initially at a uniform temperature $T_i$ and exposed to convection from all surfaces to a medium at $T_a$.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Infinite cylinder</strong></td>
<td>$\theta(x,t) = \theta_{cy} (x,t)$</td>
</tr>
<tr>
<td><strong>Semi-infinite cylinder</strong></td>
<td>$\theta(x,y,z,t) = \theta_{cy} (x,t) \theta_{semi-inf} (y,t)$</td>
</tr>
<tr>
<td><strong>Short cylinder</strong></td>
<td>$\theta(x,r,t) = \theta_{cy} (r,t) \theta_{wall} (x,t)$</td>
</tr>
<tr>
<td><strong>Semi-infinite medium</strong></td>
<td>$\theta(x,t) = \theta_{semi-inf} (x,t)$</td>
</tr>
<tr>
<td><strong>Quarter-infinite medium</strong></td>
<td>$\theta(x,y,z,t) = \theta_{semi-inf} (x,t) \theta_{semi-inf} (y,t) \theta_{semi-inf} (z,t)$</td>
</tr>
<tr>
<td><strong>Infinite plate (or plane wall)</strong></td>
<td>$\theta(x,t) = \theta_{wall} (x,t)$</td>
</tr>
<tr>
<td><strong>Semi-infinite plate</strong></td>
<td>$\theta(x,y,t) = \theta_{wall} (x,t) \theta_{semi-inf} (y,t)$</td>
</tr>
<tr>
<td><strong>Quarter-infinite plate</strong></td>
<td>$\theta(x,y,z,t) = \theta_{wall} (x,t) \theta_{wall} (y,t) \theta_{semi-inf} (z,t)$</td>
</tr>
<tr>
<td><strong>Infinite rectangular bar</strong></td>
<td>$\theta(x,y,t) = \theta_{wall} (x,t) \theta_{wall} (y,t)$</td>
</tr>
<tr>
<td><strong>Semi-infinite rectangular bar</strong></td>
<td>$\theta(x,y,z,t) = \theta_{wall} (x,t) \theta_{wall} (y,t) \theta_{semi-inf} (z,t)$</td>
</tr>
<tr>
<td><strong>Rectangular parallelepiped</strong></td>
<td>$\theta(x,y,z,t) = \theta_{wall} (x,t) \theta_{wall} (y,t) \theta_{wall} (z,t)$</td>
</tr>
</tbody>
</table>
Transient heat transfer for a two-dimensional geometry formed by the intersection of two one-dimensional geometries 1 and 2 is

\[
\frac{Q}{Q_{\text{max}}^{\text{total, 2D}}} = \left( \frac{Q}{Q_{\text{max}}^1} \right)_1 + \left( \frac{Q}{Q_{\text{max}}^2} \right)_2 \left[ 1 - \left( \frac{Q}{Q_{\text{max}}^1} \right)_1 \right]
\]  

(4-28)

Transient heat transfer for a three-dimensional body formed by the intersection of three one-dimensional bodies 1, 2, and 3 is given by

\[
\frac{Q}{Q_{\text{max}}^{\text{total, 3D}}} = \left( \frac{Q}{Q_{\text{max}}^1} \right)_1 + \left( \frac{Q}{Q_{\text{max}}^2} \right)_2 \left[ 1 - \left( \frac{Q}{Q_{\text{max}}^1} \right)_1 \right] \\
+ \left( \frac{Q}{Q_{\text{max}}^3} \right)_3 \left[ 1 - \left( \frac{Q}{Q_{\text{max}}^1} \right)_1 \right] \left[ 1 - \left( \frac{Q}{Q_{\text{max}}^2} \right)_2 \right]
\]  

(4-29)

The use of the product solution in transient two- and three-dimensional heat conduction problems is illustrated in the following examples.

**EXAMPLE 4–7  Cooling of a Short Brass Cylinder**

A short brass cylinder of diameter \(D = 10\) cm and height \(H = 12\) cm is initially at a uniform temperature \(T_i = 120^\circ\text{C}\). The cylinder is now placed in atmospheric air at 25\(^\circ\text{C}\), where heat transfer takes place by convection, with a heat transfer coefficient of \(h = 60\) W/m\(^2\) \(\cdot\) \(^\circ\text{C}\). Calculate the temperature at (a) the center of the cylinder and (b) the center of the top surface of the cylinder 15 min after the start of the cooling.

**SOLUTION** A short cylinder is allowed to cool in atmospheric air. The temperatures at the centers of the cylinder and the top surface are to be determined.

**Assumptions**

1. Heat conduction in the short cylinder is two-dimensional, and thus the temperature varies in both the axial \(x\)- and the radial \(r\)-directions.
2. The thermal properties of the cylinder and the heat transfer coefficient are constant.
3. The Fourier number is \(\tau > 0.2\) so that the one-term approximate solutions are applicable.

**Properties**  The properties of brass at room temperature are \(k = 110\) W/m \(\cdot\) \(^\circ\text{C}\) and \(\alpha = 33.9 \times 10^{-6}\) m\(^2\)/s (Table A-3). More accurate results can be obtained by using properties at average temperature.

**Analysis**  (a) This short cylinder can physically be formed by the intersection of a long cylinder of radius \(r_o = 5\) cm and a plane wall of thickness \(2L = 12\) cm, as shown in Fig. 4–28. The dimensionless temperature at the center of the plane wall is determined from Figure 4–13a to be

\[
\tau = \frac{aL^2}{\alpha} = \frac{\left(3.39 \times 10^{-5}\right) \text{m}^2/\text{s}(900\text{ s})}{\left(0.06\text{ m}\right)^2} = 8.48
\]

\[
\frac{1}{Bi} = \frac{k}{hL} = \frac{110\text{ W/m} \cdot \text{\(^\circ\text{C}\)}}{(60\text{ W/m}^2 \cdot \text{\(^\circ\text{C}\)}}(0.06\text{ m}) = 30.6
\]

\[
\theta_{\text{wall}}(0, t) = \frac{T(0, t) - T_o}{T_i - T_o} = 0.8
\]
Similarly, at the center of the cylinder, we have

\[
\tau = \frac{a t}{r_o^2} = \frac{(3.39 \times 10^{-2} \text{ m}^2/\text{s})(900 \text{ s})}{(0.05 \text{ m})^2} = 12.2
\]

\[
\frac{1}{\text{Bi}} = \frac{k}{h r_o} = \frac{110 \text{ W/m} \cdot \text{°C}}{(60 \text{ W/m}^2 \cdot \text{°C})(0.05 \text{ m})} = 36.7
\]

Therefore,

\[
\frac{\tau}{T_i - T_\infty} = \frac{\theta_{\text{cyl}}(0, t)}{T_i - T_\infty} = 0.5
\]

\[
\frac{1}{\text{Bi}} \frac{\theta_{\text{wall}}(0, t)}{T_i - T_\infty} = \frac{\theta_{\text{cyl}}(0, t) \theta_{\text{cyl}}(0, t)}{T_i - T_\infty} = 0.5
\]

and

\[
\frac{T_i - T_\infty}{T_i - T_\infty} = 0.4(T_i - T_\infty) = 25 + 0.4(120 - 25) = 63\text{°C}
\]

This is the temperature at the center of the short cylinder, which is also the center of both the long cylinder and the plate.

(b) The center of the top surface of the cylinder is still at the center of the long cylinder \(r = 0\), but at the outer surface of the plane wall \(x = L\). Therefore, we first need to find the surface temperature of the wall. Noting that \(x = L = 0.06 \text{ m}\),

\[
\frac{x}{L} = \frac{0.06 \text{ m}}{0.06 \text{ m}} = 1
\]

\[
\frac{1}{\text{Bi}} = \frac{k}{h L} = \frac{110 \text{ W/m} \cdot \text{°C}}{(60 \text{ W/m}^2 \cdot \text{°C})(0.06 \text{ m})} = 30.6
\]

Then

\[
\theta_{\text{wall}}(L, t) = \frac{T(L, t) - T_\infty}{T_i - T_\infty} = \frac{T(L, t) - T_\infty}{T_i - T_\infty} = 0.98 \times 0.8 = 0.784
\]

Therefore,

\[
\frac{T(L, 0, t) - T_\infty}{T_i - T_\infty} = \frac{T(L, 0, t) - T_\infty}{T_i - T_\infty} = 0.392
\]

and

\[
T(L, 0, t) = T_\infty + 0.392(T_i - T_\infty) = 25 + 0.392(120 - 25) = 62.2\text{°C}
\]

which is the temperature at the center of the top surface of the cylinder.

---

**EXAMPLE 4–8**  **Heat Transfer from a Short Cylinder**

Determine the total heat transfer from the short brass cylinder \((\rho = 8530 \text{ kg/m}^3, C_p = 0.380 \text{ kJ/kg} \cdot \text{°C})\) discussed in Example 4–7.
We first determine the maximum heat that can be transferred from the cylinder, which is the sensible energy content of the cylinder relative to its environment:

\[ m = \rho V = \rho \pi r^2 L = (8530 \text{ kg/m}^3)\pi(0.05 \text{ m})^2(0.06 \text{ m}) = 4.02 \text{ kg} \]

\[ Q_{\text{max}} = mC_p(T_i - T_w) = (4.02 \text{ kg})(0.380 \text{ kJ/kg \cdot } ^\circ \text{C})(120 - 25)^\circ \text{C} = 145.1 \text{ kJ} \]

Then we determine the dimensionless heat transfer ratios for both geometries. For the plane wall, it is determined from Fig. 4–13 to be

\[ \text{Bi} = \frac{1}{1/\text{Bi}} = \frac{1}{30.6} = 0.0327 \]

\[ \frac{h \alpha t}{k} = \text{Bi} \tau = (0.0327)(8.48) = 0.0091 \]

\[ \left( \frac{Q}{Q_{\text{max}}/\text{plane wall}} \right) = 0.23 \]

Similarly, for the cylinder, we have

\[ \text{Bi} = \frac{1}{1/\text{Bi}} = \frac{1}{36.7} = 0.0272 \]

\[ \frac{h \alpha t}{k} = \text{Bi} \tau = (0.0272)(12.2) = 0.0090 \]

\[ \left( \frac{Q}{Q_{\text{max}}/\text{cylinder}} \right) = 0.47 \]

Then the heat transfer ratio for the short cylinder is, from Eq. 4–28,

\[ \left( \frac{Q}{Q_{\text{max}}/\text{short cyl}} \right) = \left( \frac{Q}{Q_{\text{max}}/1} \right) + \left( \frac{Q}{Q_{\text{max}}/2} \right) \left[ 1 - \left( \frac{Q}{Q_{\text{max}}/1} \right) \right] \]

\[ = 0.23 + 0.47(1 - 0.23) = 0.592 \]

Therefore, the total heat transfer from the cylinder during the first 15 min of cooling is

\[ Q = 0.592Q_{\text{max}} = 0.592 \times (145.1 \text{ kJ}) = 85.9 \text{ kJ} \]

---

**EXAMPLE 4–9 Cooling of a Long Cylinder by Water**

A semi-infinite aluminum cylinder of diameter \( D = 20 \text{ cm} \) is initially at a uniform temperature \( T_i = 200^\circ \text{C} \). The cylinder is now placed in water at 15°C where heat transfer takes place by convection, with a heat transfer coefficient of \( h = 120 \text{ W/m}^2 \cdot ^\circ \text{C} \). Determine the temperature at the center of the cylinder 15 cm from the end surface 5 min after the start of the cooling.

**SOLUTION** A semi-infinite aluminum cylinder is cooled by water. The temperature at the center of the cylinder 15 cm from the end surface is to be determined.

**Assumptions** 1 Heat conduction in the semi-infinite cylinder is two-dimensional, and thus the temperature varies in both the axial \( x \)- and the radial \( r \)-directions. 2 The thermal properties of the cylinder and the heat transfer coefficient are constant. 3 The Fourier number is \( \tau > 0.2 \) so that the one-term approximate solutions are applicable.
Properties The properties of aluminum at room temperature are \( k = 237 \) W/m·°C and \( \alpha = 9.71 \times 10^{-6} \) m²/s (Table A-3). More accurate results can be obtained by using properties at average temperature.

Analysis This semi-infinite cylinder can physically be formed by the intersection of an infinite cylinder of radius \( r_o = 10 \) cm and a semi-infinite medium, as shown in Fig. 4–29.

We will solve this problem using the one-term solution relation for the cylinder and the analytic solution for the semi-infinite medium. First we consider the infinitely long cylinder and evaluate the Biot number:

\[
Bi = \frac{hr_o}{k} = \frac{(120 \text{ W/m}^2 \cdot \text{°C})(0.1 \text{ m})}{237 \text{ W/m} \cdot \text{°C}} = 0.05
\]

The coefficients \( \lambda_1 \) and \( A_1 \) for a cylinder corresponding to this \( Bi \) are determined from Table 4–1 to be \( \lambda_1 = 0.3126 \) and \( A_1 = 1.0124 \). The Fourier number in this case is

\[
\tau = \frac{\alpha t}{r_o^2} = \frac{(9.71 \times 10^{-6} \text{ m}^2/\text{s})(5 \times 60 \text{ s})}{(0.1 \text{ m})^2} = 2.91 > 0.2
\]

and thus the one-term approximation is applicable. Substituting these values into Eq. 4–14 gives

\[
\theta_0 = \theta_{cyl}(0, t) = A_1 e^{-\lambda_1 t} = 1.0124 e^{-0.3126(2,91)} = 0.762
\]

The solution for the semi-infinite solid can be determined from

\[
1 - \theta_{semi-inf}(x, t) = \text{erfc} \left( \frac{x}{2 \sqrt{\alpha t}} \right) - \exp \left( \frac{hx}{k} + \frac{h^2 \alpha t}{k^2} \right) \left[ \text{erfc} \left( \frac{x}{2 \sqrt{\alpha t}} + \frac{hx}{k} \right) \right]
\]

First we determine the various quantities in parentheses:

\[
\xi = \frac{x}{2 \sqrt{\alpha t}} = \frac{0.15 \text{ m}}{2 \sqrt{(9.71 \times 10^{-6} \text{ m}^2/\text{s})(5 \times 60 \text{ s})}} = 0.44
\]

\[
h \sqrt{\alpha t} = \frac{(120 \text{ W/m}^2 \cdot \text{°C}) \sqrt{9.71 \times 10^{-5} \text{ m}^2/\text{s}(300 \text{ s})}}{237 \text{ W/m} \cdot \text{°C}} = 0.086
\]

\[
\frac{hx}{k} = \frac{(120 \text{ W/m}^2 \cdot \text{°C})(0.15 \text{ m})}{237 \text{ W/m} \cdot \text{°C}} = 0.0759
\]

\[
\frac{h^2 \alpha t}{k^2} = \left( \frac{h \sqrt{\alpha t}}{k} \right)^2 = (0.086)^2 = 0.0074
\]

Substituting and evaluating the complementary error functions from Table 4–3,

\[
\theta_{semi-inf}(x, t) = 1 - \text{erfc} \left( 0.44 + \exp (0.0759 + 0.0074) \right) \text{erfc} (0.44 + 0.086)
\]

\[
= 1 - 0.5338 + \exp (0.0833) \times 0.457
\]

\[
= 0.963
\]

Now we apply the product solution to get

\[
\left( \frac{T(x, 0, t) - T_{\infty}}{T_j - T_{\infty}} \right)_{\text{semi-infinite}} = \theta_{semi-inf}(x, t) \theta_{cyl}(0, t) = 0.963 \times 0.762 = 0.734
\]
and

\[ T(x, 0, t) = T_n + 0.734(T_i - T_n) = 15 + 0.734(200 - 15) = 151^\circ C \]

which is the temperature at the center of the cylinder 15 cm from the exposed bottom surface.

**EXAMPLE 4–10  Refrigerating Steaks while Avoiding Frostbite**

In a meat processing plant, 1-in.-thick steaks initially at 75°F are to be cooled in the racks of a large refrigerator that is maintained at 5°F (Fig. 4–30). The steaks are placed close to each other, so that heat transfer from the 1-in.-thick edges is negligible. The entire steak is to be cooled below 45°F, but its temperature is not to drop below 35°F at any point during refrigeration to avoid “frostbite.” The convection heat transfer coefficient and thus the rate of heat transfer from the steak can be controlled by varying the speed of a circulating fan inside. Determine the heat transfer coefficient \( h \) that will enable us to meet both temperature constraints while keeping the refrigeration time to a minimum. The steak can be treated as a homogeneous layer having the properties \( \rho = 74.9 \) lbm/ft\(^3\), \( C_p = 0.98 \) Btu/lbm \( \cdot ^\circ F \), \( k = 0.26 \) Btu/h \( \cdot \) ft \( \cdot ^\circ F \), and \( \alpha = 0.0035 \) ft\(^2\)/h.

**SOLUTION**  Steaks are to be cooled in a refrigerator maintained at 5°F. The heat transfer coefficient that will allow cooling the steaks below 45°F while avoiding frostbite is to be determined.

**Assumptions** 1 Heat conduction through the steaks is one-dimensional since the steaks form a large layer relative to their thickness and there is thermal symmetry about the center plane. 2 The thermal properties of the steaks and the heat transfer coefficient are constant. 3 The Fourier number is \( \tau > 0.2 \) so that the one-term approximate solutions are applicable.

**Properties**  The properties of the steaks are as given in the problem statement.

**Analysis**  The lowest temperature in the steak will occur at the surfaces and the highest temperature at the center at a given time, since the inner part will be the last place to be cooled. In the limiting case, the surface temperature at \( x = L = 0.5 \) in. from the center will be 35°F, while the midplane temperature is 45°F in an environment at 5°F. Then, from Fig. 4–13b, we obtain

\[
\left\{ \begin{align*}
\frac{x}{L} &= \frac{0.5 \text{ in.}}{0.5 \text{ in.}} = 1 \\
\frac{T(L, t) - T_n}{T_m - T_n} &= \frac{35 - 5}{45 - 5} = 0.75
\end{align*} \right\} \quad \frac{1}{Bi} = \frac{k}{hL} = 1.5
\]

which gives

\[ h = \frac{1}{1.5} \frac{k}{L} = \frac{0.26 \text{ Btu/h \cdot ft \cdot } ^\circ F}{1.5(0.5/12 \text{ ft})} = 4.16 \text{ Btu/h \cdot ft}^2 \cdot ^\circ F \]

**Discussion**  The convection heat transfer coefficient should be kept below this value to satisfy the constraints on the temperature of the steak during refrigeration. We can also meet the constraints by using a lower heat transfer coefficient, but doing so would extend the refrigeration time unnecessarily.
The restrictions that are inherent in the use of Heisler charts and the one-term solutions (or any other analytical solutions) can be lifted by using the numerical methods discussed in Chapter 5.

**TOPIC OF SPECIAL INTEREST**

*Refrigeration and Freezing of Foods*

**Control of Microorganisms in Foods**

Microorganisms such as *bacteria*, *yeasts*, *molds*, and *viruses* are widely encountered in air, water, soil, living organisms, and unprocessed food items, and cause *off-flavors* and *odors*, *slime production*, *changes* in the texture and appearances, and the eventual *spoilage* of foods. Holding perishable foods at warm temperatures is the primary cause of spoilage, and the prevention of food spoilage and the premature degradation of quality due to microorganisms is the largest application area of refrigeration. The first step in controlling microorganisms is to understand what they are and the factors that affect their transmission, growth, and destruction.

Of the various kinds of microorganisms, *bacteria* are the prime cause for the spoilage of foods, especially moist foods. Dry and acidic foods create an undesirable environment for the growth of bacteria, but not for the growth of yeasts and molds. *Molds* are also encountered on moist surfaces, cheese, and spoiled foods. Specific *viruses* are encountered in certain animals and humans, and poor sanitation practices such as keeping processed foods in the same area as the uncooked ones and being careless about hand-washing can cause the contamination of food products.

When *contamination* occurs, the microorganisms start to adapt to the new environmental conditions. This initial slow or no-growth period is called the *lag phase*, and the shelf life of a food item is directly proportional to the length of this phase (Fig. 4–31). The adaptation period is followed by an *exponential growth* period during which the population of microorganisms can double two or more times every hour under favorable conditions unless drastic sanitation measures are taken. The depletion of nutrients and the accumulation of toxins slow down the growth and start the *death* period.

The *rate of growth* of microorganisms in a food item depends on the characteristics of the food itself such as the chemical structure, pH level, presence of inhibitors and competing microorganisms, and water activity as well as the environmental conditions such as the temperature and relative humidity of the environment and the air motion (Fig. 4–32).

Microorganisms need *food* to grow and multiply, and their nutritional needs are readily provided by the carbohydrates, proteins, minerals, and vitamins in a food. Different types of microorganisms have different nutritional needs, and the types of nutrients in a food determine the types of microorganisms that may dwell on them. The preservatives added to the

*This section can be skipped without a loss of continuity.
food may also inhibit the growth of certain microorganisms. Different kinds of microorganisms that exist compete for the same food supply, and thus the composition of microorganisms in a food at any time depends on the initial make-up of the microorganisms.

All living organisms need water to grow, and microorganisms cannot grow in foods that are not sufficiently moist. Microbiological growth in refrigerated foods such as fresh fruits, vegetables, and meats starts at the exposed surfaces where contamination is most likely to occur. Fresh meat in a package left in a room will spoil quickly, as you may have noticed. A meat carcass hung in a controlled environment, on the other hand, will age healthily as a result of dehydration on the outer surface, which inhibits microbiological growth there and protects the carcass.

Microorganism growth in a food item is governed by the combined effects of the characteristics of the food and the environmental factors. We cannot do much about the characteristics of the food, but we certainly can alter the environmental conditions to more desirable levels through heating, cooling, ventilating, humidification, dehumidification, and control of the oxygen levels. The growth rate of microorganisms in foods is a strong function of temperature, and temperature control is the single most effective mechanism for controlling the growth rate.

Microorganisms grow best at “warm” temperatures, usually between 20 and 60°C. The growth rate declines at high temperatures, and death occurs at still higher temperatures, usually above 70°C for most microorganisms. Cooling is an effective and practical way of reducing the growth rate of microorganisms and thus extending the shelf life of perishable foods. A temperature of 4°C or lower is considered to be a safe refrigeration temperature. Sometimes a small increase in refrigeration temperature may cause a large increase in the growth rate, and thus a considerable decrease in shelf life of the food (Fig. 4–33). The growth rate of some microorganisms, for example, doubles for each 3°C rise in temperature.

Another factor that affects microbiological growth and transmission is the relative humidity of the environment, which is a measure of the water content of the air. High humidity in cold rooms should be avoided since condensation that forms on the walls and ceiling creates the proper environment for mold growth and buildups. The drip of contaminated condensate onto food products in the room poses a potential health hazard.

Different microorganisms react differently to the presence of oxygen in the environment. Some microorganisms such as molds require oxygen for growth, while some others cannot grow in the presence of oxygen. Some grow best in low-oxygen environments, while others grow in environments regardless of the amount of oxygen. Therefore, the growth of certain microorganisms can be controlled by controlling the amount of oxygen in the environment. For example, vacuum packaging inhibits the growth of microorganisms that require oxygen. Also, the storage life of some fruits can be extended by reducing the oxygen level in the storage room.

Microorganisms in food products can be controlled by (1) preventing contamination by following strict sanitation practices, (2) inhibiting growth by altering the environmental conditions, and (3) destroying the organisms by heat treatment or chemicals. The best way to minimize contamination
in food processing areas is to use fine air filters in ventilation systems to capture the *dust particles* that transport the bacteria in the air. Of course, the filters must remain dry since microorganisms can grow in wet filters. Also, the ventilation system must maintain a positive pressure in the food processing areas to prevent any airborne contaminants from entering inside by infiltration. The elimination of *condensation* on the walls and the ceiling of the facility and the diversion of *plumbing* condensation drip pans of refrigerators to the drain system are two other preventive measures against contamination. Drip systems must be cleaned regularly to prevent microbiological growth in them. Also, any *contact* between raw and cooked food products should be minimized, and cooked products must be stored in rooms with positive pressures. Frozen foods must be kept at $-18^\circ C$ or below, and utmost care should be exercised when food products are packaged after they are frozen to avoid contamination during packaging.

The growth of microorganisms is best controlled by keeping the *temperature* and *relative humidity* of the environment in the desirable range. Keeping the relative humidity below 60 percent, for example, prevents the growth of all microorganisms on the surfaces. Microorganisms can be destroyed by *heating* the food product to high temperatures (usually above $70^\circ C$), by treating them with *chemicals*, or by exposing them to *ultraviolet light* or solar radiation.

Distinction should be made between *survival* and *growth* of microorganisms. A particular microorganism that may not grow at some low temperature may be able to survive at that temperature for a very long time (Fig. 4–34). Therefore, freezing is not an effective way of killing microorganisms. In fact, some microorganism cultures are preserved by freezing them at very low temperatures. The *rate of freezing* is also an important consideration in the refrigeration of foods since some microorganisms adapt to low temperatures and grow at those temperatures when the cooling rate is very low.

**Refrigeration and Freezing of Foods**

The *storage life* of fresh perishable foods such as meats, fish, vegetables, and fruits can be extended by several days by storing them at temperatures just above freezing, usually between 1 and $4^\circ C$. The storage life of foods can be extended by several months by freezing and storing them at sub-freezing temperatures, usually between $-18$ and $-35^\circ C$, depending on the particular food (Fig. 4–35).

Refrigeration *slows down* the chemical and biological processes in foods, and the accompanying deterioration and loss of quality and nutrients. Sweet corn, for example, may lose half of its initial sugar content in one day at $21^\circ C$, but only 5 percent of it at $0^\circ C$. Fresh asparagus may lose 50 percent of its vitamin C content in one day at $20^\circ C$, but in 12 days at $0^\circ C$. Refrigeration also extends the shelf life of products. The first appearance of unsightly yellowing of broccoli, for example, may be delayed by three or more days by refrigeration.

Early attempts to freeze food items resulted in poor-quality products because of the large ice crystals that formed. It was determined that the *rate of freezing* has a major effect on the size of ice crystals and the quality, texture, and nutritional and sensory properties of many foods. During *slow*
freezing, ice crystals can grow to a large size, whereas during fast freezing a large number of ice crystals start forming at once and are much smaller in size. Large ice crystals are not desirable since they can puncture the walls of the cells, causing a degradation of texture and a loss of natural juices during thawing. A crust forms rapidly on the outer layer of the product and seals the juices, aromatics, and flavoring agents. The product quality is also affected adversely by temperature fluctuations of the storage room.

The ordinary refrigeration of foods involves cooling only without any phase change. The freezing of foods, on the other hand, involves three stages: cooling to the freezing point (removing the sensible heat), freezing (removing the latent heat), and further cooling to the desired subfreezing temperature (removing the sensible heat of frozen food), as shown in Figure 4–36.

**Beef Products**

Meat carcasses in slaughterhouses should be cooled as fast as possible to a uniform temperature of about 1.7°C to reduce the growth rate of microorganisms that may be present on carcass surfaces, and thus minimize spoilage. The right level of temperature, humidity, and air motion should be selected to prevent excessive shrinkage, toughening, and discoloration.

The deep body temperature of an animal is about 39°C, but this temperature tends to rise a couple of degrees in the midsections after slaughter as a result of the heat generated during the biological reactions that occur in the cells. The temperature of the exposed surfaces, on the other hand, tends to drop as a result of heat losses. The thickest part of the carcass is the round, and the center of the round is the last place to cool during chilling. Therefore, the cooling of the carcass can best be monitored by inserting a thermometer deep into the central part of the round.

About 70 percent of the beef carcass is water, and the carcass is cooled mostly by evaporative cooling as a result of moisture migration toward the surface where evaporation occurs. But this shrinking translates into a loss of salable mass that can amount to 2 percent of the total mass during an overnight chilling. To prevent excessive loss of mass, carcasses are usually washed or sprayed with water prior to cooling. With adequate care, spray chilling can eliminate carcass cooling shrinkage almost entirely.

The average total mass of dressed beef, which is normally split into two sides, is about 300 kg, and the average specific heat of the carcass is about 3.14 kJ/kg · °C (Table 4–5). The chilling room must have a capacity equal to the daily kill of the slaughterhouse, which may be several hundred. A beef carcass is washed before it enters the chilling room and absorbs a large amount of water (about 3.6 kg) at its surface during the washing process. This does not represent a net mass gain, however, since it is lost by dripping or evaporation in the chilling room during cooling. Ideally, the carcass does not lose or gain any net weight as it is cooled in the chilling room. However, it does lose about 0.5 percent of the total mass in the holding room as it continues to cool. The actual product loss is determined by first weighing the dry carcass before washing and then weighing it again after it is cooled.

The refrigerated air temperature in the chilling room of beef carcasses must be sufficiently high to avoid freezing and discoloration on the outer

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**TABLE 4–5**

<table>
<thead>
<tr>
<th>Thermal properties of beef</th>
<th>Typical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quantity</td>
<td></td>
</tr>
<tr>
<td>Average density</td>
<td>1070 kg/m³</td>
</tr>
<tr>
<td>Specific heat:</td>
<td></td>
</tr>
<tr>
<td>Above freezing</td>
<td>3.14 kJ/kg · °C</td>
</tr>
<tr>
<td>Below freezing</td>
<td>1.70 kJ/kg · °C</td>
</tr>
<tr>
<td>Freezing point</td>
<td>-2.7°C</td>
</tr>
<tr>
<td>Latent heat of fusion</td>
<td>249 kJ/kg</td>
</tr>
<tr>
<td>Thermal conductivity</td>
<td>0.41 W/m · °C</td>
</tr>
<tr>
<td></td>
<td>(at 6°C)</td>
</tr>
</tbody>
</table>
surfaces of the carcass. This means a long residence time for the massive beef carcasses in the chilling room to cool to the desired temperature. Beef carcasses are only partially cooled at the end of an overnight stay in the chilling room. The temperature of a beef carcass drops to 1.7 to 7°C at the surface and to about 15°C in mid parts of the round in 10 h. It takes another day or two in the holding room maintained at 1 to 2°C to complete chilling and temperature equalization. But hog carcasses are fully chilled during that period because of their smaller size. The air circulation in the holding room is kept at minimum levels to avoid excessive moisture loss and discoloration. The refrigeration load of the holding room is much smaller than that of the chilling room, and thus it requires a smaller refrigeration system.

Beef carcasses intended for distant markets are shipped the day after slaughter in refrigerated trucks, where the rest of the cooling is done. This practice makes it possible to deliver fresh meat long distances in a timely manner.

The variation in temperature of the beef carcass during cooling is given in Figure 4–37. Initially, the cooling process is dominated by sensible heat transfer. Note that the average temperature of the carcass is reduced by about 28°C (from 36 to 8°C) in 20 h. The cooling rate of the carcass could be increased by lowering the refrigerated air temperature and increasing the air velocity, but such measures also increase the risk of surface freezing.

Most meats are judged on their tenderness, and the preservation of tenderness is an important consideration in the refrigeration and freezing of meats. Meat consists primarily of bundles of tiny muscle fibers bundled together inside long strings of connective tissues that hold it together. The tenderness of a certain cut of beef depends on the location of the cut, the age, and the activity level of the animal. Cuts from the relatively inactive mid-backbone section of the animal such as short loins, sirloin, and prime ribs are more tender than the cuts from the active parts such as the legs and the neck (Fig. 4–38). The more active the animal, the more the connective tissue, and the tougher the meat. The meat of an older animal is more flavorful, however, and is preferred for stewing since the toughness of the meat does not pose a problem for moist-heat cooking such as boiling. The
protein collagen, which is the main component of the connective tissue, softens and dissolves in hot and moist environments and gradually transforms into gelatin, and tenderizes the meat.

The old saying “one should either cook an animal immediately after slaughter or wait at least two days” has a lot of truth in it. The biomechanical reactions in the muscle continue after the slaughter until the energy supplied to the muscle to do work diminishes. The muscle then stiffens and goes into rigor mortis. This process begins several hours after the animal is slaughtered and continues for 12 to 36 h until an enzymatic action sets in and tenderizes the connective tissue, as shown in Figure 4–39. It takes about seven days to complete tenderization naturally in storage facilities maintained at 2°C. Electrical stimulation also causes the meat to be tender. To avoid toughness, fresh meat should not be frozen before rigor mortis has passed.

You have probably noticed that steaks are tender and rather tasty when they are hot but toughen as they cool. This is because the gelatin that formed during cooking thickens as it cools, and meat loses its tenderness. So it is no surprise that first-class restaurants serve their steak on hot thick plates that keep the steaks warm for a long time. Also, cooking softens the connective tissue but toughens the tender muscle fibers. Therefore, barbecuing on low heat for a long time results in a tough steak.

Variety meats intended for long-term storage must be frozen rapidly to reduce spoilage and preserve quality. Perhaps the first thought that comes to mind to freeze meat is to place the meat packages into the freezer and wait. But the freezing time is too long in this case, especially for large boxes. For example, the core temperature of a 4–40 cm-deep box containing 32 kg of variety meat can be as high as 16°C 24 h after it is placed into a −30°C freezer. The freezing time of large boxes can be shortened considerably by adding some dry ice into it.

A more effective method of freezing, called quick chilling, involves the use of lower air temperatures, −40 to −30°C, with higher velocities of 2.5 m/s to 5 m/s over the product (Fig. 4–40). The internal temperature should be lowered to −4°C for products to be transferred to a storage freezer and to −18°C for products to be shipped immediately. The rate of freezing depends on the package material and its insulating properties, the thickness of the largest box, the type of meat, and the capacity of the refrigeration system. Note that the air temperature will rise excessively during initial stages of freezing and increase the freezing time if the capacity of the system is inadequate. A smaller refrigeration system will be adequate if dry ice is to be used in packages. Shrinkage during freezing varies from about 0.5 to 1 percent.

Although the average freezing point of lean meat can be taken to be −2°C with a latent heat of 249 kJ/kg, it should be remembered that freezing occurs over a temperature range, with most freezing occurring between −1 and −4°C. Therefore, cooling the meat through this temperature range and removing the latent heat takes the most time during freezing.

Meat can be kept at an internal temperature of −2 to −1°C for local use and storage for under a week. Meat must be frozen and stored at much lower temperatures for long-term storage. The lower the storage temperature, the longer the storage life of meat products, as shown in Table 4–6.
The internal temperature of carcasses entering the cooling sections varies from 38 to 41°C for hogs and from 37 to 39°C for lambs and calves. It takes about 15 h to cool the hogs and calves to the recommended temperature of 3 to 4°C. The cooling-room temperature is maintained at −1 to 0°C and the temperature difference between the refrigerant and the cooling air is kept at about 6°C. Air is circulated at a rate of about 7 to 12 air changes per hour. Lamb carcasses are cooled to an internal temperature of 1 to 2°C, which takes about 12 to 14 h, and are held at that temperature with 85 to 90 percent relative humidity until shipped or processed. The recommended rate of air circulation is 50 to 60 air changes per hour during the first 4 to 6 h, which is reduced to 10 to 12 changes per hour afterward.

Freezing does not seem to affect the flavor of meat much, but it affects the quality in several ways. The rate and temperature of freezing may influence color, tenderness, and drip. Rapid freezing increases tenderness and reduces the tissue damage and the amount of drip after thawing. Storage at low freezing temperatures causes significant changes in animal fat. Frozen pork experiences more undesirable changes during storage because of its fat structure, and thus its acceptable storage period is shorter than that of beef, veal, or lamb.

Meat storage facilities usually have a refrigerated shipping dock where the orders are assembled and shipped out. Such docks save valuable storage space from being used for shipping purposes and provide a more acceptable working environment for the employees. Packing plants that ship whole or half carcasses in bulk quantities may not need a shipping dock; a load-out door is often adequate for such cases.

A refrigerated shipping dock, as shown in Figure 4–41, reduces the refrigeration load of freezers or coolers and prevents temperature fluctuations in the storage area. It is often adequate to maintain the shipping docks at 4 to 7°C for the coolers and about 1.5°C for the freezers. The dew point of the dock air should be below the product temperature to avoid condensation on the surface of the products and loss of quality. The rate of airflow through the loading doors and other openings is proportional to the square root of the temperature difference, and thus reducing the temperature difference at the opening by half by keeping the shipping dock at the average temperature reduces the rate of airflow into the dock and thus into the freezer by \(1 - \sqrt{0.5} \approx 0.3\), or 30 percent. Also, the air that flows into the freezer is already cooled to about 1.5°C by the refrigeration unit of the dock, which represents about 50 percent of the cooling load of the incoming air. Thus, the net effect of the refrigerated shipping dock is a reduction of the infiltration load of the freezer by about 65 percent since \(1 - 0.7 \times 0.5 = 0.65\). The net gain is equal to the difference between the reduction of the infiltration load of the freezer and the refrigeration load of the shipping dock. Note that the dock refrigerators operate at much higher temperatures (1.5°C instead of about −23°C), and thus they consume much less power for the same amount of cooling.

**Poultry Products**

Poultry products can be preserved by ice-chilling to 1 to 2°C or deep chilling to about −2°C for short-term storage, or by freezing them to −18°C or
below for long-term storage. Poultry processing plants are completely automated, and the small size of the birds makes continuous conveyor line operation feasible.

The birds are first electrically stunned before cutting to prevent struggling. Following a 90- to 120-s bleeding time, the birds are scalded by immersing them into a tank of warm water, usually at 51 to 55°C, for up to 120 s to loosen the feathers. Then the feathers are removed by feather-picking machines, and the eviscerated carcass is washed thoroughly before chilling. The internal temperature of the birds ranges from 24 to 35°C after washing, depending on the temperatures of the ambient air and the washing water as well as the extent of washing.

To control the microbial growth, the USDA regulations require that poultry be chilled to 4°C or below in less than 4 h for carcasses of less than 1.8 kg, in less than 6 h for carcasses of 1.8 to 3.6 kg, and in less than 8 h for carcasses more than 3.6 kg. Meeting these requirements today is not difficult since the slow air chilling is largely replaced by the rapid immersion chilling in tanks of slush ice. Immersion chilling has the added benefit that it not only prevents dehydration, but it causes a net absorption of water and thus increases the mass of salable product. Cool air chilling of unpacked poultry can cause a moisture loss of 1 to 2 percent, while water immersion chilling can cause a moisture absorption of 4 to 15 percent (Fig. 4–42). Water spray chilling can cause a moisture absorption of up to 4 percent. Most water absorbed is held between the flesh and the skin and the connective tissues in the skin. In immersion chilling, some soluble solids are lost from the carcass to the water, but the loss has no significant effect on flavor.

Many slush ice tank chillers today are replaced by continuous flow-type immersion slush ice chillers. Continuous slush ice-chillers can reduce the internal temperature of poultry from 32 to 4°C in about 30 minutes at a rate up to 10,000 birds per hour. Ice requirements depend on the inlet and exit temperatures of the carcass and the water, but 0.25 kg of ice per kg of carcass is usually adequate. However, bacterial contamination such as salmonella remains a concern with this method, and it may be necessary to chloride the water to control contamination.

Tenderness is an important consideration for poultry products just as it is for red meat, and preserving tenderness is an important consideration in the cooling and freezing of poultry. Birds cooked or frozen before passing through rigor mortis remain very tough. Natural tenderization begins soon after slaughter and is completed within 24 h when birds are held at 4°C. Tenderization is rapid during the first three hours and slows down thereafter. Immersion in hot water and cutting into the muscle adversely affect tenderization. Increasing the scalding temperature or the scalding time has been observed to increase toughness, and decreasing the scalding time has been observed to increase tenderness. The beating action of mechanical feather-picking machines causes considerable toughening. Therefore, it is recommended that any cutting be done after tenderization. Cutting up the bird into pieces before natural tenderization is completed reduces tenderness considerably. Therefore, it is recommended that any cutting be done after tenderization. Rapid chilling of poultry can also have a toughening

FIGURE 4–42
Air chilling causes dehydration and thus weight loss for poultry, whereas immersion chilling causes a weight gain as a result of water absorption.
effect. It is found that the tenderization process can be speeded up considerably by a patented electrical stunning process.

Poultry products are highly perishable, and thus they should be kept at the lowest possible temperature to maximize their shelf life. Studies have shown that the populations of certain bacteria double every 36 h at −2°C, 14 h at 0°C, 7 h at 5°C, and less than 1 h at 25°C (Fig. 4–43). Studies have also shown that the total bacterial counts on birds held at 2°C for 14 days are equivalent to those held at 10°C for 5 days or 24°C for 1 day. It has also been found that birds held at −1°C had 8 days of additional shelf life over those held at 4°C.

The growth of microorganisms on the surfaces of the poultry causes the development of an off-odor and bacterial slime. The higher the initial amount of bacterial contamination, the faster the sliming occurs. Therefore, good sanitation practices during processing such as cleaning the equipment frequently and washing the carcasses are as important as the storage temperature in extending shelf life.

Poultry must be frozen rapidly to ensure a light, pleasing appearance. Poultry that is frozen slowly appears dark and develops large ice crystals that damage the tissue. The ice crystals formed during rapid freezing are small. Delaying freezing of poultry causes the ice crystals to become larger. Rapid freezing can be accomplished by forced air at temperatures of −23 to −40°C and velocities of 1.5 to 5 m/s in air-blast tunnel freezers. Most poultry is frozen this way. Also, the packaged birds freeze much faster on open shelves than they do in boxes. If poultry packages must be frozen in boxes, then it is very desirable to leave the boxes open or to cut holes on the boxes in the direction of airflow during freezing. For best results, the blast tunnel should be fully loaded across its cross-section with even spacing between the products to assure uniform airflow around all sides of the packages. The freezing time of poultry as a function of refrigerated air temperature is given in Figure 4–44. Thermal properties of poultry are given in Table 4–7.

Other freezing methods for poultry include sandwiching between cold plates, immersion into a refrigerated liquid such as glycol or calcium chloride brine, and cryogenic cooling with liquid nitrogen. Poultry can be frozen in several hours by cold plates. Very high freezing rates can be obtained by immersing the packaged birds into a low-temperature brine. The freezing time of birds in −29°C brine can be as low as 20 min, depending on the size of the bird (Fig. 4–45). Also, immersion freezing produces a very appealing light appearance, and the high rates of heat transfer make continuous line operation feasible. It also has lower initial and maintenance costs than forced air, but leaks into the packages through some small holes or cracks remain a concern. The convection heat transfer coefficient is 17 W/m² · °C for air at −29°C and 2.5 m/s whereas it is 170 W/m² · °C for sodium chloride brine at −18°C and a velocity of 0.02 m/s. Sometimes liquid nitrogen is used to crust freeze the poultry products to −73°C. The freezing is then completed with air in a holding room at −23°C.

Properly packaged poultry products can be stored frozen for up to about a year at temperatures of −18°C or lower. The storage life drops considerably at higher (but still below-freezing) temperatures. Significant changes

\[
\text{Storage life (days)}
\]

Note: Freezing time is the time required for temperature to fall from 0 to −4°C. The values are for 2.3 to 3.6 kg chickens with initial temperature of 0 to 2°C and with air velocity of 2.3 to 2.8 m/s.

**FIGURE 4–44**

The variation of freezing time of poultry with air temperature (from van der Berg and Lentz, Ref. 11).
HEAT TRANSFER

Heat transfer occur in flavor and juiciness when poultry is frozen for too long, and a stale rancid odor develops. Frozen poultry may become dehydrated and experience freezer burn, which may reduce the eye appeal of the product and cause toughening of the affected area. Dehydration and thus freezer burn can be controlled by humidification, lowering the storage temperature, and packaging the product in essentially impermeable film. The storage life can be extended by packing the poultry in an oxygen-free environment. The bacterial counts in precooked frozen products must be kept at safe levels since bacteria may not be destroyed completely during the reheating process at home.

Frozen poultry can be thawed in ambient air, water, refrigerator, or oven without any significant difference in taste. Big birds like turkey should be thawed safely by holding it in a refrigerator at 2 to 4°C for two to four days, depending on the size of the bird. They can also be thawed by immersing them into cool water in a large container for 4 to 6 h, or holding them in a paper bag. Care must be exercised to keep the bird’s surface cool to minimize microbiological growth when thawing in air or water.

### TABLE 4–7
Thermal properties of poultry

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Typical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average density:</td>
<td></td>
</tr>
<tr>
<td>Muscle</td>
<td>1070 kg/m³</td>
</tr>
<tr>
<td>Skin</td>
<td>1030 kg/m³</td>
</tr>
<tr>
<td>Specific heat:</td>
<td></td>
</tr>
<tr>
<td>Above freezing</td>
<td>2.94 kJ/kg · °C</td>
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<tr>
<td>Below freezing</td>
<td>1.55 kJ/kg · °C</td>
</tr>
<tr>
<td>Freezing point</td>
<td>−2.8°C</td>
</tr>
<tr>
<td>Latent heat of fusion</td>
<td>247 kJ/kg</td>
</tr>
<tr>
<td>Thermal conductivity: (in W/m · °C)</td>
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</tr>
<tr>
<td>Breast muscle</td>
<td>0.502 at 20°C</td>
</tr>
<tr>
<td></td>
<td>1.384 at −20°C</td>
</tr>
<tr>
<td></td>
<td>1.506 at −40°C</td>
</tr>
<tr>
<td>Dark muscle</td>
<td>1.557 at −40°C</td>
</tr>
</tbody>
</table>

### FIGURE 4–45
The variation of temperature of the breast of 6.8-kg turkeys initially at 1°C with depth during immersion cooling at −29°C (from van der Berg and Lentz, Ref. 11).

### EXAMPLE 4–5  Chilling of Beef Carcasses in a Meat Plant

The chilling room of a meat plant is 18 m × 20 m × 5.5 m in size and has a capacity of 450 beef carcasses. The power consumed by the fans and the lights of the chilling room are 26 and 3 kW, respectively, and the room gains heat through its envelope at a rate of 13 kW. The average mass of beef carcasses is 285 kg. The carcasses enter the chilling room at 36°C after they are washed to facilitate evaporative cooling and are cooled to 15°C in 10 h. The water is expected to evaporate at a rate of 0.080 kg/s. The air enters the evaporator section of the refrigeration system at 0.7°C and leaves at −2°C. The air side of the evaporator is heavily finned, and the overall heat transfer coefficient of the evaporator based on the air side is 20 W/m² · °C. Also, the average temperature difference between the air and the refrigerant in the evaporator is 5.5°C.
Determine (a) the refrigeration load of the chilling room, (b) the volume flow rate of air, and (c) the heat transfer surface area of the evaporator on the air side, assuming all the vapor and the fog in the air freezes in the evaporator.

**SOLUTION**  
The chilling room of a meat plant with a capacity of 450 beef carcasses is considered. The cooling load, the airflow rate, and the heat transfer area of the evaporator are to be determined.

**Assumptions**  
1. Water evaporates at a rate of 0.080 kg/s.  
2. All the moisture in the air freezes in the evaporator.

**Properties**  
The heat of fusion and the heat of vaporization of water at 0°C are 333.7 kJ/kg and 2501 kJ/kg (Table A-9). The density and specific heat of air at 0°C are 1.292 kg/m³ and 1.006 kJ/kg · °C (Table A-15). Also, the specific heat of beef carcass is determined from the relation in Table A-7b to be

\[
C_p = 1.68 + 2.51 \times \text{(water content)} = 1.68 + 2.51 \times 0.58 = 3.14 \text{ kJ/kg} \cdot ^\circ \text{C}
\]

**Analysis**  
(a) A sketch of the chilling room is given in Figure 4–46. The amount of beef mass that needs to be cooled per unit time is

\[
m_{\text{beef}} = \frac{\text{Total beef mass cooled}}{\text{(Cooling time)}} = \frac{450 \text{ carcasses}(285 \text{ kg/carcass})(10 \times 3600 \text{ s})}{3600 \text{ s}} = 3.56 \text{ kg/s}
\]

The product refrigeration load can be viewed as the energy that needs to be removed from the beef carcass as it is cooled from 36 to 15°C at a rate of 3.56 kg/s and is determined to be

\[
\dot{Q}_{\text{beef}} = (mC_p\Delta T)_{\text{beef}} = (3.56 \text{ kg/s})(3.14 \text{ kJ/kg} \cdot ^\circ \text{C})(36 - 15)^\circ \text{C} = 235 \text{ kW}
\]

Then the total refrigeration load of the chilling room becomes

\[
\dot{Q}_{\text{total, chillroom}} = \dot{Q}_{\text{beef}} + \dot{Q}_{\text{fan}} + \dot{Q}_{\text{lights}} + \dot{Q}_{\text{heat gain}} = 235 + 26 + 3 + 13 = 277 \text{ kW}
\]

The amount of carcass cooling due to evaporative cooling of water is

\[
\dot{Q}_{\text{beef, evaporative}} = (mh_{fg})_{\text{water}} = (0.080 \text{ kg/s})(2490 \text{ kJ/kg}) = 199 \text{ kW}
\]

which is 199/235 = 85 percent of the total product cooling load. The remaining 15 percent of the heat is transferred by convection and radiation.

(b) Heat is transferred to air at the rate determined above, and the temperature of the air rises from −2°C to 0.7°C as a result. Therefore, the mass flow rate of air is

\[
m_{\text{air}} = \frac{\dot{Q}_{\text{air}}}{(C_p\Delta T_{\text{air}})} = \frac{277 \text{ kW}}{(1.006 \text{ kJ/kg} \cdot ^\circ \text{C})(0.7 - (-2)^\circ \text{C})} = 102.0 \text{ kg/s}
\]

Then the volume flow rate of air becomes

\[
V_{\text{air}} = \frac{m_{\text{air}}}{\rho_{\text{air}}} = \frac{102 \text{ kg/s}}{1.292 \text{ kg/m}^3} = 78.9 \text{ m}^3/\text{s}
\]

(c) Normally the heat transfer load of the evaporator is the same as the refrigeration load. But in this case the water that enters the evaporator as a liquid is
frozen as the temperature drops to $-2^\circ$C, and the evaporator must also remove the latent heat of freezing, which is determined from

$$Q_{\text{freezing}} = (m h_{\text{latent, water}}) = (0.080 \text{ kg/s})(334 \text{ kJ/kg}) = 27 \text{ kW}$$

Therefore, the total rate of heat removal at the evaporator is

$$\dot{Q}_{\text{evaporator}} = \dot{Q}_{\text{total, chill room}} + \dot{Q}_{\text{freezing}} = 277 + 27 = 304 \text{ kW}$$

Then the heat transfer surface area of the evaporator on the air side is determined from

$$A = \frac{\dot{Q}_{\text{evaporator}}}{U \Delta T} = \frac{304,000 \text{ W}}{(20 \text{ W/m}^2 \cdot ^\circ\text{C})(5.5^\circ\text{C})} = 2764 \text{ m}^2$$

Obviously, a finned surface must be used to provide such a large surface area on the air side.

**SUMMARY**

In this chapter we considered the variation of temperature with time as well as position in one- or multidimensional systems. We first considered the *lumped systems* in which the temperature varies with time but remains uniform throughout the system at any time. The temperature of a lumped body of arbitrary shape of mass $m$, volume $V$, surface area $A_s$, density $\rho$, and specific heat $C_p$ initially at a uniform temperature $T_i$ that is exposed to convection at time $t = 0$ in a medium at temperature $T_m$ with a heat transfer coefficient $h$ is expressed as

$$\frac{T(t) - T_m}{T_i - T_m} = e^{-bt}$$

where

$$b = \frac{h A_s}{\rho C_p V} = \frac{h}{\rho C_p L_c} \quad (1/\text{s})$$

is a positive quantity whose dimension is $(\text{time})^{-1}$. This relation can be used to determine the temperature $T(t)$ of a body at time $t$ or, alternately, the time $t$ required for the temperature to reach a specified value $T_i$. Once the temperature $T(t)$ at time $t$ is available, the *rate* of convection heat transfer between the body and its environment at that time can be determined from Newton’s law of cooling as

$$\dot{Q}(t) = h A_s [T(t) - T_m] \quad (\text{W})$$

The total amount of heat transfer between the body and the surrounding medium over the time interval $t = 0$ to $t$ is simply the change in the energy content of the body,

$$Q = m C_p [T(t) - T_i] \quad (\text{kJ})$$

The amount of heat transfer reaches its upper limit when the body reaches the surrounding temperature $T_m$. Therefore, the *maximum* heat transfer between the body and its surroundings is

$$Q_{\text{max}} = m C_p (T_m - T_i) \quad (\text{kJ})$$

The error involved in lumped system analysis is negligible when

$$\text{Bi} = \frac{h L_c}{k} < 0.1$$

where Bi is the *Biot number* and $L_c = V/A_s$ is the *characteristic length*.

When the lumped system analysis is not applicable, the variation of temperature with position as well as time can be determined using the *transient temperature charts* given in Figs. 4–13, 4–14, 4–15, and 4–23 for a large plane wall, a long cylinder, a sphere, and a semi-infinite medium, respectively. These charts are applicable for one-dimensional heat transfer in those geometries. Therefore, their use is limited to situations in which the body is initially at a uniform temperature, all surfaces are subjected to the same thermal conditions, and the body does not involve any heat generation. These charts can also be used to determine the total heat transfer from the body up to a specified time $t$. 
Using a one-term approximation, the solutions of one-dimensional transient heat conduction problems are expressed analytically as

Plane wall: \[ \frac{Q}{Q_{\text{max,wall}}} = 1 - \theta_{h,\text{wall}} \sin \frac{\lambda_1}{\lambda_1}, \]

Cylinder: \[ \frac{Q}{Q_{\text{max,cyl}}} = 1 - 2 \theta_{h,cyl} \frac{J_1(\lambda_1)}{\lambda_1}, \]

Sphere: \[ \frac{Q}{Q_{\text{max,sph}}} = 1 - 3 \theta_{h,sph} \frac{\sin \lambda_1 - \lambda_1 \cos \lambda_3}{\lambda_1^2}. \]

The analytic solution for one-dimensional transient heat conduction in a semi-infinite solid subjected to convection is given by

\[ \frac{T(x, t) - T_i}{T_e - T_i} = \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right) - \exp \left( \frac{hx}{k} + \frac{h^2\alpha t}{k^2} \right) \left[ \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} + \frac{h\sqrt{\alpha t}}{k} \right) \right], \]

where the quantity \( \text{erfc} (\xi) \) is the complementary error function. For the special case of \( h \to \infty \), the surface temperature \( T_s \) becomes equal to the fluid temperature \( T_e \), and the above equation reduces to

\[ \frac{T(x, t) - T_i}{T_e - T_i} = \text{erfc} \left( \frac{x}{2\sqrt{\alpha t}} \right) \quad (T_i = \text{constant}) \]

Using a clever superposition principle called the product solution these charts can also be used to construct solutions for the two-dimensional transient heat conduction problems encountered in geometries such as a short cylinder, a long rectangular bar, or a semi-infinite cylinder or plate, and even three-dimensional problems associated with geometries such as a rectangular prism or a semi-infinite rectangular bar, provided that all surfaces of the solid are subjected to convection to the same fluid at temperature \( T_e \), with the same convection heat transfer coefficient \( h \), and the body involves no heat generation. The solution in such multidimensional geometries can be expressed as the product of the solutions for the one-dimensional geometries whose intersection is the multi-dimensional geometry.

The total heat transfer to or from a multidimensional geometry can also be determined by using the one-dimensional values. The transient heat transfer for a two-dimensional geometry formed by the intersection of two one-dimensional geometries 1 and 2 is

\[ \left( \frac{Q}{Q_{\text{max,2D}}} \right) = \frac{Q}{Q_{\text{max,1}}} + \frac{Q}{Q_{\text{max,2}}} \left[ 1 - \frac{Q}{Q_{\text{max,1}}} \right] \]

Transient heat transfer for a three-dimensional body formed by the intersection of three one-dimensional bodies 1, 2, and 3 is given by

\[ \left( \frac{Q}{Q_{\text{max,3D}}} \right) = \frac{Q}{Q_{\text{max,1}}} + \frac{Q}{Q_{\text{max,2}}} + \frac{Q}{Q_{\text{max,3}}} \left[ 1 - \frac{Q}{Q_{\text{max,1}}} \right] \left[ 1 - \frac{Q}{Q_{\text{max,2}}} \right] \]

REFERENCES AND SUGGESTED READING

1. ASHRAE. Handbook of Fundamentals. SI version.

2. ASHRAE. Handbook of Fundamentals. SI version.


**PROBLEMS***

**Lumped System Analysis**

4–1C What is lumped system analysis? When is it applicable?

4–2C Consider heat transfer between two identical hot solid bodies and the air surrounding them. The first solid is being cooled by a fan while the second one is allowed to cool naturally. For which solid is the lumped system analysis more likely to be applicable? Why?

4–3C Consider heat transfer between two identical hot solid bodies and their environments. The first solid is dropped in a large container filled with water, while the second one is allowed to cool naturally in the air. For which solid is the lumped system analysis more likely to be applicable? Why?

4–4C Consider a hot baked potato on a plate. The temperature of the potato is observed to drop by 4°C during the first minute. Will the temperature drop during the second minute be less than, equal to, or more than 4°C? Why?

4–5C Consider a potato being baked in an oven that is maintained at a constant temperature. The temperature of the potato is observed to rise by 5°C during the first minute. Will the temperature rise during the second minute be less than, equal to, or more than 5°C? Why?

4–6C What is the physical significance of the Biot number? Is the Biot number more likely to be larger for highly conducting solids or poorly conducting ones?

4–7C Consider two identical 4-kg pieces of roast beef. The first piece is baked as a whole, while the second is baked after being cut into two equal pieces in the same oven. Will there be any difference between the cooking times of the whole and cut roasts? Why?

4–8C Consider a sphere and a cylinder of equal volume made of copper. Both the sphere and the cylinder are initially at the same temperature and are exposed to convection in the same environment. Which do you think will cool faster, the cylinder or the sphere? Why?

4–9C In what medium is the lumped system analysis more likely to be applicable: in water or in air? Why?

4–10C For which solid is the lumped system analysis more likely to be applicable: an actual apple or a golden apple of the same size? Why?

4–11C For which kind of bodies made of the same material is the lumped system analysis more likely to be applicable: slender ones or well-rounded ones of the same volume? Why?

4–12 Obtain relations for the characteristic lengths of a large plane wall of thickness \(2l\), a very long cylinder of radius \(r_o\), and a sphere of radius \(r_o\).

4–13 Obtain a relation for the time required for a lumped system to reach the average temperature \(\frac{1}{2} (T_i + T_e)\), where \(T_i\) is the initial temperature and \(T_e\) is the temperature of the environment.

4–14 The temperature of a gas stream is to be measured by a thermocouple whose junction can be approximated as a 1.2-mm-diameter sphere. The properties of the junction are \(k = 35\) W/m \(\cdot\) °C, \(\rho = 8500\) kg/m³, and \(C_p = 320\) J/kg \(\cdot\) °C, and the heat transfer coefficient between the junction and the gas is \(h = 65\) W/m² \(\cdot\) °C. Determine how long it will take for the thermocouple to read 99 percent of the initial temperature difference. *Answer: 38.5 s*

4–15E In a manufacturing facility, 2-in.-diameter brass balls \((k = 64.1\) Btu/h \(\cdot\) ft \(\cdot\) °F, \(\rho = 532\) lbm/ft³, and \(C_p = 0.092\) Btu/lbm \(\cdot\) °F) initially at 250°F are quenched in a water bath at 120°F for a period of 2 min at a rate of 120 balls per minute. If the convection heat transfer coefficient is 42 Btu/h \(\cdot\) ft² \(\cdot\) °F, determine (a) the temperature of the balls after quenching and (b) the rate at which heat needs to be removed from the water in order to keep its temperature constant at 120°F.
4–16E Repeat Problem 4–15E for aluminum balls.

4–17 To warm up some milk for a baby, a mother pours milk into a thin-walled glass whose diameter is 6 cm. The height of the milk in the glass is 7 cm. She then places the glass into a large pan filled with hot water at 60°C. The milk is stirred constantly, so that its temperature is uniform at all times. If the heat transfer coefficient between the water and the glass is 120 W/m²·°C, determine how long it will take for the milk to warm up from 3°C to 38°C. Take the properties of the milk to be the same as those of water. Can the milk in this case be treated as a lumped system? Why? Answer: 5.8 min

4–18 Repeat Problem 4–17 for the case of water also being stirred, so that the heat transfer coefficient is doubled to 240 W/m²·°C.

4–19E During a picnic on a hot summer day, all the cold drinks disappeared quickly, and the only available drinks were those at the ambient temperature of 80°F. In an effort to cool a 12-fluid-oz drink in a can, which is 5 in. high and has a diameter of 2.5 in., a person grabs the can and starts shaking it in the iced water of the chest at 32°F. The temperature of the drink can be assumed to be uniform at all times, and the heat transfer coefficient between the iced water and the aluminum can is 30 Btu/h·ft²·°F. Using the properties of water for the drink, estimate how long it will take for the canned drink to cool to 45°F.

4–20 Consider a 1000-W iron whose base plate is made of 0.5-cm-thick aluminum alloy 2024-T6 (ρ = 2770 kg/m³, Cp = 875 J/kg·°C, α = 7.3 × 10⁻⁵ m²/s). The base plate has a surface area of 0.03 m². Initially, the iron is in thermal equilibrium with the ambient air at 22°C. Taking the heat transfer coefficient at the surface of the base plate to be 12 W/m²·°C and assuming 85 percent of the heat generated in the resistance wires is transferred to the plate, determine how long it will take for the plate temperature to reach 140°C. Is it realistic to assume the plate temperature to be uniform at all times?

4–21 Reconsider Problem 4–20. Using EES (or other) software, investigate the effects of the heat transfer coefficient and the final plate temperature on the time it will take for the plate to reach this temperature. Let the heat transfer coefficient vary from 5 W/m²·°C to 25 W/m²·°C and the temperature from 30°C to 200°C. Plot the time as functions of the heat transfer coefficient and the temperature, and discuss the results.

4–22 Stainless steel ball bearings (ρ = 8085 kg/m³, k = 15.1 W/m·°C, Cp = 0.480 kJ/kg·°C, and α = 3.91 × 10⁻⁶ m²/s) having a diameter of 1.2 cm are to be quenched in water. The balls leave the oven at a uniform temperature of 900°C and are exposed to air at 30°C for a while before they are dropped into the water. If the temperature of the balls is not to fall below 850°C prior to quenching and the heat transfer coefficient in the air is 125 W/m²·°C, determine how long they can stand in the air before being dropped into the water. Answer: 3.7 s

4–23 Carbon steel balls (ρ = 7833 kg/m³, k = 54 W/m·°C, Cp = 0.465 kJ/kg·°C, and α = 1.474 × 10⁻⁶ m²/s) 8 mm in
diameter are annealed by heating them first to 900°C in a furnace and then allowing them to cool slowly to 100°C in ambient air at 35°C. If the average heat transfer coefficient is 75 W/m² · °C, determine how long the annealing process will take. If 2500 balls are to be annealed per hour, determine the total rate of heat transfer from the balls to the ambient air.

4–24 Reconsider Problem 4–23. Using EES (or other) software, investigate the effect of the initial temperature of the balls on the annealing time and the total rate of heat transfer. Let the temperature vary from 500°C to 1000°C. Plot the time and the total rate of heat transfer as a function of the initial temperature, and discuss the results.

4–25 An electronic device dissipating 30 W has a mass of 20 g, a specific heat of 850 J/kg · °C, and a surface area of 5 cm². The device is lightly used, and it is on for 5 min and then off for several hours, during which it cools to the ambient temperature of 25°C. Taking the heat transfer coefficient to be 12 W/m² · °C, determine the temperature of the device at the end of the 5-min operating period. What would your answer be if the device were attached to an aluminum heat sink having a mass of 200 g and a surface area of 80 cm²? Assume the device and the heat sink to be nearly isothermal.

Transient Heat Conduction in Large Plane Walls, Long Cylinders, and Spheres with Spatial Effects

4–26C What is an infinitely long cylinder? When is it proper to treat an actual cylinder as being infinitely long, and when is it not? For example, is it proper to use this model when finding the temperatures near the bottom or top surfaces of a cylinder? Explain.

4–27C Can the transient temperature charts in Fig. 4–13 for a plane wall exposed to convection on both sides be used for a plane wall with one side exposed to convection while the other side is insulated? Explain.

4–28C Why are the transient temperature charts prepared using nondimensionalized quantities such as the Biot and Fourier numbers instead of the actual variables such as thermal conductivity and time?

4–29C What is the physical significance of the Fourier number? Will the Fourier number for a specified heat transfer problem double when the time is doubled?

4–30C How can we use the transient temperature charts when the surface temperature of the geometry is specified instead of the temperature of the surrounding medium and the convection heat transfer coefficient?

4–31C A body at an initial temperature of $T_i$ is brought into a medium at a constant temperature of $T_m$. How can you determine the maximum possible amount of heat transfer between the body and the surrounding medium?

4–32C The Biot number during a heat transfer process between a sphere and its surroundings is determined to be 0.02. Would you use lumped system analysis or the transient temperature charts when determining the midpoint temperature of the sphere? Why?

4–33 A student calculates that the total heat transfer from a spherical copper ball of diameter 15 cm initially at 200°C and its environment at a constant temperature of 25°C during the first 20 min of cooling is 4520 kJ. Is this result reasonable? Why?

4–34 An ordinary egg can be approximated as a 5.5-cm-diameter sphere whose properties are roughly $k = 0.6 \, \text{W/m} \cdot \text{°C}$ and $\alpha = 0.14 \times 10^{-6} \, \text{m}^2/\text{s}$. The egg is initially at a uniform temperature of 8°C and is dropped into boiling water at 97°C. Taking the convection heat transfer coefficient to be $h = 1400 \, \text{W/m}^2 \cdot \text{°C}$, determine how long it will take for the center of the egg to reach 70°C.

4–35 Reconsider Problem 4–34. Using EES (or other) software, investigate the effect of the final center temperature of the egg on the time it will take for the center to reach this temperature. Let the temperature vary from 50°C to 95°C. Plot the time versus the temperature, and discuss the results.

4–36 In a production facility, 3-cm-thick large brass plates ($k = 110 \, \text{W/m} \cdot \text{°C}$, $\rho = 8530 \, \text{kg/m}^3$, $C_p = 380 \, \text{J/kg} \cdot \text{°C}$, and $\alpha = 33.9 \times 10^{-6} \, \text{m}^2/\text{s}$) that are initially at a uniform temperature of 25°C are heated by passing them through an oven maintained at 700°C. The plates remain in the oven for a period of 10 min. Taking the convection heat transfer coefficient to be $h = 80 \, \text{W/m}^2 \cdot \text{°C}$, determine the surface temperature of the plates when they come out of the oven.
4–37 Reconsider Problem 4–36. Using EES (or other) software, investigate the effects of the temperature of the oven and the heating time on the final surface temperature of the plates. Let the oven temperature vary from 500°C to 900°C and the time from 2 min to 30 min. Plot the surface temperature as the functions of the oven temperature and the time, and discuss the results.

4–38 A long 35-cm-diameter cylindrical shaft made of stainless steel 304 (k = 14.9 W/m · °C, ρ = 7900 kg/m³, Cₚ = 477 J/kg · °C, and α = 3.95 × 10⁻⁶ m²/s) comes out of an oven at a uniform temperature of 400°C. The shaft is then allowed to cool slowly in a chamber at 150°C with an average convection heat transfer coefficient of h = 60 W/m² · °C. Determine the temperature at the center of the shaft 20 min after the start of the cooling process. Also, determine the heat transfer per unit length of the shaft during this time period.

Answers: 390°C, 16,015 kJ/m

4–39 Reconsider Problem 4–38. Using EES (or other) software, investigate the effect of the cooling time on the final center temperature of the shaft and the amount of heat transfer. Let the time vary from 5 min to 60 min. Plot the center temperature and the heat transfer as a function of the time, and discuss the results.

4–40E Long cylindrical AISI stainless steel rods (k = 7.74 Btu/h · ft · °F and α = 0.135 ft²/h) of 4-in. diameter are heat-treated by drawing them at a velocity of 10 ft/min through a 30-ft-long oven maintained at 1700°F. The heat transfer coefficient in the oven is 20 Btu/h · ft² · °F. If the rods enter the oven at 85°F, determine their centerline temperature when they leave.

![FIGURE P4–40E](Image)

4–41 In a meat processing plant, 2-cm-thick steaks (k = 0.45 W/m · °C and α = 0.91 × 10⁻⁷ m²/s) that are initially at 25°C are to be cooled by passing them through a refrigeration room at −11°C. The heat transfer coefficient on both sides of the steaks is 9 W/m² · °C. If both surfaces of the steaks are to be cooled to 2°C, determine how long the steaks should be kept in the refrigeration room.

4–42 A long cylindrical wood log (k = 0.17 W/m · °C and α = 1.28 × 10⁻⁷ m²/s) is 10 cm in diameter and is initially at a uniform temperature of 10°C. It is exposed to hot gases at 500°C in a fireplace with a heat transfer coefficient of 13.6 W/m² · °C on the surface. If the ignition temperature of the wood is 420°C, determine how long it will be before the log ignites.

4–43 In Betty Crocker’s Cookbook, it is stated that it takes 2 h 45 min to roast a 3.2-kg rib initially at 4.5°C “rare” in an oven maintained at 163°C. It is recommended that a meat thermometer be used to monitor the cooking, and the rib is considered rare done when the thermometer inserted into the center of the thickest part of the meat registers 60°C. The rib can be treated as a homogeneous spherical object with the properties ρ = 1200 kg/m³, Cₚ = 4.1 kJ/kg · °C, k = 0.45 W/m · °C, and α = 0.91 × 10⁻⁷ m²/s. Determine (a) the heat transfer coefficient at the surface of the rib, (b) the temperature of the outer surface of the rib when it is done, and (c) the amount of heat transferred to the rib. (d) Using the values obtained, predict how long it will take to roast this rib to “medium” level, which occurs when the innermost temperature of the rib reaches 71°C. Compare your result to the listed value of 3 h 20 min.

If the roast rib is to be set on the counter for about 15 min before it is sliced, it is recommended that the rib be taken out of the oven when the thermometer registers about 4°C below the indicated value because the rib will continue cooking even after it is taken out of the oven. Do you agree with this recommendation?

Answers: (a) 156.9 W/m² · °C, (b) 159.5°C, (c) 1629 kJ, (d) 3.0 h

4–44 Repeat Problem 4–43 for a roast rib that is to be “well-done” instead of “rare.” A rib is considered to be well-done when its center temperature reaches 77°C, and the roasting in this case takes about 4 h 15 min.

4–45 For heat transfer purposes, an egg can be considered to be a 5.5-cm-diameter sphere having the properties of water. An egg that is initially at 8°C is dropped into the boiling water at 100°C. The heat transfer coefficient at the surface of the egg is estimated to be 800 W/m² · °C. If the egg is considered cooked when its center temperature reaches 60°C, determine how long the egg should be kept in the boiling water.
4–46 Repeat Problem 4–45 for a location at 1610-m elevation such as Denver, Colorado, where the boiling temperature of water is 94.4°C.

4–47 The author and his 6-year-old son have conducted the following experiment to determine the thermal conductivity of a hot dog. They first boiled water in a large pan and measured the temperature of the boiling water to be 94°C, which is not surprising, since they live at an elevation of about 1650 m in Reno, Nevada. They then took a hot dog that is 12.5 cm long and 2.2 cm in diameter and inserted a thermocouple into the midpoint of the hot dog and another thermocouple just under the skin. They waited until both thermocouples read 20°C, which is the ambient temperature. They then dropped the hot dog into boiling water and observed the changes in both temperatures. Exactly 2 min after the hot dog was dropped into the boiling water, they recorded the center and the surface temperatures to be 59°C and 88°C, respectively. The density of the hot dog can be taken to be 980 kg/m³, which is slightly less than the density of water, since the hot dog was observed to be floating in water while being almost completely immersed. The specific heat of a hot dog can be taken to be 3900 J/kg°C, which is slightly less than that of water, since a hot dog is mostly water. Using transient temperature charts, determine (a) the thermal diffusivity of the hot dog, (b) the thermal conductivity of the hot dog, and (c) the convection heat transfer coefficient.

Answers: (a) $2.02 \times 10^{-7}$ m²/s, (b) 0.771 W/m°C, (c) 467 W/m²°C.

4–48 Using the data and the answers given in Problem 4–47, determine the center and the surface temperatures of the hot dog 4 min after the start of the cooking. Also determine the amount of heat transferred to the hot dog.

4–49E In a chicken processing plant, whole chickens averaging 5 lb each and initially at 72°F are to be cooled in the racks of a large refrigerator that is maintained at 5°F. The entire chicken is to be cooled below 45°F, but the temperature of the chicken is not to drop below 35°F at any point during refrigeration. The convection heat transfer coefficient and thus the rate of heat transfer from the chicken can be controlled by varying the speed of a circulating fan inside. Determine the heat transfer coefficient that will enable us to meet both temperature constraints while keeping the refrigeration time to a minimum. The chicken can be treated as a homogeneous spherical object having the properties $\rho = 74.9$ lbm/ft³, $C_p = 0.98$ Btu/lbm °F, $k = 0.26$ Btu/h·ft·°F, and $\alpha = 0.0035$ ft²/h.

4–50 A person puts a few apples into the freezer at −15°C to cool them quickly for guests who are about to arrive. Initially, the apples are at a uniform temperature of 20°C, and the heat transfer coefficient on the surfaces is 8 W/m²°C. Treating the apples as 9-cm-diameter spheres and taking their properties to be $\rho = 840$ kg/m³, $C_p = 3.81$ kJ/kg°C, $k = 0.418$ W/m°C, and $\alpha = 1.3 \times 10^{-7}$ m²/s, determine the center and surface temperatures of the apples in 1 h. Also, determine the amount of heat transfer from each apple.

4–51 Reconsider Problem 4–50. Using EES (or other) software, investigate the effect of the initial temperature of the apples on the final center and surface temperatures and the amount of heat transfer. Let the initial temperature vary from 2°C to 30°C. Plot the center temperature, the surface temperature, and the amount of heat transfer as a function of the initial temperature, and discuss the results.

4–52 Citrus fruits are very susceptible to cold weather, and extended exposure to subfreezing temperatures can destroy them. Consider an 8-cm-diameter orange that is initially at −15°C. The orange can be treated as a homogeneous object having the properties $\rho = 926$ kg/m³, $C_p = 0.95$ kJ/kg°C, and $k = 0.25$ W/m°C, and $\alpha = 1.3 \times 10^{-7}$ m²/s.
15°C. A cold front moves in one night, and the ambient temperature suddenly drops to −6°C, with a heat transfer coefficient of 15 W/m² · °C. Using the properties of water for the orange and assuming the ambient conditions to remain constant for 4 h before the cold front moves out, determine if any part of the orange will freeze that night.

4–53 An 8-cm-diameter potato (ρ = 1100 kg/m³, Cp = 3900 J/kg · °C, k = 0.6 W/m · °C, and α = 1.4 × 10⁻⁷ m²/s) that is initially at a uniform temperature of 25°C is baked in an oven at 170°C until a temperature sensor inserted to the center of the potato indicates a reading of 70°C. The potato is then taken out of the oven and wrapped in thick towels so that almost no heat is lost from the baked potato. Assuming the heat transfer coefficient in the oven to be 25 W/m² · °C, determine (a) how long the potato is baked in the oven and (b) the final equilibrium temperature of the potato after it is wrapped.

4–54 White potatoes (k = 0.50 W/m · °C and α = 0.13 × 10⁻⁶ m²/s) that are initially at a uniform temperature of 25°C and have an average diameter of 6 cm are to be cooled by refrigerated air at 2°C flowing at a velocity of 4 m/s. The average heat transfer coefficient between the potatoes and the air is experimentally determined to be 19 W/m² · °C. Determine how long it will take for the center temperature of the potatoes to drop to 4°C. Also, determine if any part of the potatoes will experience chilling injury during this process.

4–55E Oranges of 2.5-in. diameter (k = 0.26 Btu/h · ft · °F and α = 1.4 × 10⁻⁶ ft²/s) initially at a uniform temperature of 78°F are to be cooled by refrigerated air at 25°F flowing at a velocity of 1 ft/s. The average heat transfer coefficient between the oranges and the air is experimentally determined to be 4.6 Btu/h · ft² · °F. Determine how long it will take for the center temperature of the oranges to drop to 40°F. Also, determine if any part of the oranges will freeze during this process.

4–56 A 65-kg beef carcass (k = 0.47 W/m · °C and α = 0.13 × 10⁻⁶ m²/s) initially at a uniform temperature of 37°C is to be cooled by refrigerated air at −6°C flowing at a velocity of 1.8 m/s. The average heat transfer coefficient between the carcass and the air is 22 W/m² · °C. Treating the carcass as a cylinder of diameter 24 cm and height 1.4 m and disregarding heat transfer from the base and top surfaces, determine how long it will take for the center temperature of the carcass to drop to 4°C. Also, determine if any part of the carcass will freeze during this process. Answer: 14.0 h

4–57 Layers of 23-cm-thick meat slabs (k = 0.47 W/m · °C and α = 0.13 × 10⁻⁶ m²/s) initially at a uniform temperature of 7°C are to be frozen by refrigerated air at −30°C flowing at a velocity of 1.4 m/s. The average heat transfer coefficient between the meat and the air is 20 W/m² · °C. Assuming the size of the meat slabs to be large relative to their thickness, determine how long it will take for the center temperature of the slabs to drop to −18°C. Also, determine the surface temperature of the meat slab at that time.

4–58E Layers of 6-in.-thick meat slabs (k = 0.26 Btu/h · ft · °F and α = 1.4 × 10⁻⁶ ft²/s) initially at a uniform temperature of 50°F are cooled by refrigerated air at 23°F to a temperature of 36°F at their center in 12 h. Estimate the average heat transfer coefficient during this cooling process.

Answer: 1.5 Btu/h · ft² · °F

4–59 Chickens with an average mass of 1.7 kg (k = 0.45 W/m · °C and α = 0.13 × 10⁻⁶ m²/s) initially at a uniform temperature of 15°C are to be chilled in agitated brine at −10°C. The average heat transfer coefficient between the chicken and the brine is determined experimentally to be 440 W/m² · °C. Taking the average density of the chicken to be 0.95 g/cm³ and treating the chicken as a spherical lump, determine the center and the surface temperatures of the chicken in 2 h and 30 min. Also, determine if any part of the chicken will freeze during this process.
**Transient Heat Conduction in Semi-Infinite Solids**

**4–60C** What is a semi-infinite medium? Give examples of solid bodies that can be treated as semi-infinite mediums for heat transfer purposes.

**4–61C** Under what conditions can a plane wall be treated as a semi-infinite medium?

**4–62C** Consider a hot semi-infinite solid at an initial temperature of \( T_i \) that is exposed to convection to a cooler medium at a constant temperature of \( T_w \), with a heat transfer coefficient of \( h \). Explain how you can determine the total amount of heat transfer from the solid up to a specified time \( t \).

**4–63** In areas where the air temperature remains below 0°C for prolonged periods of time, the freezing of water in underground pipes is a major concern. Fortunately, the soil remains relatively warm during those periods, and it takes weeks for the subfreezing temperatures to reach the water mains in the ground. Thus, the soil effectively serves as an insulation to protect the water from the freezing atmospheric temperatures in winter.

The ground at a particular location is covered with snow pack at 8°C for a continuous period of 60 days, and the average soil properties at that location are \( k = 0.35 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 0.15 \times 10^{-5} \text{ m}^2/\text{s} \). Assuming an initial uniform temperature of 8°C for the ground, determine the minimum burial depth to prevent the water pipes from freezing.

**4–64** The soil temperature in the upper layers of the earth varies with the variations in the atmospheric conditions. Before a cold front moves in, the earth at a location is initially at a uniform temperature of 10°C. Then the area is subjected to a temperature of -10°C and high winds that resulted in a convection heat transfer coefficient of 40 W/m²·°C on the earth’s surface for a period of 10 h. Taking the properties of the soil at that location to be \( k = 0.9 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 1.6 \times 10^{-5} \text{ m}^2/\text{s} \), determine the soil temperature at distances 0, 10, 20, and 50 cm from the earth’s surface at the end of this 10-h period.

**4–65** Reconsider Problem 4–64. Using EES (or other) software, plot the soil temperature as a function of the distance from the earth’s surface as the distance varies from 0 m to 1 m, and discuss the results.

**4–66E** The walls of a furnace are made of 1.5-ft-thick concrete (\( k = 0.64 \text{ Btu/h} \cdot \text{ ft} \cdot \text{°F} \) and \( \alpha = 0.023 \text{ ft}^2/\text{h} \)). Initially, the furnace and the surrounding air are in thermal equilibrium at 70°F. The furnace is then fired, and the inner surfaces of the furnace are subjected to hot gases at 1800°F with a very large heat transfer coefficient. Determine how long it will take for the temperature of the outer surface of the furnace walls to rise to 70.1°F. Answer: 181 min

**4–67** A thick wood slab (\( k = 0.17 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 1.28 \times 10^{-7} \text{ m}^2/\text{s} \)) that is initially at a uniform temperature of 25°C is exposed to hot gases at 550°C for a period of 5 minutes. The heat transfer coefficient between the gases and the wood slab is 35 W/m²·°C. If the ignition temperature of the wood is 450°C, determine if the wood will ignite.

**4–68** A large cast iron container (\( k = 52 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 1.70 \times 10^{-7} \text{ m}^2/\text{s} \)) with 5-cm-thick walls is initially at a uniform temperature of 0°C and is filled with ice at 0°C. Now the outer surfaces of the container are exposed to hot water at 60°C with a very large heat transfer coefficient. Determine how long it will be before the ice inside the container starts melting. Also, taking the heat transfer coefficient on the inner surface of the container to be 250 W/m²·°C, determine the rate of heat transfer to the ice through a 1.2-m-wide and 2-m-high section of the wall when steady operating conditions are reached. Assume the ice starts melting when its inner surface temperature rises to 0.1°C.

**FIGURE P4–68**

**FIGURE P4–64**

**Transient Heat Conduction in Multidimensional Systems**

**4–69C** What is the product solution method? How is it used to determine the transient temperature distribution in a two-dimensional system?

**4–70C** How is the product solution used to determine the variation of temperature with time and position in three-dimensional systems?

**4–71C** A short cylinder initially at a uniform temperature \( T_i \) is subjected to convection from all of its surfaces to a medium at temperature \( T_w \). Explain how you can determine the temperature of the midpoint of the cylinder at a specified time \( t \).

**4–72C** Consider a short cylinder whose top and bottom surfaces are insulated. The cylinder is initially at a uniform temperature \( T_i \) and is subjected to convection from its side surface.
to a medium at temperature \( T_a \) with a heat transfer coefficient of \( h \). Is the heat transfer in this short cylinder one- or two-dimensional? Explain.

4–73 A short brass cylinder (\( \rho = 8530 \text{ kg/m}^3, C_p = 0.389 \text{ kJ/kg} \cdot \text{°C}, k = 110 \text{ W/m} \cdot \text{°C}, \) and \( \alpha = 3.39 \times 10^{-5} \text{ m}^2/\text{s} \)) of diameter \( D = 8 \text{ cm} \) and height \( H = 15 \text{ cm} \) is initially at a uniform temperature of \( T_i = 150\text{°C} \). The cylinder is now placed in atmospheric air at \( 20\text{°C} \), where heat transfer takes place by convection with a heat transfer coefficient of \( h = 40 \text{ W/m}^2 \cdot \text{°C} \). Calculate \( a \) the center temperature of the cylinder, \( b \) the center temperature of the top surface of the cylinder, and \( c \) the total heat transfer from the cylinder 15 min after the start of the cooling.

FIGURE P4–73

4–74 Reconsider Problem 4–73. Using EES (or other) software, investigate the effect of the cooling time on the center temperature of the cylinder, the center temperature of the top surface of the cylinder, and the total heat transfer. Let the time vary from 5 min to 60 min. Plot the center temperature of the cylinder, the center temperature of the top surface, and the total heat transfer as a function of the time, and discuss the results.

4–75 A semi-infinite aluminum cylinder (\( k = 237 \text{ W/m} \cdot \text{°C} \), \( \alpha = 9.71 \times 10^{-5} \text{ m}^2/\text{s} \)) of diameter \( D = 15 \text{ cm} \) is initially at a uniform temperature of \( T_i = 150\text{°C} \). The cylinder is now placed in water at \( 10\text{°C} \), where heat transfer takes place by convection with a heat transfer coefficient of \( h = 140 \text{ W/m}^2 \cdot \text{°C} \). Determine the temperature at the center of the cylinder 5 cm from the end surface 8 min after the start of cooling.

4–76E A hot dog can be considered to be a cylinder 5 in. long and 0.8 in. in diameter whose properties are \( \rho = 61.2 \text{ lbm/ft}^3, C_p = 0.93 \text{ Btu/lbm} \cdot \text{°F}, k = 0.44 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F}, \) and \( \alpha = 0.0077 \text{ ft}^2/\text{h} \). A hot dog initially at 40°F is dropped into boiling water at 212°F. If the heat transfer coefficient at the surface of the hot dog is estimated to be 120 Btu/h \cdot ft^2 \cdot °F, determine the center temperature of the hot dog after 5, 10, and 15 min by treating the hot dog as \( a \) a finite cylinder and \( b \) an infinitely long cylinder.

4–77E Repeat Problem 4–76E for a location at 5300 ft elevation such as Denver, Colorado, where the boiling temperature of water is 202°F.

4–78 A 5-cm-high rectangular ice block (\( k = 2.22 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 0.124 \times 10^{-7} \text{ m}^2/\text{s} \)) initially at \(-20\text{°C} \) is placed on a table on its square base \( 4 \times 4 \text{ cm} \) in size in a room at \( 18\text{°C} \). The heat transfer coefficient on the exposed surfaces of the ice block is \( 12 \text{ W/m}^2 \cdot \text{°C} \). Disregarding any heat transfer from the base to the table, determine how long it will be before the ice block starts melting. Where on the ice block will the first liquid droplets appear?

FIGURE P4–78

4–79 Reconsider Problem 4–78. Using EES (or other) software, investigate the effect of the initial temperature of the ice block on the time period before the ice block starts melting. Let the initial temperature vary from \(-26\text{°C} \) to \(-4\text{°C} \). Plot the time versus the initial temperature, and discuss the results.

4–80 A 2-cm-high cylindrical ice block (\( k = 2.22 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 0.124 \times 10^{-7} \text{ m}^2/\text{s} \)) is placed on a table on its base of diameter \( 2 \text{ cm} \) in a room at \( 20\text{°C} \). The heat transfer coefficient on the exposed surfaces of the ice block is \( 13 \text{ W/m}^2 \cdot \text{°C} \), and heat transfer from the base of the ice block to the table is negligible. If the ice block is not to start melting at any point for at least 2 h, determine what the initial temperature of the ice block should be.

4–81 Consider a cubic block whose sides are 5 cm long and a cylindrical block whose height and diameter are also 5 cm. Both blocks are initially at 20°C and are made of granite (\( k = 2.5 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 1.15 \times 10^{-7} \text{ m}^2/\text{s} \)). Now both blocks are exposed to hot gases at 500°C in a furnace on all of their surfaces with a heat transfer coefficient of \( 40 \text{ W/m}^2 \cdot \text{°C} \). Determine the center temperature of each geometry after 10, 20, and 60 min.

4–82 Repeat Problem 4–81 with the heat transfer coefficient at the top and the bottom surfaces of each block being doubled to \( 80 \text{ W/m}^2 \cdot \text{°C} \).
A 20-cm-long cylindrical aluminum block ($\rho = 2702$ kg/m$^3$, $C_p = 0.896$ kJ/kg · °C, $k = 236$ W/m · °C, and $\alpha = 9.75 \times 10^{-5}$ m$^2$/s), 15 cm in diameter, is initially at a uniform temperature of 20°C. The block is to be heated in a furnace at 1200°C until its center temperature rises to 300°C. If the heat transfer coefficient on all surfaces of the block is 80 W/m$^2$ · °C, determine how long the block should be kept in the furnace. Also, determine the amount of heat transfer from the aluminum block if it is allowed to cool in the room until its temperature drops to 20°C throughout.

Repeat Problem 4–83 for the case where the aluminum block is inserted into the furnace on a low-conductivity material so that the heat transfer to or from the bottom surface of the block is negligible.

6–85 Reconsider Problem 4–83. Using EES (or other) software, investigate the effect of the final center temperature of the block on the heating time and the amount of heat transfer. Let the final center temperature vary from 50°C to 1000°C. Plot the time and the heat transfer as a function of the final center temperature, and discuss the results.

**Special Topic: Refrigeration and Freezing of Foods**

What are the common kinds of microorganisms? What undesirable changes do microorganisms cause in foods?

How does refrigeration prevent or delay the spoilage of foods? Why does freezing extend the storage life of foods for months?

What are the environmental factors that affect the growth rate of microorganisms in foods?

What is the effect of cooking on the microorganisms in foods? Why is it important that the internal temperature of a roast in an oven be raised above 70°C?

How can the contamination of foods with microorganisms be prevented or minimized? How can the growth of microorganisms in foods be retarded? How can the microorganisms in foods be destroyed?

How does (a) the air motion and (b) the relative humidity of the environment affect the growth of microorganisms in foods?

The cooling of a beef carcass from 37°C to 5°C with refrigerated air at 0°C in a chilling room takes about 48 h. To reduce the cooling time, it is proposed to cool the carcass with refrigerated air at 10°C. How would you evaluate this proposal?

Consider the freezing of packaged meat in boxes with refrigerated air. How do (a) the temperature of air, (b) the velocity of air, (c) the capacity of the refrigeration system, and (d) the size of the meat boxes affect the freezing time?

How does the rate of freezing affect the tenderness, color, and the drip of meat during thawing?

It is claimed that beef can be stored for up to two years at −23°C but no more than one year at −12°C. Is this claim reasonable? Explain.

What is a refrigerated shipping dock? How does it reduce the refrigeration load of the cold storage rooms?

How does immersion chilling of poultry compare to forced-air chilling with respect to (a) cooling time, (b) moisture loss of poultry, and (c) microbial growth?

What is the proper storage temperature of frozen poultry? What are the primary methods of freezing for poultry?

What are the factors that affect the quality of frozen fish?

The chilling room of a meat plant is 15 m × 18 m × 5.5 m in size and has a capacity of 350 beef carcasses. The power consumed by the fans and the lights in the chilling room are 22 and 2 kW, respectively, and the room gains heat through its envelope at a rate of 11 kW. The average mass of beef carcasses is 280 kg. The carcasses enter the chilling room at 35°C, after they are washed to facilitate evaporative cooling, and are cooled to 16°C in 12 h. The air enters the chilling room at −2.2°C and leaves at 0.5°C. Determine (a) the refrigeration load of the chilling room and (b) the volume flow rate of air. The average specific heats of beef carcasses and air are 3.14 and 1.0 kJ/kg · °C, respectively, and the density of air can be taken to be 1.28 kg/m$^3$.

Turkeys with a water content of 64 percent that are initially at 1°C and have a mass of about 7 kg are to be frozen by submerging them into brine at −29°C. Using Figure 4–45, determine how long it will take to reduce the temperature of the turkey breast at a depth of 3.8 cm to −18°C. If the temperature at a depth of 3.8 cm in the breast represents the average
temperature of the turkey, determine the amount of heat transfer per turkey assuming (a) the entire water content of the turkey is frozen and (b) only 90 percent of the water content of the turkey is frozen at −18°C. Take the specific heats of turkey to be 2.98 and 1.65 kJ/kg · °C above and below the freezing point of −2.8°C, respectively, and the latent heat of fusion of turkey to be 214 kJ/kg.

**Answers:** (a) 1753 kJ, (b) 1617 kJ

4–102E Consider two 2-cm-thick large steel plates (\(k = 43 \text{ W/m · °C}\) and \(\alpha = 1.17 \times 10^{-5} \text{ m}^2/\text{s}\)) that were put on top of each other while wet and left outside during a cold winter night at −15°C. The next day, a worker needs one of the plates, but the plates are stuck together because the freezing of the water between the two plates has bonded them together. In an effort to melt the ice between the plates and separate them, the worker takes a large hairdryer and blows hot air at 50°C all over the exposed surface of the plate on the top. The convection heat transfer coefficient at the top surface is estimated to be 40 W/m² · °C. Determine how long the worker must keep blowing hot air before the two plates separate.

**Answer:** 482 s

4–106 Consider a curing kiln whose walls are made of 30-cm-thick concrete whose properties are \(k = 0.9 \text{ W/m · °C}\) and \(\alpha = 0.23 \times 10^{-5} \text{ m}^2/\text{s}\). Initially, the kiln and its walls are in equilibrium with the surroundings at 2°C. Then all the doors are closed and the kiln is heated by steam so that the temperature of the inner surface of the walls is raised to 42°C and is maintained at that level for 3 h. The curing kiln is then opened and exposed to the atmospheric air after the steam flow is turned off. If the outer surfaces of the walls of the kiln were insulated, would it save any energy that day during the period the kiln was used for curing for 3 h only, or would it make no difference? Base your answer on calculations.

**Answer:**

4–107 The water main in the cities must be placed at sufficient depth below the earth’s surface to avoid freezing during extended periods of subfreezing temperatures. Determine the minimum depth at which the water main must be placed at a location where the soil is initially at 15°C and the earth’s surface temperature under the worst conditions is expected to remain at −10°C for a period of 75 days. Take the properties of soil at that location to be \(k = 0.7 \text{ W/m · °C}\) and \(\alpha = 1.4 \times 10^{-5} \text{ m}^2/\text{s}\).

**Answer:** 7.05 m

4–108 A hot dog can be considered to be a 12-cm-long cylinder whose diameter is 2 cm and whose properties are \(p = 980 \text{ kg/m}^3\), \(C_p = 3.9 \text{ kJ/kg · °C}\), \(k = 0.76 \text{ W/m · °C}\), and

**Figure P4–108**

**Review Problems**

4–105 Consider two 2-cm-thick large steel plates (\(k = 43 \text{ W/m · °C}\) and \(\alpha = 1.17 \times 10^{-5} \text{ m}^2/\text{s}\)) that were put on top of each other while wet and left outside during a cold winter night at −15°C. The next day, a worker needs one of the plates, but the plates are stuck together because the freezing of the water

**Figure P4–104**

**Air**

**Meat**

**−12°C**

**10 cm**
\[ \alpha = 2 \times 10^{-7} \text{ m}^2/\text{s} \]. A hot dog initially at 5°C is dropped into boiling water at 100°C. The heat transfer coefficient at the surface of the hot dog is estimated to be 600 W/m²·°C. If the hot dog is considered cooked when its center temperature reaches 80°C, determine how long it will take to cook it in the boiling water.

**4–109** A long roll of 2-m-wide and 0.5-cm-thick 1-Mn manganese steel plate coming off a furnace at 820°C is to be quenched in an oil bath \((C_p = 2.0 \text{ kJ/kg} \cdot \text{°C})\) at 45°C. The metal sheet is moving at a steady velocity of 10 m/min, and the oil bath is 5 m long. Taking the convection heat transfer coefficient on both sides of the plate to be 860 W/m²·°C, determine the temperature of the sheet metal when it leaves the oil bath. Also, determine the required rate of heat removal from the oil to keep its temperature constant at 45°C.

**4–110E** In *Betty Crocker’s Cookbook*, it is stated that it takes 5 h to roast a 14-lb stuffed turkey initially at 40°F in an oven maintained at 325°F. It is recommended that a meat thermometer be used to monitor the cooking, and the turkey is considered done when the thermometer inserted deep into the thickest part of the breast or thigh without touching the bone registers 185°F. The turkey can be treated as a homogeneous spherical object with the properties \(\rho = 75 \text{ lbm/ft}^3\), \(C_p = 0.98 \text{ Btu/lbm} \cdot \text{°F}\), \(k = 0.26 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F}\), and \(\alpha = 0.0035 \text{ ft}^2/\text{h}\). Assuming the tip of the thermometer is at one-third radial distance from the center of the turkey, determine (a) the average heat transfer coefficient at the surface of the turkey, (b) the temperature of the skin of the turkey when it is done, and (c) the total amount of heat transferred to the turkey in the oven. Will the reading of the thermometer be more or less than 185°F 5 min after the turkey is taken out of the oven?

**4–111** During a fire, the trunks of some dry oak trees \((k = 0.17 \text{ W/m} \cdot \text{°C} \text{ and } \alpha = 1.28 \times 10^{-7} \text{ m}^2/\text{s})\) that are initially at a uniform temperature of 30°C are exposed to hot gases at 520°C for a period of 5 h, with a heat transfer coefficient of 65 W/m²·°C on the surface. The ignition temperature of the trees is 410°C. Treating the trunks of the trees as long cylindrical rods of diameter 20 cm, determine if these dry trees will ignite as the fire sweeps through them.

**4–112** We often cut a watermelon in half and put it into the freezer to cool it quickly. But usually we forget to check on it and end up having a watermelon with a frozen layer on the top. To avoid this potential problem a person wants to set the timer such that it will go off when the temperature of the exposed surface of the watermelon drops to 3°C.

Consider a 30-cm-diameter spherical watermelon that is cut into two equal parts and put into a freezer at \(-12^\circ\text{C}\). Initially, the entire watermelon is at a uniform temperature of 25°C. In order to measure the temperatures of...
In desert climates, rainfall is not a common occurrence since the raindrops often evaporate before they reach the ground. Consider a raindrop that is initially at a temperature of 50°C and has a diameter of 5 mm. Determine how long it will take for the diameter of the raindrop to reduce to 3 mm as it falls through ambient air at 18°C with a heat transfer coefficient of 400 W/m²·°C. The water temperature can be assumed to remain constant and uniform at 5°C at all times.

Consider a plate of thickness 1 in., a long cylinder of diameter 1 in., and a sphere of diameter 1 in., all initially at 400°F and all made of bronze (k = 15.0 Btu/h·ft·°F and α = 0.333 ft²/h). Now all three of these geometries are exposed to cool air at 75°F on all of their surfaces, with a heat transfer coefficient of 7 Btu/h·ft²·°F. Determine the center temperature of each geometry after 5, 10, and 30 min. Explain why the center temperature of the sphere is always the lowest.

A watermelon initially at 35°C is to be cooled by dropping it into a lake at 15°C. After 4 h and 40 min of cooling, the center temperature of watermelon is measured to be 20°C. Treating the watermelon as a 20-cm-diameter sphere and using the properties k = 0.618 W/m·°C, α = 0.15 × 10⁻⁶ m²/s, ρ = 995 kg/m³, and Cp = 4.18 kJ/kg·°C, determine the average heat transfer coefficient and the surface temperature of watermelon at the end of the cooling period.

10-cm-thick large food slabs tightly wrapped by thin paper are to be cooled in a refrigeration room maintained at 0°C. The heat transfer coefficient on the box surfaces is 25 W/m²·°C and the boxes are to be kept in the refrigeration room for a period of 6 h. If the initial temperature of the boxes is 30°C determine the center temperature of the boxes if the boxes contain (a) margarine (k = 0.233 W/m·°C and α = 0.11 × 10⁻⁶ m²/s), (b) white cake (k = 0.082 W/m·°C and α = 0.10 × 10⁻⁶ m²/s), and (c) chocolate cake (k = 0.106 W/m·°C and α = 0.12 × 10⁻⁶ m²/s).

A 30-cm-diameter, 3.5-m-high cylindrical column of a house made of concrete (k = 0.79 W/m·°C, α = 5.94 × 10⁻⁷ m²/s, ρ = 1600 kg/m³, and Cp = 0.84 kJ/kg·°C) cooled to 16°C during a cold night is heated again during the day by being exposed to ambient air at an average temperature of 28°C with an average heat transfer coefficient of 14 W/m²·°C. Determine (a) how long it will take for the column surface temperature to rise to 27°C, (b) the amount of heat transfer until the center temperature reaches 28°C, and (c) the amount of heat transfer until the surface temperature reaches 27°C.

Long aluminum wires of diameter 3 mm (ρ = 2702 kg/m³, Cp = 0.896 kJ/kg·°C, k = 236 W/m·°C, and α = 9.75 × 10⁻⁷ m²/s) are extruded at a temperature of 350°C and exposed to atmospheric air at 30°C with a heat transfer coefficient of 35 W/m²·°C. (a) Determine how long it will take for the wire temperature to drop to 50°C. (b) If the wire is extruded at a velocity of 10 m/min, determine how far the wire travels after extrusion by the time its temperature drops to 50°C. What change in the cooling process would you propose to shorten this distance? (c) Assuming the aluminum wire leaves the extrusion room at 50°C, determine the rate of heat transfer from the wire to the extrusion room.

Answers: (a) 144 s, (b) 24 m, (c) 856 W
HEAT TRANSFER

4–123 Repeat Problem 4–122 for a copper wire (\( \rho = 8950 \text{ kg/m}^3 \), \( C_p = 0.383 \text{ kJ/kg} \cdot \text{°C} \), \( k = 386 \text{ W/m} \cdot \text{°C} \), and \( \alpha = 1.13 \times 10^{-4} \text{ m}^2/\text{s} \)).

4–124 Consider a brick house (\( k = 0.72 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 0.45 \times 10^{-6} \text{ m}^2/\text{s} \)) whose walls are 10 m long, 3 m high, and 0.3 m thick. The heater of the house broke down one night, and the entire house, including its walls, was observed to be 5°C throughout in the morning. The outdoors warmed up as the day progressed, but no change was felt in the house, which was tightly sealed. Assuming the outer surface temperature of the house to remain constant at 15°C, determine how long it would take for the temperature of the inner surfaces of the walls to rise to 5.1°C.

4–125 A 40-cm-thick brick wall (\( k = 0.72 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 1.6 \times 10^{-7} \text{ m}^2/\text{s} \)) is heated to an average temperature of 18°C by the heating system and the solar radiation incident on it during the day. During the night, the outer surface of the wall is exposed to cold air at 2°C with an average heat transfer coefficient of 20 W/m² · °C, determine the wall temperatures at distances 15, 30, and 40 cm from the outer surface for a period of 2 hours.

4–126 Consider the engine block of a car made of cast iron (\( k = 52 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 1.7 \times 10^{-5} \text{ m}^2/\text{s} \)). The engine can be considered to be a rectangular block whose sides are 80 cm, 40 cm, and 40 cm. The engine is at a temperature of 150°C when it is turned off. The engine is then exposed to atmospheric air at 17°C with a heat transfer coefficient of 6 W/m² · °C. Determine (a) the center temperature of the top surface whose sides are 80 cm and 40 cm and (b) the corner temperature after 45 min of cooling.

4–127 A man is found dead in a room at 16°C. The surface temperature on his waist is measured to be 23°C and the heat transfer coefficient is estimated to be 9 W/m² · °C. Modeling the body as 28-cm diameter, 1.80-m-long cylinder, estimate how long it has been since he died. Take the properties of the body to be \( k = 0.62 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 0.15 \times 10^{-6} \text{ m}^2/\text{s} \), and assume the initial temperature of the body to be 36°C.

Computer, Design, and Essay Problems

4–128 Conduct the following experiment at home to determine the combined convection and radiation heat transfer coefficient at the surface of an apple exposed to the room air. You will need two thermometers and a clock. First, weigh the apple and measure its diameter. You may measure its volume by placing it in a large measuring cup halfway filled with water, and measuring the change in volume when it is completely immersed in the water. Refrigerate the apple overnight so that it is at a uniform temperature in the morning and measure the air temperature in the kitchen. Then take the apple out and stick one of the thermometers to its middle and the other just under the skin. Record both temperatures every 5 min for an hour. Using these two temperatures, calculate the heat transfer coefficient for each interval and take their average. The result is the combined convection and radiation heat transfer coefficient for this heat transfer process. Using your experimental data, also calculate the thermal conductivity and thermal diffusivity of the apple and compare them to the values given above.

4–129 Repeat Problem 4–128 using a banana instead of an apple. The thermal properties of bananas are practically the same as those of apples.

4–130 Conduct the following experiment to determine the time constant for a can of soda and then predict the temperature of the soda at different times. Leave the soda in the refrigerator overnight. Measure the air temperature in the kitchen and the temperature of the soda while it is still in the refrigerator by taping the sensor of the thermometer to the outer surface of the can. Then take the soda out and measure its temperature again in 5 min. Using these values, calculate the exponent \( b \). Using this \( b \)-value, predict the temperatures of the soda in 10, 15, 20, 30, and 60 min and compare the results with the actual temperature measurements. Do you think the lumped system analysis is valid in this case?

4–131 Citrus trees are very susceptible to cold weather, and extended exposure to subfreezing temperatures can destroy the crop. In order to protect the trees from occasional cold fronts with subfreezing temperatures, tree growers in Florida usually install water sprinklers on the trees. When the temperature drops below a certain level, the sprinklers spray water on the trees and their fruits to protect them against the damage the subfreezing temperatures can cause. Explain the basic mechanism behind this protection measure and write an essay on how the system works in practice.
So far we have mostly considered relatively simple heat conduction problems involving simple geometries with simple boundary conditions because only such simple problems can be solved analytically. But many problems encountered in practice involve complicated geometries with complex boundary conditions or variable properties and cannot be solved analytically. In such cases, sufficiently accurate approximate solutions can be obtained by computers using a numerical method.

Analytical solution methods such as those presented in Chapter 2 are based on solving the governing differential equation together with the boundary conditions. They result in solution functions for the temperature at every point in the medium. Numerical methods, on the other hand, are based on replacing the differential equation by a set of algebraic equations for the unknown temperatures at n selected points in the medium, and the simultaneous solution of these equations results in the temperature values at those discrete points.

There are several ways of obtaining the numerical formulation of a heat conduction problem, such as the finite difference method, the finite element method, the boundary element method, and the energy balance (or control volume) method. Each method has its own advantages and disadvantages, and each is used in practice. In this chapter we will use primarily the energy balance approach since it is based on the familiar energy balances on control volumes instead of heavy mathematical formulations, and thus it gives a better physical feel for the problem. Besides, it results in the same set of algebraic equations as the finite difference method. In this chapter, the numerical formulation and solution of heat conduction problems are demonstrated for both steady and transient cases in various geometries.
5–1 WHY NUMERICAL METHODS?

The ready availability of high-speed computers and easy-to-use powerful software packages has had a major impact on engineering education and practice in recent years. Engineers in the past had to rely on analytical skills to solve significant engineering problems, and thus they had to undergo a rigorous training in mathematics. Today’s engineers, on the other hand, have access to a tremendous amount of computation power under their fingertips, and they mostly need to understand the physical nature of the problem and interpret the results. But they also need to understand how calculations are performed by the computers to develop an awareness of the processes involved and the limitations, while avoiding any possible pitfalls.

In Chapter 2 we solved various heat conduction problems in various geometries in a systematic but highly mathematical manner by (1) deriving the governing differential equation by performing an energy balance on a differential volume element, (2) expressing the boundary conditions in the proper mathematical form, and (3) solving the differential equation and applying the boundary conditions to determine the integration constants. This resulted in a solution function for the temperature distribution in the medium, and the solution obtained in this manner is called the analytical solution of the problem. For example, the mathematical formulation of one-dimensional steady heat conduction in a sphere of radius \( r_0 \) whose outer surface is maintained at a uniform temperature of \( T_1 \) with uniform heat generation at a rate of \( \dot{g}_0 \) was expressed as (Fig. 5–1)

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dT}{dr} \right) + \frac{\dot{g}_0}{k} = 0
\]

\[
\frac{dT(0)}{dr} = 0 \quad \text{and} \quad T(r_0) = T_1
\]

whose (analytical) solution is

\[
T(r) = T_1 + \frac{\dot{g}_0}{6k} (r_0^3 - r^3)
\]

This is certainly a very desirable form of solution since the temperature at any point within the sphere can be determined simply by substituting the \( r \)-coordinate of the point into the analytical solution function above. The analytical solution of a problem is also referred to as the exact solution since it satisfies the differential equation and the boundary conditions. This can be verified by substituting the solution function into the differential equation and the boundary conditions. Further, the rate of heat flow at any location within the sphere or its surface can be determined by taking the derivative of the solution function \( T(r) \) and substituting it into Fourier’s law as

\[
\dot{Q}(r) = -kA \frac{dT}{dr} = -k(4\pi r^2) \left( \frac{\dot{g}_0}{3k} \right) = \frac{4\pi \dot{g}_0 r^3}{3}
\]

The analysis above did not require any mathematical sophistication beyond the level of simple integration, and you are probably wondering why anyone would ask for something else. After all, the solutions obtained are exact and
easy to use. Besides, they are instructive since they show clearly the functional dependence of temperature and heat transfer on the independent variable \( r \). Well, there are several reasons for searching for alternative solution methods.

1 Limitations

Analytical solution methods are limited to highly simplified problems in simple geometries (Fig. 5–2). The geometry must be such that its entire surface can be described mathematically in a coordinate system by setting the variables equal to constants. That is, it must fit into a coordinate system perfectly with nothing sticking out or in. In the case of one-dimensional heat conduction in a solid sphere of radius \( r_0 \), for example, the entire outer surface can be described by \( r = r_0 \). Likewise, the surfaces of a finite solid cylinder of radius \( r_0 \) and height \( H \) can be described by \( r = r_0 \) for the side surface and \( z = 0 \) and \( z = H \) for the bottom and top surfaces, respectively. Even minor complications in geometry can make an analytical solution impossible. For example, a spherical object with an extrusion like a handle at some location is impossible to handle analytically since the boundary conditions in this case cannot be expressed in any familiar coordinate system.

Even in simple geometries, heat transfer problems cannot be solved analytically if the thermal conditions are not sufficiently simple. For example, the consideration of the variation of thermal conductivity with temperature, the variation of the heat transfer coefficient over the surface, or the radiation heat transfer on the surfaces can make it impossible to obtain an analytical solution. Therefore, analytical solutions are limited to problems that are simple or can be simplified with reasonable approximations.

2 Better Modeling

We mentioned earlier that analytical solutions are exact solutions since they do not involve any approximations. But this statement needs some clarification. Distinction should be made between an actual real-world problem and the mathematical model that is an idealized representation of it. The solutions we get are the solutions of mathematical models, and the degree of applicability of these solutions to the actual physical problems depends on the accuracy of the model. An “approximate” solution of a realistic model of a physical problem is usually more accurate than the “exact” solution of a crude mathematical model (Fig. 5–3).

When attempting to get an analytical solution to a physical problem, there is always the tendency to oversimplify the problem to make the mathematical model sufficiently simple to warrant an analytical solution. Therefore, it is common practice to ignore any effects that cause mathematical complications such as nonlinearities in the differential equation or the boundary conditions. So it comes as no surprise that nonlinearities such as temperature dependence of thermal conductivity and the radiation boundary conditions are seldom considered in analytical solutions. A mathematical model intended for a numerical solution is likely to represent the actual problem better. Therefore, the numerical solution of engineering problems has now become the norm rather than the exception even when analytical solutions are available.
3 Flexibility

Engineering problems often require extensive parametric studies to understand the influence of some variables on the solution in order to choose the right set of variables and to answer some “what-if” questions. This is an iterative process that is extremely tedious and time-consuming if done by hand. Computers and numerical methods are ideally suited for such calculations, and a wide range of related problems can be solved by minor modifications in the code or input variables. Today it is almost unthinkable to perform any significant optimization studies in engineering without the power and flexibility of computers and numerical methods.

4 Complications

Some problems can be solved analytically, but the solution procedure is so complex and the resulting solution expressions so complicated that it is not worth all that effort. With the exception of steady one-dimensional or transient lumped system problems, all heat conduction problems result in partial differential equations. Solving such equations usually requires mathematical sophistication beyond that acquired at the undergraduate level, such as orthogonality, eigenvalues, Fourier and Laplace transforms, Bessel and Legendre functions, and infinite series. In such cases, the evaluation of the solution, which often involves double or triple summations of infinite series at a specified point, is a challenge in itself (Fig. 5–4). Therefore, even when the solutions are available in some handbooks, they are intimidating enough to scare prospective users away.

5 Human Nature

As human beings, we like to sit back and make wishes, and we like our wishes to come true without much effort. The invention of TV remote controls made us feel like kings in our homes since the commands we give in our comfortable chairs by pressing buttons are immediately carried out by the obedient TV sets. After all, what good is cable TV without a remote control. We certainly would love to continue being the king in our little cubicle in the engineering office by solving problems at the press of a button on a computer (until they invent a remote control for the computers, of course). Well, this might have been a fantasy yesterday, but it is a reality today. Practically all engineering offices today are equipped with high-powered computers with sophisticated software packages, with impressive presentation-style colorful output in graphical and tabular form (Fig. 5–5). Besides, the results are as accurate as the analytical results for all practical purposes. The computers have certainly changed the way engineering is practiced.

The discussions above should not lead you to believe that analytical solutions are unnecessary and that they should be discarded from the engineering curriculum. On the contrary, insight to the physical phenomena and engineering wisdom is gained primarily through analysis. The “feel” that engineers develop during the analysis of simple but fundamental problems serves as an invaluable tool when interpreting a huge pile of results obtained from a computer when solving a complex problem. A simple analysis by hand for a limiting case can be used to check if the results are in the proper range. Also,

Analytical solution:

\[
\frac{T(r, z) - T_0}{T_0 - T_s} = \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n} \sinh \lambda_n (L - z) \sinh (\lambda_n L)
\]

where \(\lambda_n\)’s are roots of \(J_0(\lambda_n r_0) = 0\)

**FIGURE 5–4**
Some analytical solutions are very complex and difficult to use.

**FIGURE 5–5**
The ready availability of high-powered computers with sophisticated software packages has made numerical solution the norm rather than the exception.
nothing can take the place of getting “ball park” results on a piece of paper during preliminary discussions. The calculators made the basic arithmetic operations by hand a thing of the past, but they did not eliminate the need for instructing grade school children how to add or multiply.

In this chapter, you will learn how to formulate and solve heat transfer problems numerically using one or more approaches. In your professional life, you will probably solve the heat transfer problems you come across using a professional software package, and you are highly unlikely to write your own programs to solve such problems. (Besides, people will be highly skeptical of the results obtained using your own program instead of using a well-established commercial software package that has stood the test of time.) The insight you will gain in this chapter by formulating and solving some heat transfer problems will help you better understand the available software packages and be an informed and responsible user.

5–2 FINITE DIFFERENCE FORMULATION OF DIFFERENTIAL EQUATIONS

The numerical methods for solving differential equations are based on replacing the differential equations by algebraic equations. In the case of the popular finite difference method, this is done by replacing the derivatives by differences. Below we will demonstrate this with both first- and second-order derivatives. But first we give a motivational example.

Consider a man who deposits his money in the amount of $A_0 = 100$ in a savings account at an annual interest rate of 18 percent, and let us try to determine the amount of money he will have after one year if interest is compounded continuously (or instantaneously). In the case of simple interest, the money will earn $18 interest, and the man will have $100 + 100 \times 0.18 = 118.00$ in his account after one year. But in the case of compounding, the interest earned during a compounding period will also earn interest for the remaining part of the year, and the year-end balance will be greater than $118.00.

For example, if the money is compounded twice a year, the balance will be $100 + 100 \times (0.18/2) = 109$ after six months, and $109 + 109 \times (0.18/2) = 118.81$ at the end of the year. We could also determine the balance $A$ directly from

$$A = A_0(1 + i)^n = (100)(1 + 0.09)^2 = 118.81$$

(5-4)

where $i$ is the interest rate for the compounding period and $n$ is the number of periods. Using the same formula, the year-end balance is determined for monthly, daily, hourly, minutely, and even secondly compounding, and the results are given in Table 5–1.

Note that in the case of daily compounding, the year-end balance will be $119.72$, which is $1.72$ more than the simple interest case. (So it is no wonder that the credit card companies usually charge interest compounded daily when determining the balance.) Also note that compounding at smaller time intervals, even at the end of each second, does not change the result, and we suspect that instantaneous compounding using “differential” time intervals $dt$ will give the same result. This suspicion is confirmed by obtaining the differential

<table>
<thead>
<tr>
<th>Compounding Period</th>
<th>Number of Periods, $n$</th>
<th>Year-End Balance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 year</td>
<td>1</td>
<td>$118.00$</td>
</tr>
<tr>
<td>6 months</td>
<td>2</td>
<td>118.81</td>
</tr>
<tr>
<td>1 month</td>
<td>12</td>
<td>119.56</td>
</tr>
<tr>
<td>1 week</td>
<td>52</td>
<td>119.68</td>
</tr>
<tr>
<td>1 day</td>
<td>365</td>
<td>119.72</td>
</tr>
<tr>
<td>1 hour</td>
<td>8760</td>
<td>119.72</td>
</tr>
<tr>
<td>1 minute</td>
<td>525,600</td>
<td>119.72</td>
</tr>
<tr>
<td>1 second</td>
<td>31,536,000</td>
<td>119.72</td>
</tr>
<tr>
<td>Instantaneous</td>
<td>$\infty$</td>
<td>119.72</td>
</tr>
</tbody>
</table>
heat transfer in a plane wall. The finite difference formulation is based on replacing the second derivatives by appropriate expressions obtained by writing the Taylor series expansion of the function.

Derivatives are the building blocks of differential equations, and thus we first give a brief review of derivatives. Consider a function $f(x)$ at a point $x$. The first derivative $f'(x)$ is the slope of the function at $x$ and is defined as

$$
\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}
$$

which is the ratio of the increment $\Delta f$ of the function to the increment $\Delta x$ of the independent variable as $\Delta x \to 0$. If we don’t take the indicated limit, we will have the following approximate relation for the derivative:

$$
\frac{df(x)}{dx} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}
$$

This approximate expression of the derivative in terms of differences is the finite difference form of the first derivative. The equation above can also be obtained by writing the Taylor series expansion of the function $f$ about the point $x$,

$$
f(x + \Delta x) = f(x) + \Delta x \frac{df(x)}{dx} + \frac{1}{2} \Delta x^2 \frac{d^2f(x)}{dx^2} + \cdots
$$

and neglecting all the terms in the expansion except the first two. The first term neglected is proportional to $\Delta x^2$, and thus the error involved in each step of this approximation is also proportional to $\Delta x^2$. However, the commutative error involved after $M$ steps in the direction of length $L$ is proportional to $\Delta x$ since $M\Delta x^2 = (L/\Delta x)\Delta x^2 = L\Delta x$. Therefore, the smaller the $\Delta x$, the smaller the error, and thus the more accurate the approximation.

Now consider steady one-dimensional heat transfer in a plane wall of thickness $L$ with heat generation. The wall is subdivided into $M$ sections of equal thickness $\Delta x = L/M$ in the $x$-direction, separated by planes passing through $M + 1$ points $0, 1, 2, \ldots, m - 1, m, m + 1, \ldots, M$ called nodes or nodal points, as shown in Figure 5–7. The $x$-coordinate of any point $m$ is simply $x_m = m\Delta x$, and the temperature at that point is simply $T(x_m) = T_m$.

The heat conduction equation involves the second derivatives of temperature with respect to the space variables, such as $d^2T/dx^2$, and the finite difference formulation is based on replacing the second derivatives by appropriate approximations.
differences. But we need to start the process with first derivatives. Using
Eq. 5–6, the first derivative of temperature \( \frac{dT}{dx} \) at the midpoints \( m - \frac{1}{2} \) and
\( m + \frac{1}{2} \) of the sections surrounding the node \( m \) can be expressed as

\[
\frac{dT}{dx} \bigg|_{m - \frac{1}{2}} = \frac{T_m - T_{m-1}}{\Delta x} \quad \text{and} \quad \frac{dT}{dx} \bigg|_{m + \frac{1}{2}} = \frac{T_{m+1} - T_m}{\Delta x}
\]  

(5-8)

Noting that the second derivative is simply the derivative of the first deriva-
tive, the second derivative of temperature at node \( m \) can be expressed as

\[
\frac{d^2T}{dx^2} \bigg|_m = \frac{\frac{dT}{dx} \bigg|_{m + \frac{1}{2}} - \frac{dT}{dx} \bigg|_{m - \frac{1}{2}}}{\Delta x} = \frac{T_{m+1} - T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}
\]

= \( T_{m-1} - 2T_m + T_{m+1} \)

\[
\frac{\Delta x^2}{5-9}
\]

which is the finite difference representation of the second derivative at a gen-
eral internal node \( m \). Note that the second derivative of temperature at a node
\( m \) is expressed in terms of the temperatures at node \( m \) and its two neighboring
nodes. Then the differential equation

\[
\frac{d^2T}{dx^2} + \frac{\dot{g}_m}{k} = 0
\]  

(5-10)

which is the governing equation for steady one-dimensional heat transfer in a
plane wall with heat generation and constant thermal conductivity, can be ex-
pressed in the finite difference form as (Fig. 5–8)

\[
\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0, \quad m = 1, 2, 3, \ldots, M - 1
\]  

(5-11)

where \( \dot{g}_m \) is the rate of heat generation per unit volume at node \( m \). If the sur-
facet temperatures \( T_0 \) and \( T_M \) are specified, the application of this equation to
each of the \( M - 1 \) interior nodes results in \( M - 1 \) equations for the determi-
nation of \( M - 1 \) unknown temperatures at the interior nodes. Solving these
equations simultaneously gives the temperature values at the nodes. If the
temperatures at the outer surfaces are not known, then we need to obtain two
more equations in a similar manner using the specified boundary conditions.
Then the unknown temperatures at \( M + 1 \) nodes are determined by solving
the resulting system of \( M + 1 \) equations in \( M + 1 \) unknowns simultaneously.

Note that the boundary conditions have no effect on the finite difference
formulation of interior nodes of the medium. This is not surprising since the
control volume used in the development of the formulation does not involve
any part of the boundary. You may recall that the boundary conditions had no
effect on the differential equation of heat conduction in the medium either.

The finite difference formulation above can easily be extended to two- or
three-dimensional heat transfer problems by replacing each second derivative
by a difference equation in that direction. For example, the finite difference
formulation for steady two-dimensional heat conduction in a region with
heat generation and constant thermal conductivity can be expressed in rectangular coordinates as (Fig. 5–9)

$$\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{\Delta x^2} + \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{\Delta y^2} + \frac{\dot{q}_{m,n}}{k} = 0 \quad (5-12)$$

for $m = 1, 2, 3, \ldots, M - 1$ and $n = 1, 2, 3, \ldots, N - 1$ at any interior node $(m, n)$. Note that a rectangular region that is divided into $M$ equal subregions in the $x$-direction and $N$ equal subregions in the $y$-direction has a total of $(M + 1)(N + 1)$ nodes, and Eq. 5–12 can be used to obtain the finite difference equations at $(M - 1)(N - 1)$ of these nodes (i.e., all nodes except those at the boundaries).

The finite difference formulation is given above to demonstrate how difference equations are obtained from differential equations. However, we will use the energy balance approach in the following sections to obtain the numerical formulation because it is more intuitive and can handle boundary conditions more easily. Besides, the energy balance approach does not require having the differential equation before the analysis.

### 5–3 ONE-DIMENSIONAL STEADY HEAT CONDUCTION

In this section we will develop the finite difference formulation of heat conduction in a plane wall using the energy balance approach and discuss how to solve the resulting equations. The energy balance method is based on subdividing the medium into a sufficient number of volume elements and then applying an energy balance on each element. This is done by first selecting the nodal points (or nodes) at which the temperatures are to be determined and then forming elements (or control volumes) over the nodes by drawing lines through the midpoints between the nodes. This way, the interior nodes remain at the middle of the elements, and the properties at the node such as the temperature and the rate of heat generation represent the average properties of the element. Sometimes it is convenient to think of temperature as varying linearly between the nodes, especially when expressing heat conduction between the elements using Fourier’s law.

To demonstrate the approach, again consider steady one-dimensional heat transfer in a plane wall of thickness $L$ with heat generation $\dot{q}(x)$ and constant conductivity $k$. The wall is now subdivided into $M$ equal regions of thickness $\Delta x = L/M$ in the $x$-direction, and the divisions between the regions are selected as the nodes. Therefore, we have $M + 1$ nodes labeled $0, 1, 2, \ldots, M - 1, m, m + 1, \ldots, M$, as shown in Figure 5–10. The $x$-coordinate of any node $m$ is simply $x_m = m\Delta x$, and the temperature at that point is $T(x_m) = T_m$. Elements are formed by drawing vertical lines through the midpoints between the nodes. Note that all interior elements represented by interior nodes are full-size elements (they have a thickness of $\Delta x$), whereas the two elements at the boundaries are half-sized.

To obtain a general difference equation for the interior nodes, consider the element represented by node $m$ and the two neighboring nodes $m - 1$ and $m + 1$. Assuming the heat conduction to be into the element on all surfaces, an energy balance on the element can be expressed as
since the energy content of a medium (or any part of it) does not change under steady conditions and thus $\Delta E_{\text{element}} = 0$. The rate of heat generation within the element can be expressed as

$$\dot{G}_{\text{element}} = \dot{g}_m V_{\text{element}} = \dot{g}_m A \Delta x$$

(5-14)

where $\dot{g}_m$ is the rate of heat generation per unit volume in W/m$^3$ evaluated at node $m$ and treated as a constant for the entire element, and $A$ is heat transfer area, which is simply the inner (or outer) surface area of the wall.

Recall that when temperature varies linearly, the steady state rate of heat conduction across a plane wall of thickness $L$ can be expressed as

$$Q_{\text{cond}} = kA \frac{T_m - T_{m-1}}{\Delta x}$$

(5-15)

where $\Delta T$ is the temperature change across the wall and the direction of heat transfer is from the high temperature side to the low temperature. In the case of a plane wall with heat generation, the variation of temperature is not linear and thus the relation above is not applicable. However, the variation of temperature between the nodes can be approximated as being linear in the determination of heat conduction across a thin layer of thickness $\Delta x$ between two nodes (Fig. 5–11). Obviously the smaller the distance $\Delta x$ between two nodes, the more accurate is this approximation. (In fact, such approximations are the reason for classifying the numerical methods as approximate solution methods. In the limiting case of $\Delta x$ approaching zero, the formulation becomes exact and we obtain a differential equation.) Noting that the direction of heat transfer on both surfaces of the element is assumed to be toward the node $m$, the rate of heat conduction at the left and right surfaces can be expressed as

$$Q_{\text{cond, left}} = kA \frac{T_m - T_{m-1}}{\Delta x} \quad \text{and} \quad Q_{\text{cond, right}} = kA \frac{T_{m+1} - T_m}{\Delta x}$$

(5-16)

Substituting Eqs. 5–14 and 5–16 into Eq. 5–13 gives

$$kA \frac{T_{m-1} - T_m}{\Delta x} + kA \frac{T_{m+1} - T_m}{\Delta x} + \dot{g}_m A \Delta x = 0$$

(5-17)

which simplifies to

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}_m}{k} = 0, \quad m = 1, 2, 3, \ldots, M - 1$$

(5-18)
which is identical to the difference equation (Eq. 5–11) obtained earlier. Again, this equation is applicable to each of the \( M - 1 \) interior nodes, and its application gives \( M - 1 \) equations for the determination of temperatures at \( M + 1 \) nodes. The two additional equations needed to solve for the \( M + 1 \) unknown nodal temperatures are obtained by applying the energy balance on the two elements at the boundaries (unless, of course, the boundary temperatures are specified).

You are probably thinking that if heat is conducted into the element from both sides, as assumed in the formulation, the temperature of the medium will have to rise and thus heat conduction cannot be steady. Perhaps a more realistic approach would be to assume the heat conduction to be into the element on the left side and out of the element on the right side. If you repeat the formulation using this assumption, you will again obtain the same result since the heat conduction term on the right side in this case will involve \( T_m - T_{m+1} \) instead of \( T_{m+1} - T_m \), which is subtracted instead of being added. Therefore, the assumed direction of heat conduction at the surfaces of the volume elements has no effect on the formulation, as shown in Figure 5–12. (Besides, the actual direction of heat transfer is usually not known.) However, it is convenient to assume heat conduction to be into the element at all surfaces and not worry about the sign of the conduction terms. Then all temperature differences in conduction relations are expressed as the temperature of the neighboring node minus the temperature of the node under consideration, and all conduction terms are added.

### Boundary Conditions

Above we have developed a general relation for obtaining the finite difference equation for each interior node of a plane wall. This relation is not applicable to the nodes on the boundaries, however, since it requires the presence of nodes on both sides of the node under consideration, and a boundary node does not have a neighboring node on at least one side. Therefore, we need to obtain the finite difference equations of boundary nodes separately. This is best done by applying an energy balance on the volume elements of boundary nodes.

Boundary conditions most commonly encountered in practice are the specified temperature, specified heat flux, convection, and radiation boundary conditions, and here we develop the finite difference formulations for them for the case of steady one-dimensional heat conduction in a plane wall of thickness \( L \) as an example. The node number at the left surface at \( x = 0 \) is 0, and at the right surface at \( x = L \) it is \( M \). Note that the width of the volume element for either boundary node is \( \Delta x / 2 \).

The specified temperature boundary condition is the simplest boundary condition to deal with. For one-dimensional heat transfer through a plane wall of thickness \( L \), the specified temperature boundary conditions on both the left and right surfaces can be expressed as (Fig. 5–13)

\[
\begin{align*}
T(0) &= T_0 = \text{Specified value} \\
T(L) &= T_m = \text{Specified value}
\end{align*}
\]

where \( T_0 \) and \( T_m \) are the specified temperatures at surfaces at \( x = 0 \) and \( x = L \), respectively. Therefore, the specified temperature boundary conditions are
incorporated by simply assigning the given surface temperatures to the boundary nodes. We do not need to write an energy balance in this case unless we decide to determine the rate of heat transfer into or out of the medium after the temperatures at the interior nodes are determined.

When other boundary conditions such as the specified heat flux, convection, radiation, or combined convection and radiation conditions are specified at a boundary, the finite difference equation for the node at that boundary is obtained by writing an energy balance on the volume element at that boundary. The energy balance is again expressed as

$$
\sum_{\text{all sides}} \dot{Q} + \dot{G}_{\text{element}} = 0
$$

(5-20)

for heat transfer under steady conditions. Again we assume all heat transfer to be into the volume element from all surfaces for convenience in formulation, except for specified heat flux since its direction is already specified. Specified heat flux is taken to be a positive quantity if into the medium and a negative quantity if out of the medium. Then the finite difference formulation at the node $m = 0$ (at the left boundary where $x = 0$) of a plane wall of thickness $L$ during steady one-dimensional heat conduction can be expressed as (Fig. 5–14)

$$
\dot{Q}_{\text{left surface}} + kA \frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0
$$

(5-21)

where $A\Delta x/2$ is the volume of the volume element (note that the boundary element has half thickness), $\dot{g}_0$ is the rate of heat generation per unit volume (in W/m$^3$) at $x = 0$, and $A$ is the heat transfer area, which is constant for a plane wall. Note that we have $\Delta x$ in the denominator of the second term instead of $\Delta x/2$. This is because the ratio in that term involves the temperature difference between nodes 0 and 1, and thus we must use the distance between those two nodes, which is $\Delta x$.

The finite difference form of various boundary conditions can be obtained from Eq. 5–21 by replacing $\dot{Q}_{\text{left surface}}$ by a suitable expression. Next this is done for various boundary conditions at the left boundary.

1. **Specified Heat Flux Boundary Condition**

$$
q_0 A + kA \frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0
$$

(5-22)

**Special case: Insulated Boundary ($\dot{q}_0 = 0$)**

$$
kA \frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0
$$

(5-23)

2. **Convection Boundary Condition**

$$
hA(T_a - T_0) + kA \frac{T_1 - T_0}{\Delta x} + \dot{g}_0(A\Delta x/2) = 0
$$

(5-24)
3. Radiation Boundary Condition

\[ hA(T_m - T_0) + \varepsilon A(T_{\text{surf}} - T_0^4) + kA \frac{T_1 - T_0}{\Delta x} + \dot{g}_0 A \Delta x/2 = 0 \]  

(5-25)

4. Combined Convection and Radiation Boundary Condition

(Fig. 5–15)

\[ hA(T_m - T_0) + \varepsilon A(T_{\text{surf}} - T_0^4) + kA \frac{T_1 - T_0}{\Delta x} + \dot{g}_0 A \Delta x/2 = 0 \]  

(5-26)

or

\[ h_{\text{combined}} A(T_m - T_0) + kA \frac{T_1 - T_0}{\Delta x} + \dot{g}_0 A \Delta x/2 = 0 \]  

(5-27)

5. Combined Convection, Radiation, and Heat Flux Boundary Condition

\[ \dot{q}_A + hA(T_m - T_0) + \varepsilon A(T_{\text{surf}} - T_0^4) + kA \frac{T_1 - T_0}{\Delta x} + \dot{g}_0 A \Delta x/2 = 0 \]  

(5-28)

6. Interface Boundary Condition  

Two different solid media A and B are assumed to be in perfect contact, and thus at the same temperature at the interface at node m (Fig. 5–16). Subscripts A and B indicate properties of media A and B, respectively.

\[ k_A A \frac{T_{m-1} - T_m}{\Delta x} + k_B A \frac{T_{m+1} - T_m}{\Delta x} + \dot{g}_{A,m} A \Delta x/2 + \dot{g}_{B,m} A \Delta x/2 = 0 \]  

(5-29)

In these relations, \( \dot{q}_0 \) is the specified heat flux in W/m², \( h \) is the convection coefficient, \( h_{\text{combined}} \) is the combined convection and radiation coefficient, \( T_m \) is the temperature of the surrounding medium, \( T_{\text{surf}} \) is the temperature of the surrounding surfaces, \( \varepsilon \) is the emissivity of the surface, and \( \sigma \) is the Stefan–Boltzmann constant. The relations above can also be used for node \( M \) on the right boundary by replacing the subscript “0” by “\( M \)” and the subscript “1” by “\( M - 1 \).”

Note that absolute temperatures must be used in radiation heat transfer calculations, and all temperatures should be expressed in K or R when a boundary condition involves radiation to avoid mistakes. We usually try to avoid the radiation boundary condition even in numerical solutions since it causes the finite difference equations to be nonlinear, which are more difficult to solve.

Treating Insulated Boundary Nodes as Interior Nodes: The Mirror Image Concept

One way of obtaining the finite difference formulation of a node on an insulated boundary is to treat insulation as “zero” heat flux and to write an energy balance, as done in Eq. 5–23. Another and more practical way is to treat the node on an insulated boundary as an interior node. Conceptually this is done
by replacing the insulation on the boundary by a mirror and considering the reflection of the medium as its extension (Fig. 5–17). This way the node next to the boundary node appears on both sides of the boundary node because of symmetry, converting it into an interior node. Then using the general formula (Eq. 5–18) for an interior node, which involves the sum of the temperatures of the adjoining nodes minus twice the node temperature, the finite difference formulation of a node \( m = 0 \) on an insulated boundary of a plane wall can be expressed as

\[
\frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2} + \frac{h}{k}T_m = 0 \quad \rightarrow \quad \frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{h}{k}T_0 = 0 \quad (5-30)
\]

which is equivalent to Eq. 5–23 obtained by the energy balance approach.

The mirror image approach can also be used for problems that possess thermal symmetry by replacing the plane of symmetry by a mirror. Alternately, we can replace the plane of symmetry by insulation and consider only half of the medium in the solution. The solution in the other half of the medium is simply the mirror image of the solution obtained.

**EXAMPLE 5–1  Steady Heat Conduction in a Large Uranium Plate**

Consider a large uranium plate of thickness \( L = 4 \) cm and thermal conductivity \( k = 28 \) W/m \( \cdot \) °C in which heat is generated uniformly at a constant rate of \( \dot{g} = 5 \times 10^6 \) W/m\(^3\). One side of the plate is maintained at 0°C by iced water while the other side is subjected to convection to an environment at \( T_e = 30°C \) with a heat transfer coefficient of \( h = 45 \) W/m\(^2\) \( \cdot \) °C, as shown in Figure 5–18. Considering a total of three equally spaced nodes in the medium, two at the boundaries and one at the middle, estimate the exposed surface temperature of the plate under steady conditions using the finite difference approach.

**SOLUTION** A uranium plate is subjected to specified temperature on one side and convection on the other. The unknown surface temperature of the plate is to be determined numerically using three equally spaced nodes.

**Assumptions** 1 Heat transfer through the wall is steady since there is no indication of any change with time. 2 Heat transfer is one-dimensional since the plate is large relative to its thickness. 3 Thermal conductivity is constant. 4 Radiation heat transfer is negligible.

**Properties** The thermal conductivity is given to be \( k = 28 \) W/m \( \cdot \) °C.

**Analysis** The number of nodes is specified to be \( M = 3 \), and they are chosen to be at the two surfaces of the plate and the midpoint, as shown in the figure. Then the nodal spacing \( \Delta x \) becomes

\[
\Delta x = \frac{L}{M - 1} = \frac{0.04 \text{ m}}{3 - 1} = 0.02 \text{ m}
\]

We number the nodes 0, 1, and 2. The temperature at node 0 is given to be \( T_0 = 0°C \), and the temperatures at nodes 1 and 2 are to be determined. This problem involves only two unknown nodal temperatures, and thus we need to have only two equations to determine them uniquely. These equations are obtained by applying the finite difference method to nodes 1 and 2.
Node 1 is an interior node, and the finite difference formulation at that node is obtained directly from Eq. 5–18 by setting $m = 1$:

$$\frac{T_0 - 2T_1 + T_2}{\Delta x^2} + \frac{\dot{g}_1}{k} = 0 \quad \rightarrow \quad \frac{0 - 2T_1 + T_3}{\Delta x^2} + \frac{\dot{g}_1}{k} = 0 \quad \rightarrow \quad 2T_1 - T_2 = \frac{\dot{g}_1 \Delta x^2}{k}$$

Node 2 is a boundary node subjected to convection, and the finite difference formulation at that node is obtained by writing an energy balance on the volume element of thickness $\Delta x/2$ at that boundary by assuming heat transfer to be into the medium at all sides:

$$hA(T_w - T_2) + kA \frac{T_1 - T_2}{\Delta x} + \dot{g}_2(\Delta x/2) = 0$$

Canceling the heat transfer area $A$ and rearranging give

$$T_1 - \left(1 + \frac{h \Delta x}{k}\right)T_2 = -\frac{h \Delta x}{k}T_w - \frac{\dot{g}_2 \Delta x^2}{2k}$$

Equations (1) and (2) form a system of two equations in two unknowns $T_1$ and $T_2$. Substituting the given quantities and simplifying gives

$$2T_1 - T_2 = 71.43 \quad \text{(in °C)}$$
$$T_1 - 1.032T_2 = -36.68 \quad \text{(in °C)}$$

This is a system of two algebraic equations in two unknowns and can be solved easily by the elimination method. Solving the first equation for $T_1$ and substituting into the second equation result in an equation in $T_2$ whose solution is

$$T_2 = 136.1°C$$

This is the temperature of the surface exposed to convection, which is the desired result. Substitution of this result into the first equation gives $T_1 = 103.8°C$, which is the temperature at the middle of the plate.

**Discussion** The purpose of this example is to demonstrate the use of the finite difference method with minimal calculations, and the accuracy of the result was not a major concern. But you might still be wondering how accurate the result obtained above is. After all, we used a mesh of only three nodes for the entire plate, which seems to be rather crude. This problem can be solved analytically as described in Chapter 2, and the analytical (exact) solution can be shown to be

$$T(x) = \frac{0.5ghL^2/k + \dot{g}L + T_0h}{hL + k} x - \frac{\dot{g}x^2}{2k}$$

Substituting the given quantities, the temperature of the exposed surface of the plate at $x = L = 0.04$ m is determined to be 136.0°C, which is almost identical to the result obtained here with the approximate finite difference method (Fig. 5–19). Therefore, highly accurate results can be obtained with numerical methods by using a limited number of nodes.

**FIGURE 5–19**
Despite being approximate in nature, highly accurate results can be obtained by numerical methods.
EXAMPLE 5–2  Heat Transfer from Triangular Fins

Consider an aluminum alloy fin \((k = 180 \text{ W/m} \cdot \text{°C})\) of triangular cross section with length \(L = 5 \text{ cm}\), base thickness \(b = 1 \text{ cm}\), and very large width \(w\) in the direction normal to the plane of paper, as shown in Figure 5–20. The base of the fin is maintained at a temperature of \(T_0 = 200\text{°C}\). The fin is losing heat to the surrounding medium at \(T_w = 25\text{°C}\) with a heat transfer coefficient of \(h = 15 \text{ W/m}^2 \cdot \text{°C}\). Using the finite difference method with six equally spaced nodes along the fin in the \(x\)-direction, determine \((a)\) the temperatures at the nodes, \((b)\) the rate of heat transfer from the fin for \(w = 1 \text{ m}\), and \((c)\) the fin efficiency.

**SOLUTION**  A long triangular fin attached to a surface is considered. The nodal temperatures, the rate of heat transfer, and the fin efficiency are to be determined numerically using six equally spaced nodes.

**Assumptions** 1 Heat transfer is steady since there is no indication of any change with time. 2 The temperature along the fin varies in the \(x\) direction only. 3 Thermal conductivity is constant. 4 Radiation heat transfer is negligible.

**Properties**  The thermal conductivity is given to be \(k = 180 \text{ W/m} \cdot \text{°C}\).

**Analysis**  \((a)\) The number of nodes in the fin is specified to be \(M = 6\), and their location is as shown in the figure. Then the nodal spacing \(\Delta x\) becomes

\[
\Delta x = \frac{L}{M - 1} = \frac{0.05 \text{ m}}{6 - 1} = 0.01 \text{ m}
\]

The temperature at node 0 is given to be \(T_0 = 200\text{°C}\), and the temperatures at the remaining five nodes are to be determined. Therefore, we need to have five equations to determine them uniquely. Nodes 1, 2, 3, and 4 are interior nodes, and the finite difference formulation for a general interior node \(m\) is obtained by applying an energy balance on the volume element of this node. Noting that heat transfer is steady and there is no heat generation in the fin and assuming heat transfer to be into the medium at all sides, the energy balance can be expressed as

\[
\sum_{\text{all sides}} \dot{Q} = 0 \rightarrow kA_{\text{left}} \frac{T_{m-1} - T_m}{\Delta x} + kA_{\text{right}} \frac{T_{m+1} - T_m}{\Delta x} + hA_{\text{conv}}(T_w - T_m) = 0
\]

Note that heat transfer areas are different for each node in this case, and using geometrical relations, they can be expressed as

\[
A_{\text{left}} = (\text{Height} \times \text{Width})_{m-\frac{1}{2}} = 2w[L - (m - 1/2)\Delta x] \tan \theta
\]

\[
A_{\text{right}} = (\text{Height} \times \text{Width})_{m+\frac{1}{2}} = 2w[L - (m + 1/2)\Delta x] \tan \theta
\]

\[
A_{\text{conv}} = 2 \times \text{Length} \times \text{Width} = 2w(\Delta x \cos \theta)
\]

Substituting,

\[
2kw[L - (m - 1/2)\Delta x] \tan \theta \frac{T_{m-1} - T_m}{\Delta x} + 2kw[L - (m + 1/2)\Delta x] \tan \theta \frac{T_{m+1} - T_m}{\Delta x} + h\frac{2w\Delta x}{\cos \theta}(T_w - T_m) = 0
\]
Dividing each term by $2kW\tan\theta/\Delta x$ gives

$$
\left[ 1 - (m + \frac{1}{2}) \frac{\Delta x}{L} \right] (T_{m-1} - T_m) + \left[ 1 - (m + \frac{1}{2}) \frac{\Delta x}{L} \right] (T_{m+1} - T_m) + \frac{h(\Delta x)^2}{kL \sin \theta} (T_m - T_m) = 0
$$

Note that

$$
\tan \theta = \frac{h}{L} = \frac{0.5 \text{ cm}}{5 \text{ cm}} = 0.1 \quad \Rightarrow \quad \theta = \tan^{-1} 0.1 = 5.71^\circ
$$

Also, $\sin 5.71^\circ = 0.0995$. Then the substitution of known quantities gives

$$
(5.5 - m)T_{m-1} - (10.00838 - 2m)T_m + (4.5 - m)T_{m+1} = -0.209
$$

Now substituting 1, 2, 3, and 4 for $m$ results in these finite difference equations for the interior nodes:

1. $m = 1: \quad -8.00838T_1 + 3.5T_2 = -900.209$  
2. $m = 2: \quad 3.5T_1 - 6.00838T_2 + 2.5T_3 = -0.209$  
3. $m = 3: \quad 2.5T_2 - 4.00838T_3 + 1.5T_4 = -0.209$  
4. $m = 4: \quad 1.5T_3 - 2.00838T_4 + 0.5T_5 = -0.209$

The finite difference equation for the boundary node 5 is obtained by writing an energy balance on the volume element of length $\Delta x/2$ at that boundary, again by assuming heat transfer to be into the medium at all sides (Fig. 5–21):

$$
kA_{\text{left}} \frac{T_4 - T_5}{\Delta x} + hA_{\text{conv}} (T_w - T_5) = 0
$$

where

$$
A_{\text{left}} = 2w \frac{\Delta x}{2} \tan \theta \quad \text{and} \quad A_{\text{conv}} = 2w \frac{\Delta x/2}{\cos \theta}
$$

Canceling $w$ in all terms and substituting the known quantities gives

$$
T_4 - 1.00838T_5 = -0.209
$$

Equations (1) through (5) form a linear system of five algebraic equations in five unknowns. Solving them simultaneously using an equation solver gives

$$
T_1 = 198.6^\circ C, \quad T_2 = 197.1^\circ C, \quad T_3 = 195.7^\circ C, \quad T_4 = 194.3^\circ C, \quad T_5 = 192.9^\circ C
$$

which is the desired solution for the nodal temperatures.

(b) The total rate of heat transfer from the fin is simply the sum of the heat transfer from each volume element to the ambient, and for $w = 1$ m it is determined from
The finite difference formulation of steady heat conduction problems usually results in a system of $N$ algebraic equations in $N$ unknown nodal temperatures that need to be solved simultaneously. When $N$ is small (such as 2 or 3), we can use the elementary elimination method to eliminate all unknowns except one and then solve for that unknown (see Example 5–1). The other unknowns are then determined by back substitution. When $N$ is large, which is usually the case, the elimination method is not practical and we need to use a more systematic approach that can be adapted to computers.

There are numerous systematic approaches available in the literature, and they are broadly classified as direct and iterative methods. The direct methods are based on a fixed number of well-defined steps that result in the solution in a systematic manner. The iterative methods, on the other hand, are based on an initial guess for the solution that is refined by iteration until a specified convergence criterion is satisfied (Fig. 5–22). The direct methods usually require a large amount of computer memory and computation time, while the iterative methods are more flexible and can handle larger systems.

The heat transfer surface area is $w\Delta x/\cos \theta$ for the boundary nodes 0 and 5, and twice as large for the interior nodes 1, 2, 3, and 4, we have

$$Q_{\text{fin}} = \sum_{n=0}^{5} \dot{Q}_{\text{element}, m} = \sum_{n=0}^{5} hA_{\text{conv}, m}(T_m - T_w)$$

Noting that the heat transfer surface area is $w\Delta x/\cos \theta$ for the boundary nodes 0 and 5, and twice as large for the interior nodes 1, 2, 3, and 4, we have

$$\dot{Q}_{\text{fin}} = h \frac{w\Delta x}{\cos \theta} [T_0 - T_w] + 2(T_1 - T_w) + 2(T_2 - T_w) + 2(T_3 - T_w) + 2(T_4 - T_w) + (T_5 - T_w)$$

$$= h \frac{w\Delta x}{\cos \theta} [T_0 + 2(T_1 + T_2 + T_3 + T_4) + T_5 - 10T_w]$$

$$= (15 \text{ W/m}^2 \cdot ^\circ \text{C}) \frac{(1 \text{ m})(0.01 \text{ m})}{\cos 5.71^\circ} \left[200 + 2 \times 785.7 + 192.9 - 10 \times 25\right]$$

$$= 258.4 \text{ W}$$

(c) If the entire fin were at the base temperature of $T_0 = 200^\circ \text{C}$, the total rate of heat transfer from the fin for $w = 1 \text{ m}$ would be

$$\dot{Q}_{\text{max}} = hA_{\text{fin}, \text{total}}(T_0 - T_w) = h(2wL/\cos \theta)(T_0 - T_w)$$

$$= (15 \text{ W/m}^2 \cdot ^\circ \text{C})[2(1 \text{ m})(0.05 \text{ m})/\cos 5.71^\circ](200 - 25)^\circ \text{C}$$

$$= 263.8 \text{ W}$$

Then the fin efficiency is determined from

$$\eta_{\text{fin}} = \frac{\dot{Q}_{\text{fin}}}{\dot{Q}_{\text{max}}} = \frac{258.4 \text{ W}}{263.8 \text{ W}} = 0.98$$

which is less than 1, as expected. We could also determine the fin efficiency in this case from the proper fin efficiency curve in Chapter 3, which is based on the analytical solution. We would read 0.98 for the fin efficiency, which is identical to the value determined above numerically.
and they are more suitable for systems with a relatively small number of equations. The computer memory requirements for iterative methods are minimal, and thus they are usually preferred for large systems. The convergence of iterative methods to the desired solution, however, may pose a problem.

5–4 = TWO-DIMENSIONAL STEADY HEAT CONDUCTION

In Section 5–3 we considered one-dimensional heat conduction and assumed heat conduction in other directions to be negligible. Many heat transfer problems encountered in practice can be approximated as being one-dimensional, but this is not always the case. Sometimes we need to consider heat transfer in other directions as well when the variation of temperature in other directions is significant. In this section we will consider the numerical formulation and solution of two-dimensional steady heat conduction in rectangular coordinates using the finite difference method. The approach presented below can be extended to three-dimensional cases.

Consider a rectangular region in which heat conduction is significant in the x- and y-directions. Now divide the x-y plane of the region into a rectangular mesh of nodal points spaced \( \Delta x \) and \( \Delta y \) apart in the x- and y-directions, respectively, as shown in Figure 5–23, and consider a unit depth of \( \Delta z = 1 \) in the z-direction. Our goal is to determine the temperatures at the nodes, and it is convenient to number the nodes and describe their position by the numbers instead of actual coordinates. A logical numbering scheme for two-dimensional problems is the double subscript notation \((m, n)\) where \( m = 0, 1, 2, \ldots, M \) is the node count in the x-direction and \( n = 0, 1, 2, \ldots, N \) is the node count in the y-direction. The coordinates of the node \((m, n)\) are simply \( x = m\Delta x \) and \( y = n\Delta y \), and the temperature at the node \((m, n)\) is denoted by \( T_{m,n} \).

Now consider a volume element of size \( \Delta x \times \Delta y \times 1 \) centered about a general interior node \((m, n)\) in a region in which heat is generated at a rate of \( g \) and the thermal conductivity \( k \) is constant, as shown in Figure 5–24. Again assuming the direction of heat conduction to be toward the node under consideration at all surfaces, the energy balance on the volume element can be expressed as

\[
\text{Rate of heat conduction at the left, top, right, and bottom surfaces} + \text{Rate of heat generation inside the element} = \text{Rate of change of the energy content of the element}
\]

or

\[
\dot{Q}_{\text{cond, left}} + \dot{Q}_{\text{cond, top}} + \dot{Q}_{\text{cond, right}} + \dot{Q}_{\text{cond, bottom}} + \dot{Q}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0 \tag{5-31}
\]

for the steady case. Again assuming the temperatures between the adjacent nodes to vary linearly and noting that the heat transfer area is \( A_x = \Delta y \times 1 = \Delta y \) in the x-direction and \( A_y = \Delta x \times 1 = \Delta x \) in the y-direction, the energy balance relation above becomes
Dividing each term by $\Delta x \times \Delta y$ and simplifying gives

$$\frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{\Delta x^2} + \frac{T_{m,n-1} - 2T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{\dot{g}_{m,n}}{k} = 0 \quad (5-33)$$

for $m = 1, 2, 3, \ldots, M - 1$ and $n = 1, 2, 3, \ldots, N - 1$. This equation is identical to Eq. 5–12 obtained earlier by replacing the derivatives in the differential equation by differences for an interior node $(m, n)$. Again a rectangular region $M$ equally spaced nodes in the $x$-direction and $N$ equally spaced nodes in the $y$-direction has a total of $(M + 1)(N + 1)$ nodes, and Eq. 5–33 can be used to obtain the finite difference equations at all interior nodes.

In finite difference analysis, usually a square mesh is used for simplicity (except when the magnitudes of temperature gradients in the $x$- and $y$-directions are very different), and thus $\Delta x$ and $\Delta y$ are taken to be the same. Then $\Delta x = \Delta y = l$, and the relation above simplifies to

$$T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{\dot{g}_{m,n}}{k} l^2 = 0 \quad (5-34)$$

That is, the finite difference formulation of an interior node is obtained by adding the temperatures of the four nearest neighbors of the node, subtracting four times the temperature of the node itself, and adding the heat generation term. It can also be expressed in this form, which is easy to remember:

$$T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\dot{g}_{\text{node}} l^2}{k} = 0 \quad (5-35)$$

When there is no heat generation in the medium, the finite difference equation for an interior node further simplifies to $T_{\text{node}} = (T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}})/4$, which has the interesting interpretation that the temperature of each interior node is the arithmetic average of the temperatures of the four neighboring nodes. This statement is also true for the three-dimensional problems except that the interior nodes in that case will have six neighboring nodes instead of four.

**Boundary Nodes**

The development of finite difference formulation of boundary nodes in two- (or three-) dimensional problems is similar to the development in the one-dimensional case discussed earlier. Again, the region is partitioned between the nodes by forming volume elements around the nodes, and an energy balance is written for each boundary node. Various boundary conditions can be handled as discussed for a plane wall, except that the volume elements in the two-dimensional case involve heat transfer in the $y$-direction as well as the $x$-direction. Insulated surfaces can still be viewed as "mirrors," and the
The finite difference formulation of a boundary node is obtained by writing an energy balance on its volume element.

\[
\dot{Q}_{\text{left}} + \dot{Q}_{\text{top}} + \dot{Q}_{\text{right}} + \dot{Q}_{\text{bottom}} + \frac{\dot{g}_V V_s}{k} = 0
\]

**FIGURE 5–25**
The finite difference formulation of a boundary node is obtained by writing an energy balance on its volume element.

**FIGURE 5–26**
Schematic for Example 5–3 and the nodal network (the boundaries of volume elements of the nodes are indicated by dashed lines).

**EXAMPLE 5–3 Steady Two-Dimensional Heat Conduction in L-Bars**
Consider steady heat transfer in an L-shaped solid body whose cross section is given in Figure 5–26. Heat transfer in the direction normal to the plane of the paper is negligible, and thus heat transfer in the body is two-dimensional. The thermal conductivity of the body is \( k = 15 \text{ W/m} \cdot \text{°C} \), and heat is generated in the body at a rate of \( g = 2 \times 10^6 \text{ W/m}^3 \). The left surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of 90°C. The entire top surface is subjected to convection to ambient air at \( T_a = 25°C \) with a convection coefficient of \( h = 80 \text{ W/m}^2 \cdot \text{°C} \), and the right surface is subjected to heat flux at a uniform rate of \( q_R = 5000 \text{ W/m}^2 \). The nodal network of the problem consists of 15 equally spaced nodes with \( \Delta x = \Delta y = 1.2 \text{ cm} \), as shown in the figure. Five of the nodes are at the bottom surface, and thus their temperatures are known. Obtain the finite difference equations at the remaining nine nodes and determine the nodal temperatures by solving them.

**SOLUTION**
Heat transfer in a long L-shaped solid bar with specified boundary conditions is considered. The nine unknown nodal temperatures are to be determined with the finite difference method.

**Assumptions**
1. Heat transfer is steady and two-dimensional, as stated. 2. Thermal conductivity is constant. 3. Heat generation is uniform. 4. Radiation heat transfer is negligible.

**Properties**
The thermal conductivity is given to be \( k = 15 \text{ W/m} \cdot \text{°C} \).

**Analysis**
We observe that all nodes are boundary nodes except node 5, which is an interior node. Therefore, we will have to rely on energy balances to obtain the finite difference equations. But first we form the volume elements by partitioning the region among the nodes equitably by drawing dashed lines between the nodes. If we consider the volume element represented by an interior node to be *full size* (i.e., \( \Delta x \times \Delta y \times 1 \)), then the element represented by a regular boundary node such as node 2 becomes *half size* (i.e., \( \Delta x \times \Delta y/2 \times 1 \)), and a corner node such as node 1 is *quarter size* (i.e., \( \Delta x/2 \times \Delta y/2 \times 1 \)). Keeping Eq. 5–36 in mind for the energy balance, the finite difference equations for each of the nine nodes are obtained as follows:

**(a) Node 1**
The volume element of this corner node is insulated on the left and subjected to convection at the top and to conduction at the right and bottom surfaces. An energy balance on this element gives [Fig. 5–27a]
\[ 0 + h \frac{\Delta x}{2} (T_x - T_1) + k \frac{\Delta y}{2} T_2 - T_1 + k \frac{\Delta x}{2} T_4 - T_1 + \frac{\Delta y}{2} \frac{\Delta y}{2} = 0 \]

Taking \( \Delta x = \Delta y = l \), it simplifies to

\[ -\left(2 + \frac{hl}{k}\right) T_1 + T_2 + T_4 = -\frac{hl}{k} T_x - \frac{\hat{g}l^2}{2k} \]

(b) Node 2. The volume element of this boundary node is subjected to convection at the top and to conduction at the right, bottom, and left surfaces. An energy balance on this element gives [Fig. 5–27b]

\[ h\Delta x(T_x - T_2) + k \frac{\Delta y}{2} T_2 - T_3 + k \frac{\Delta x}{2} T_5 - T_3 + \frac{\Delta y}{2} \frac{\Delta y}{2} = 0 \]

Taking \( \Delta x = \Delta y = l \), it simplifies to

\[ T_1 - \left(4 + \frac{2hl}{k}\right) T_2 + 2T_3 = -\frac{2hl}{k} T_x - \frac{\hat{g}l^2}{2k} \]

(c) Node 3. The volume element of this corner node is subjected to convection at the top and right surfaces and to conduction at the bottom and left surfaces. An energy balance on this element gives [Fig. 5–28a]

\[ h\left(\frac{\Delta x}{2} + \frac{\Delta y}{2}\right)(T_x - T_3) + k \frac{\Delta x}{2} T_6 - T_3 + \frac{\Delta y}{2} \frac{\Delta y}{2} + \hat{g}_3 \frac{\Delta y}{2} = 0 \]

Taking \( \Delta x = \Delta y = l \), it simplifies to

\[ T_2 - \left(2 + \frac{2hl}{k}\right) T_3 + T_6 = -\frac{2hl}{k} T_x - \frac{\hat{g}l^2}{2k} \]

(d) Node 4. This node is on the insulated boundary and can be treated as an interior node by replacing the insulation by a mirror. This puts a reflected image of node 5 to the left of node 4. Noting that \( \Delta x = \Delta y = l \), the general interior node relation for the steady two-dimensional case (Eq. 5–35) gives [Fig. 5–28b]

\[ T_3 + T_1 + T_5 + T_{10} - 4T_4 + \frac{\hat{g}l^2}{k} = 0 \]

or, noting that \( T_{10} = 90^\circ \text{C} \),

\[ T_1 - 4T_4 + 2T_3 = -90 - \frac{\hat{g}l^2}{k} \]

(e) Node 5. This is an interior node, and noting that \( \Delta x = \Delta y = l \), the finite difference formulation of this node is obtained directly from Eq. 5–35 to be [Fig. 5–29a]

\[ T_4 + T_2 + T_6 + T_{11} - 4T_5 + \frac{\hat{g}l^2}{k} = 0 \]
or, noting that \( T_{11} = 90^\circ C \),
\[
T_2 + T_4 - 4T_3 + T_6 = -90 - \frac{\dot{\ell}d^2}{k}
\]

(f) **Node 6.** The volume element of this inner corner node is subjected to convection on the L-shaped exposed surfaces and to conduction at other surfaces. An energy balance on this element gives [Fig. 5–29b]
\[
h \left( \frac{\Delta x}{2} + \frac{\Delta y}{2} \right) (T_2 - T_0) + k \frac{\Delta y}{2} T_7 - T_6 + k \Delta x \frac{T_{12} - T_6}{\Delta y} + k \Delta y \frac{T_5 - T_6}{\Delta x} + \dot{g}_{b} \frac{3 \Delta x \Delta y}{4} = 0
\]

Taking \( \Delta x = \Delta y = l \) and noting that \( T_{12} = 90^\circ C \), it simplifies to
\[
T_1 + 2T_3 - \left( 6 + \frac{2hl}{k} \right) T_0 + T_7 = -180 - \frac{2hl}{k} T_6 - \frac{3\dot{\ell}d^2}{2k}
\]

(g) **Node 7.** The volume element of this boundary node is subjected to convection at the top and to conduction at the right, bottom, and left surfaces. An energy balance on this element gives [Fig. 5–30a]
\[
h \Delta x (T_2 - T_3) + k \frac{\Delta y}{2} T_7 - T_3 + k \Delta x T_{13} - T_7
\]
\[
+ k \frac{\Delta y}{2} T_{11} - T_7 + \dot{g}_{b} \frac{\Delta y}{2} = 0
\]

Taking \( \Delta x = \Delta y = l \) and noting that \( T_{13} = 90^\circ C \), it simplifies to
\[
T_6 - \left( 4 + \frac{2hl}{k} \right) T_0 + T_8 = -180 - \frac{2hl}{k} T_3 - \frac{\dot{\ell}d^2}{k}
\]

(h) **Node 8.** This node is identical to Node 7, and the finite difference formulation of this node can be obtained from that of Node 7 by shifting the node numbers by \( 1 \) (i.e., replacing subscript \( m \) by \( m + 1 \)). It gives
\[
T_7 - \left( 4 + \frac{2hl}{k} \right) T_0 + T_9 = -180 - \frac{2hl}{k} T_5 - \frac{\dot{\ell}d^2}{k}
\]

(i) **Node 9.** The volume element of this corner node is subjected to convection at the top surface, to heat flux at the right surface, and to conduction at the bottom and left surfaces. An energy balance on this element gives [Fig. 5–30b]
\[
h \frac{\Delta x}{2} (T_2 - T_0) + q_{b} \frac{\Delta y}{2} + k \frac{\Delta x}{2} T_{15} - T_0 + k \frac{\Delta y}{2} T_8 - T_9 + \dot{g}_{b} \frac{\Delta x \Delta y}{2} = 0
\]

Taking \( \Delta x = \Delta y = l \) and noting that \( T_{15} = 90^\circ C \), it simplifies to
\[
T_8 - \left( 2 + \frac{hl}{k} \right) T_0 = -90 - \frac{\dot{\ell}d^2}{k} - \frac{hl}{k} T_2 - \frac{\dot{\ell}d^2}{2k}
\]
Irregular Boundaries

In problems with simple geometries, we can fill the entire region using simple volume elements such as strips for a plane wall and rectangular elements for two-dimensional conduction in a rectangular region. We can also use cylindrical or spherical shell elements to cover the cylindrical and spherical bodies entirely. However, many geometries encountered in practice such as turbine blades or engine blocks do not have simple shapes, and it is difficult to fill such geometries having irregular boundaries with simple volume elements.

A practical way of dealing with such geometries is to replace the irregular geometry by a series of simple volume elements, as shown in Figure 5–31.

This completes the development of finite difference formulation for this problem. Substituting the given quantities, the system of nine equations for the determination of nine unknown nodal temperatures becomes

\[ -2.064T_1 + T_2 + T_3 = -11.2 \]
\[ T_1 - 4.128T_2 + T_3 + 2T_5 = -22.4 \]
\[ T_2 - 2.128T_3 + T_6 = -12.8 \]
\[ T_1 - 4T_4 + 2T_5 = -109.2 \]
\[ T_2 + T_4 - 4T_5 + T_6 = -109.2 \]
\[ T_3 + 2T_5 - 6.128T_6 + T_7 = -212.0 \]
\[ T_6 - 4.128T_7 + T_8 = -202.4 \]
\[ T_7 - 4.128T_8 + T_9 = -202.4 \]
\[ T_8 - 2.064T_9 = -105.2 \]

which is a system of nine algebraic equations with nine unknowns. Using an equation solver, its solution is determined to be

\begin{align*}
T_1 &= 112.1^\circ\text{C} & T_2 &= 110.8^\circ\text{C} & T_3 &= 106.6^\circ\text{C} \\
T_4 &= 109.4^\circ\text{C} & T_5 &= 108.1^\circ\text{C} & T_6 &= 103.2^\circ\text{C} \\
T_7 &= 97.3^\circ\text{C} & T_8 &= 96.3^\circ\text{C} & T_9 &= 97.6^\circ\text{C}
\end{align*}

Note that the temperature is the highest at node 1 and the lowest at node 8. This is consistent with our expectations since node 1 is the farthest away from the bottom surface, which is maintained at 90°C and has one side insulated, and node 8 has the largest exposed area relative to its volume while being close to the surface at 90°C.

**Irregular Boundaries**

In problems with simple geometries, we can fill the entire region using simple volume elements such as strips for a plane wall and rectangular elements for two-dimensional conduction in a rectangular region. We can also use cylindrical or spherical shell elements to cover the cylindrical and spherical bodies entirely. However, many geometries encountered in practice such as turbine blades or engine blocks do not have simple shapes, and it is difficult to fill such geometries having irregular boundaries with simple volume elements.

A practical way of dealing with such geometries is to replace the irregular geometry by a series of simple volume elements, as shown in Figure 5–31. This simple approach is often satisfactory for practical purposes, especially when the nodes are closely spaced near the boundary. More sophisticated approaches are available for handling irregular boundaries, and they are commonly incorporated into the commercial software packages.

**EXAMPLE 5–4**  Heat Loss through Chimneys

Hot combustion gases of a furnace are flowing through a square chimney made of concrete \((k = 1.4 \text{ W/m} \cdot \text{°C})\). The flow section of the chimney is 20 cm \(\times\) 20 cm, and the thickness of the wall is 20 cm. The average temperature of the
hot gases in the chimney is $T_i = 300\, ^\circ C$, and the average convection heat transfer coefficient inside the chimney is $h_i = 70\, W/m^2 \cdot ^\circ C$. The chimney is losing heat from its outer surface to the ambient air at $T_o = 20\, ^\circ C$ by convection with a heat transfer coefficient of $h_o = 21\, W/m^2 \cdot ^\circ C$ and to the sky by radiation. The emissivity of the outer surface of the wall is $\varepsilon = 0.9$, and the effective sky temperature is estimated to be $260\, K$. Using the finite difference method with $\Delta x = \Delta y = 10\, cm$ and taking full advantage of symmetry, determine the temperatures at the nodal points of a cross section and the rate of heat loss for a 1-m-long section of the chimney.

**SOLUTION** Heat transfer through a square chimney is considered. The nodal temperatures and the rate of heat loss per unit length are to be determined with the finite difference method.

**Assumptions**
1. Heat transfer is steady since there is no indication of change with time.
2. Heat transfer through the chimney is two-dimensional since the height of the chimney is large relative to its cross section, and thus heat conduction through the chimney in the axial direction is negligible. It is tempting to simplify the problem further by considering heat transfer in each wall to be one-dimensional, which would be the case if the walls were thin and thus the corner effects were negligible. This assumption cannot be justified in this case since the walls are very thick and the corner sections constitute a considerable portion of the chimney structure.
3. Thermal conductivity is constant.

**Properties** The properties of chimney are given to be $k = 1.4\, W/m \cdot ^\circ C$ and $\varepsilon = 0.9$.

**Analysis** The cross section of the chimney is given in Figure 5–32. The most striking aspect of this problem is the apparent symmetry about the horizontal and vertical lines passing through the midpoint of the chimney as well as the diagonal axes, as indicated on the figure. Therefore, we need to consider only one-eighth of the geometry in the solution whose nodal network consists of nine equally spaced nodes.

No heat can cross a symmetry line, and thus symmetry lines can be treated as insulated surfaces and thus “mirrors” in the finite difference formulation. Then the nodes in the middle of the symmetry lines can be treated as interior nodes by using mirror images. Six of the nodes are boundary nodes, so we will have to write energy balances to obtain their finite difference formulations. First we partition the region among the nodes equitably by drawing dashed lines between the nodes through the middle. Then the region around a node surrounded by the boundary or the dashed lines represents the volume element of the node. Considering a unit depth and using the energy balance approach for the boundary nodes (again assuming all heat transfer into the volume element for convenience) and the formula for the interior nodes, the finite difference equations for the nine nodes are determined as follows:

(a) Node 1. On the inner boundary, subjected to convection, Figure 5–33a

$$0 + h_i \frac{\Delta x}{2} (T_i - T_1) + k \frac{\Delta y}{2} (T_3 - T_1) + k \frac{\Delta x}{2} (T_5 - T_1) + 0 = 0$$

Taking $\Delta x = \Delta y = l$, it simplifies to

$$-\left(2 + \frac{h_i l}{k}\right) T_1 + T_2 + T_3 = -\frac{h_i l}{k} T_i$$
(b) Node 2. On the inner boundary, subjected to convection, Figure 5–33b
\[ k \frac{\Delta y}{2} \frac{T_1 - T_2}{\Delta x} + h_i \frac{\Delta x}{2} (T_i - T_2) + 0 + k \frac{\Delta y}{2} \frac{T_1 - T_2}{\Delta y} = 0 \]

Taking \( \Delta x = \Delta y = l \), it simplifies to
\[ T_1 - \left( 3 + \frac{h_i l}{k} \right) T_2 + 2 T_4 = -\frac{h_i l}{k} T_i \]

(c) Nodes 3, 4, and 5. (Interior nodes, Fig. 5–34)

Node 3: \( T_4 + T_1 + T_4 + T_6 - 4 T_3 = 0 \)
Node 4: \( T_3 + T_2 + T_3 + T_7 - 4 T_4 = 0 \)
Node 5: \( T_4 + T_4 + T_8 + T_6 - 4 T_5 = 0 \)

(d) Node 6. (On the outer boundary, subjected to convection and radiation)
\[ 0 + k \frac{\Delta x}{2} \frac{T_5 - T_6}{\Delta y} + k \frac{\Delta y}{2} \frac{T_6 - T_8}{\Delta x} + h_o \frac{\Delta x}{2} (T_o - T_6) + \varepsilon \sigma \frac{\Delta x}{2} (T_{sky}^4 - T_6^4) = 0 \]

Taking \( \Delta x = \Delta y = l \), it simplifies to
\[ T_2 + T_3 - \left( 2 + \frac{h_o l}{k} \right) T_6 = -\frac{h_o l}{k} T_o - \frac{\varepsilon \sigma l}{k} (T_{sky}^4 - T_6^4) \]

(e) Node 7. (On the outer boundary, subjected to convection and radiation, Fig. 5–35)
\[ k \frac{\Delta y}{2} \frac{T_6 - T_7}{\Delta x} + k \frac{\Delta x}{2} \frac{T_7 - T_8}{\Delta y} + k \frac{\Delta y}{2} \frac{T_8 - T_7}{\Delta x} + h_o \frac{\Delta x}{2} (T_o - T_7) + \varepsilon \sigma \frac{\Delta x}{2} (T_{sky}^4 - T_7^4) = 0 \]

Taking \( \Delta x = \Delta y = l \), it simplifies to
\[ 2T_4 + T_6 - \left( 4 + \frac{2h_o l}{k} \right) T_7 + T_8 = -\frac{2h_o l}{k} T_o - \frac{2\varepsilon \sigma l}{k} (T_{sky}^4 - T_7^4) \]

(f) Node 8. Same as Node 7, except shift the node numbers up by 1 (replace 4 by 5, 6 by 7, 7 by 8, and 8 by 9 in the last relation)
\[ 2T_5 + T_7 - \left( 4 + \frac{2h_o l}{k} \right) T_8 + T_9 = -\frac{2h_o l}{k} T_o - \frac{2\varepsilon \sigma l}{k} (T_{sky}^4 - T_8^4) \]

(g) Node 9. (On the outer boundary, subjected to convection and radiation, Fig. 5–35)
\[ k \frac{\Delta y}{2} \frac{T_8 - T_9}{\Delta x} + 0 + h_o \frac{\Delta x}{2} (T_o - T_9) + \varepsilon \sigma \frac{\Delta x}{2} (T_{sky}^4 - T_9^4) = 0 \]
Taking $\Delta x = \Delta y = l$, it simplifies to

$$T_8 - \left( 1 + \frac{h_x l}{k} \right) T_9 = - \frac{h_y l}{k} T_o - \frac{\varepsilon a l}{k} (T_9 - T_8)$$

This problem involves radiation, which requires the use of absolute temperature, and thus all temperatures should be expressed in Kelvin. Alternately, we could use °C for all temperatures provided that the four temperatures in the radiation terms are expressed in the form $(T + 273)\text{K}$. Substituting the given quantities, the system of nine equations for the determination of nine unknown nodal temperatures in a form suitable for use with the Gauss-Seidel iteration method becomes

\begin{align*}
T_1 &= (T_2 + T_3 + 2865)/7 \\
T_2 &= (T_1 + 2T_4 + 2865)/8 \\
T_3 &= (T_1 + 2T_4 + T_6)/4 \\
T_4 &= (T_2 + T_1 + T_5 + T_7)/4 \\
T_5 &= (2T_4 + T_3)/4 \\
T_6 &= (T_2 + T_3 + 456.2 - 0.3645 \times 10^{-9} T_8^4)/3.5 \\
T_7 &= (2T_4 + T_6 + T_8 + 912.4 - 0.729 \times 10^{-9} T_7^4)/7 \\
T_8 &= (2T_4 + T_7 + T_9 + 912.4 - 0.729 \times 10^{-9} T_8^4)/7 \\
T_9 &= (T_8 + 456.2 - 0.3645 \times 10^{-9} T_9^4)/2.5
\end{align*}

which is a system of nonlinear equations. Using an equation solver, its solution is determined to be

\begin{align*}
T_1 &= 545.7 \text{K} = 272.6\text{°C} & T_2 &= 529.2 \text{K} = 256.1\text{°C} & T_3 &= 425.2 \text{K} = 152.1\text{°C} \\
T_4 &= 411.2 \text{K} = 138.0\text{°C} & T_5 &= 362.1 \text{K} = 89.0\text{°C} & T_6 &= 332.9 \text{K} = 59.7\text{°C} \\
T_7 &= 328.1 \text{K} = 54.9\text{°C} & T_8 &= 313.1 \text{K} = 39.9\text{°C} & T_9 &= 296.5 \text{K} = 23.4\text{°C}
\end{align*}

The variation of temperature in the chimney is shown in Figure 5–36.

Note that the temperatures are highest at the inner wall (but less than 300°C) and lowest at the outer wall (but more than 260 K), as expected.

The average temperature at the outer surface of the chimney weighed by the surface area is

$$T_{\text{wall, out}} = \frac{(0.5T_4 + T_7 + T_8 + 0.5T_6)}{0.5 + 1 + 1 + 0.5} = \frac{0.5 \times 332.9 + 328.1 + 313.1 + 0.5 \times 296.5}{3} = 318.6 \text{K}$$

Then the rate of heat loss through the 1-m-long section of the chimney can be determined approximately from
5–5 TRANSIENT HEAT CONDUCTION

So far in this chapter we have applied the finite difference method to steady heat transfer problems. In this section we extend the method to solve transient problems.

We applied the finite difference method to steady problems by discretizing the problem in the space variables and solving for temperatures at discrete points called the nodes. The solution obtained is valid for any time since under steady conditions the temperatures do not change with time. In transient problems, however, the temperatures change with time as well as position, and thus the finite difference solution of transient problems requires discretization in time in addition to discretization in space, as shown in Figure 5–37. This is done by selecting a suitable time step $\Delta t$ and solving for the unknown nodal temperatures repeatedly for each $\Delta t$ until the solution at the desired time is obtained. For example, consider a hot metal object that is taken out of the oven at an initial temperature of $T_i$ at time $t = 0$ and is allowed to cool in ambient air. If a time step of $\Delta t = 5$ min is chosen, the determination of the temperature distribution in the metal piece after 3 h requires the determination of the temperatures at $3 \times 60/5 = 36$ times, or in 36 time steps. Therefore, the computation time of this problem will be 36 times that of a steady problem. Choosing a smaller $\Delta t$ will increase the accuracy of the solution, but it will also increase the computation time.

In transient problems, the superscript $i$ is used as the index or counter of time steps, with $i = 0$ corresponding to the specified initial condition. In the case of the hot metal piece discussed above, $i = 1$ corresponds to $t = 1 \times \Delta t = 5$ min, $i = 2$ corresponds to $t = 2 \times \Delta t = 10$ min, and a general

\[
\dot{Q}_{\text{chimney}} = h_o A_o (T_{\text{wall, out}} - T_o) + \varepsilon \sigma A_o (T^4_{\text{wall, out}} - T^4_{\text{sky}})
\]

\[
= (21 \text{ W/m}^2 \cdot \text{K})[4 \times (0.6 \text{ m})(1 \text{ m})][318.6 - 293] \text{K}
\]

\[
+ 0.9(5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)
\]

\[
[4 \times (0.6 \text{ m})(1 \text{ m})][318.6 \text{ K}^4 - (260 \text{ K})^4]
\]

\[
= 1291 + 702 = 1993 \text{ W}
\]

We could also determine the heat transfer by finding the average temperature of the inner wall, which is $(272.6 + 256.1)/2 = 264.4^\circ \text{C}$, and applying Newton's law of cooling at that surface:

\[
\dot{Q}_{\text{chimney}} = h_i A_i (T_i - T_{\text{wall, in}})
\]

\[
= (70 \text{ W/m}^2 \cdot \text{K})[4 \times (0.2 \text{ m})(1 \text{ m})][300 - 264.4] \text{C} = 1994 \text{ W}
\]

The difference between the two results is due to the approximate nature of the numerical analysis.

**Discussion** We used a relatively crude numerical model to solve this problem to keep the complexities at a manageable level. The accuracy of the solution obtained can be improved by using a finer mesh and thus a greater number of nodes. Also, when radiation is involved, it is more accurate (but more laborious) to determine the heat losses for each node and add them up instead of using the average temperature.

We could also determine the heat transfer by finding the average temperature of the inner wall, which is $(272.6 + 256.1)/2 = 264.4^\circ \text{C}$, and applying Newton's law of cooling at that surface:

\[
\dot{Q}_{\text{chimney}} = h_o A_o (T_{\text{wall, out}} - T_o) + \varepsilon \sigma A_o (T^4_{\text{wall, out}} - T^4_{\text{sky}})
\]

\[
= (21 \text{ W/m}^2 \cdot \text{K})[4 \times (0.6 \text{ m})(1 \text{ m})][318.6 - 293] \text{K}
\]

\[
+ 0.9(5.67 \times 10^{-8} \text{ W/m}^2 \cdot \text{K}^4)
\]

\[
[4 \times (0.6 \text{ m})(1 \text{ m})][318.6 \text{ K}^4 - (260 \text{ K})^4]
\]

\[
= 1291 + 702 = 1993 \text{ W}
\]
time step \( i \) corresponds to \( t_i = i \Delta t \). The notation \( T_m^i \) is used to represent the temperature at the node \( m \) at time step \( i \).

The formulation of transient heat conduction problems differs from that of steady ones in that the transient problems involve an additional term representing the change in the energy content of the medium with time. This additional term appears as a first derivative of temperature with respect to time in the differential equation, and as a change in the internal energy content during \( \Delta t \) in the energy balance formulation. The nodes and the volume elements in transient problems are selected as they are in the steady case, and, again assuming all heat transfer is into the element for convenience, the energy balance on a volume element during a time interval \( \Delta t \) can be expressed as

\[
\text{Heat transferred into the volume element from all of its surfaces during } \Delta t + \text{Heat generated within the volume element during } \Delta t = \text{The change in the energy content of the volume element during } \Delta t
\]

or

\[
\Delta t \sum_{\text{All sides}} \dot{Q} + \Delta t \dot{G}_{\text{element}} = \Delta E_{\text{element}} \tag{5-37}
\]

where the rate of heat transfer \( \dot{Q} \) normally consists of conduction terms for interior nodes, but may involve convection, heat flux, and radiation for boundary nodes.

Noting that \( \Delta E_{\text{element}} = mC\Delta T = \rho V_{\text{element}} C \Delta T \), where \( \rho \) is density and \( C \) is the specific heat of the element, dividing the earlier relation by \( \Delta t \) gives

\[
\sum_{\text{All sides}} \dot{Q} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = \rho V_{\text{element}} C \frac{\Delta T}{\Delta t} \tag{5-38}
\]

or, for any node \( m \) in the medium and its volume element,

\[
\sum_{\text{All sides}} \dot{Q} + \dot{G}_{\text{element}} = \rho V_{\text{element}} C \frac{T_{m+1}^i - T_m^i}{\Delta t} \tag{5-39}
\]

where \( T_m^i \) and \( T_{m+1}^i \) are the temperatures of node \( m \) at times \( t_i = i \Delta t \) and \( t_{i+1} = (i + 1) \Delta t \), respectively, and \( T_{m+1}^i - T_m^i \) represents the temperature change of the node during the time interval \( \Delta t \) between the time steps \( i \) and \( i + 1 \) (Fig. 5–38).

Note that the ratio \( (T_{m+1}^i - T_m^i)/\Delta t \) is simply the finite difference approximation of the partial derivative \( \partial T/\partial t \) that appears in the differential equations of transient problems. Therefore, we would obtain the same result for the finite difference formulation if we followed a strict mathematical approach instead of the energy balance approach used above. Also note that the finite difference formulations of steady and transient problems differ by the single term on the right side of the equal sign, and the format of that term remains the same in all coordinate systems regardless of whether heat transfer is one-, two-, or three-dimensional. For the special case of \( T_{m+1}^i = T_m^i \) (i.e., no change in temperature with time), the formulation reduces to that of steady case, as expected.
The nodal temperatures in transient problems normally change during each time step, and you may be wondering whether to use temperatures at the previous time step \( i \) or the new time step \( i + 1 \) for the terms on the left side of Eq. 5–39. Well, both are reasonable approaches and both are used in practice. The finite difference approach is called the explicit method in the first case and the implicit method in the second case, and they are expressed in the general form as (Fig. 5–39)

**Explicit method:**
\[
\sum_{\text{All sides}} \dot{Q}^i + \dot{G}_{\text{element}}^i = \rho V_{\text{element}} C \frac{T_{m+1}^i - T_m^i}{\Delta t}
\]

**Implicit method:**
\[
\sum_{\text{All sides}} \dot{Q}^{i+1} + \dot{G}_{\text{element}}^{i+1} = \rho V_{\text{element}} C \frac{T_{m+1}^{i+1} - T_m^i}{\Delta t}
\]

It appears that the time derivative is expressed in *forward difference* form in the explicit case and *backward difference* form in the implicit case. Of course, it is also possible to mix the two fundamental formulations of Eqs. 5–40 and 5–41 and come up with more elaborate formulations, but such formulations offer little insight and are beyond the scope of this text. Note that both formulations are simply expressions between the nodal temperatures before and after a time interval and are based on determining the new temperatures \( T_{m+1}^i \) using the previous temperatures \( T_m^i \). The explicit and implicit formulations given here are quite general and can be used in any coordinate system regardless of the dimension of heat transfer. The volume elements in multidimensional cases simply have more surfaces and thus involve more terms in the summation.

The explicit and implicit methods have their advantages and disadvantages, and one method is not necessarily better than the other one. Next you will see that the explicit method is easy to implement but imposes a limit on the allowable time step to avoid instabilities in the solution, and the implicit method requires the nodal temperatures to be solved simultaneously for each time step but imposes no limit on the magnitude of the time step. We will limit the discussion to one- and two-dimensional cases to keep the complexities at a manageable level, but the analysis can readily be extended to three-dimensional cases and other coordinate systems.

**Transmit Heat Conduction in a Plane Wall**

Consider transient one-dimensional heat conduction in a plane wall of thickness \( L \) with heat generation \( g(x, t) \) that may vary with time and position and constant conductivity \( k \) with a mesh size of \( \Delta x = L/M \) and nodes 0, 1, 2, . . . , \( M \) in the \( x \)-direction, as shown in Figure 5–40. Noting that the volume element of a general interior node \( m \) involves heat conduction from two sides and the volume of the element is \( V_{\text{element}} = A \Delta x \), the transient finite difference formulation for an interior node can be expressed on the basis of Eq. 5–39 as

\[
kA \frac{T_{m-1}^i - T_m^i}{\Delta x} + kA \frac{T_{m+1}^i - T_m^i}{\Delta x} + g_m A \Delta x = \rho A \Delta x C \frac{T_{m+1}^{i+1} - T_m^i}{\Delta t}
\]

**FIGURE 5–39**
The formulation of explicit and implicit methods differs at the time step (previous or new) at which the heat transfer and heat generation terms are expressed.

**FIGURE 5–40**
The nodal points and volume elements for the transient finite difference formulation of one-dimensional conduction in a plane wall.
Canceling the surface area \( A \) and multiplying by \( \Delta x/k \), it simplifies to

\[
T_{m-1} - 2T_m + T_{m+1} + \frac{g_m \Delta x^2}{k} = \frac{\Delta x^2}{\alpha \Delta t} (T_{m+1}^{i+1} - T_m^i) \quad (5-43)
\]

where \( \alpha = k/\rho C \) is the thermal diffusivity of the wall material. We now define a dimensionless mesh Fourier number as

\[
\tau = \frac{\alpha \Delta t}{\Delta x^2} \quad (5-44)
\]

Then Eq. 5–43 reduces to

\[
T_{m-1} - 2T_m + T_{m+1} + \frac{g_m \Delta x^2}{k} = \frac{T_{m+1}^{i+1} - T_m^i}{\tau} \quad (5-45)
\]

Note that the left side of this equation is simply the finite difference formulation of the problem for the steady case. This is not surprising since the formulation must reduce to the steady case for \( T_{m+1}^{i+1} = T_m^i \). Also, we are still not committed to explicit or implicit formulation since we did not indicate the time step on the left side of the equation. We now obtain the explicit finite difference formulation by expressing the left side at time step \( i \) as

\[
T_{i-1}^i - 2T_m^i + T_{m+1}^i + \frac{g_m^i \Delta x^2}{k} = \frac{T_{m+1}^{i+1} - T_m^i}{\tau} \quad \text{(explicit)} \quad (5-46)
\]

This equation can be solved explicitly for the new temperature \( T_m^{i+1} \) and thus the name explicit method to give

\[
T_m^{i+1} = \tau(T_{m+1}^{i+1} + T_m^{i+1}) + (1 - 2\tau) T_m^i + \frac{g_m^i \Delta x^2}{k} \quad (5-47)
\]

for all interior nodes \( m = 1, 2, 3, \ldots, M - 1 \) in a plane wall. Expressing the left side of Eq. 5–45 at time step \( i + 1 \) instead of \( i \) would give the implicit finite difference formulation as

\[
T_{m-1}^{i+1} - 2T_m^{i+1} + T_{m+1}^{i+1} + \frac{g_m^{i+1} \Delta x^2}{k} = \frac{T_{m+1}^{i+1} - T_m^{i+1}}{\tau} \quad \text{(implicit)} \quad (5-48)
\]

which can be rearranged as

\[
\tau T_m^{i+1} - (1 + 2\tau) T_m^{i+1} + \tau T_m^{i+1} + \frac{g_m^{i+1} \Delta x^2}{k} + T_m^i = 0 \quad (5-49)
\]

The application of either the explicit or the implicit formulation to each of the \( M - 1 \) interior nodes gives \( M - 1 \) equations. The remaining two equations are obtained by applying the same method to the two boundary nodes unless, of course, the boundary temperatures are specified as constants (invariant with time). For example, the formulation of the convection boundary condition at the left boundary (node 0) for the explicit case can be expressed as (Fig. 5–41)

\[
hA(T_m - T_0) + kA \frac{T_1 - T_0}{\Delta x} + g_0 A \frac{\Delta x}{2} = \rho A \frac{\Delta x}{2} C \frac{T_0^{i+1} - T_0^i}{\Delta t} \quad (5-50)
\]
which simplifies to

\[ T_i^{n+1} = \left( 1 - 2\tau - 2\tau \frac{h \Delta x}{k} \right) T_i^n + 2\tau T_i^n + \frac{h \Delta x}{k} T_{i-1}^n + \tau \frac{g \Delta x^2}{k} \]  

(5-51)

Note that in the case of no heat generation and \( \tau = 0.5 \), the explicit finite difference formulation for a general interior node reduces to \( T_i^{n+1} = (T_{i-1}^n + T_{i+1}^n)/2 \), which has the interesting interpretation that the temperature of an interior node at the new time step is simply the average of the temperatures of its neighboring nodes at the previous time step.

Once the formulation (explicit or implicit) is complete and the initial condition is specified, the solution of a transient problem is obtained by marching in time using a step size of \( \Delta t \) as follows: select a suitable time step \( \Delta t \) and determine the nodal temperatures from the initial condition. Taking the initial temperatures as the previous solution \( T_m^0 \) at \( t = 0 \), obtain the new solution \( T_m^{i+1} \) at all nodes at time \( t = \Delta t \) using the transient finite difference relations. Now using the solution just obtained at \( t = \Delta t \) as the previous solution \( T_m^i \), obtain the new solution \( T_m^{i+1} \) at \( t = 2\Delta t \) using the same relations. Repeat the process until the solution at the desired time is obtained.

### Stability Criterion for Explicit Method: Limitation on \( \Delta t \)

The explicit method is easy to use, but it suffers from an undesirable feature that severely restricts its utility: the explicit method is not unconditionally stable, and the largest permissible value of the time step \( \Delta t \) is limited by the stability criterion. If the time step \( \Delta t \) is not sufficiently small, the solutions obtained by the explicit method may oscillate wildly and diverge from the actual solution. To avoid such divergent oscillations in nodal temperatures, the value of \( \Delta t \) must be maintained below a certain upper limit established by the stability criterion. It can be shown mathematically or by a physical argument based on the second law of thermodynamics that the stability criterion is satisfied if the coefficients of all \( T_i^m \) in the \( T_i^{m+1} \) expressions (called the primary coefficients) are greater than or equal to zero for all nodes \( m \) (Fig. 5–42). Of course, all the terms involving \( T_i^m \) for a particular node must be grouped together before this criterion is applied.

Different equations for different nodes may result in different restrictions on the size of the time step \( \Delta t \), and the criterion that is most restrictive should be used in the solution of the problem. A practical approach is to identify the equation with the smallest primary coefficient since it is the most restrictive and to determine the allowable values of \( \Delta t \) by applying the stability criterion to that equation only. A \( \Delta t \) value obtained this way will also satisfy the stability criterion for all other equations in the system.

For example, in the case of transient one-dimensional heat conduction in a plane wall with specified surface temperatures, the explicit finite difference equations for all the nodes (which are interior nodes) are obtained from Eq. 5–47. The coefficient of \( T_i^m \) in the \( T_i^{m+1} \) expression is \( 1 - 2\tau \), which is independent of the node number \( m \), and thus the stability criterion for all nodes in this case is \( 1 - 2\tau \geq 0 \) or

\[ \tau = \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2} \quad \text{(interior nodes, one-dimensional heat transfer in rectangular coordinates)} \]  

(5-52)
When the material of the medium and thus its thermal diffusivity $\alpha$ is known and the value of the mesh size $\Delta x$ is specified, the largest allowable value of the time step $\Delta t$ can be determined from this relation. For example, in the case of a brick wall ($\alpha = 0.45 \times 10^{-6} \text{ m}^2/\text{s}$) with a mesh size of $\Delta x = 0.01 \text{ m}$, the upper limit of the time step is

$$\Delta t \leq \frac{1}{2} \frac{\Delta x^2}{\alpha} = \frac{(0.01 \text{ m})^2}{2(0.45 \times 10^{-6} \text{ m}^2/\text{s})} = 111 \text{ s} = 1.85 \text{ min}$$

The boundary nodes involving convection and/or radiation are more restrictive than the interior nodes and thus require smaller time steps. Therefore, the most restrictive boundary node should be used in the determination of the maximum allowable time step $\Delta t$ when a transient problem is solved with the explicit method.

To gain a better understanding of the stability criterion, consider the explicit finite difference formulation for an interior node of a plane wall (Eq. 5–47) for the case of no heat generation,

$$T_{m+1}^i = \tau(T_{m-1}^i + T_{m+1}^i) + (1 - 2\tau)T_m^i$$

Assume that at some time step $i$ the temperatures $T_{m-1}^i$ and $T_{m+1}^i$ are equal but less than $T_m^i$ (say, $T_{m-1}^i = T_{m+1}^i = 50^\circ\text{C}$ and $T_m^i = 80^\circ\text{C}$). At the next time step, we expect the temperature of node $m$ to be between the two values (say, 70$^\circ\text{C}$). However, if the value of $\tau$ exceeds 0.5 (say, $\tau = 1$), the temperature of node $m$ at the next time step will be less than the temperature of the neighboring nodes (it will be 20$^\circ\text{C}$), which is physically impossible and violates the second law of thermodynamics (Fig. 5–43). Requiring the new temperature of node $m$ to remain above the temperature of the neighboring nodes is equivalent to requiring the value of $\tau$ to remain below 0.5.

The implicit method is unconditionally stable, and thus we can use any time step we please with that method (of course, the smaller the time step, the better the accuracy of the solution). The disadvantage of the implicit method is that it results in a set of equations that must be solved simultaneously for each time step. Both methods are used in practice.

### EXAMPLE 5–5

**Transient Heat Conduction in a Large Uranium Plate**

Consider a large uranium plate of thickness $L = 4 \text{ cm}$, thermal conductivity $k = 28 \text{ W/m} \cdot ^\circ\text{C}$, and thermal diffusivity $\alpha = 12.5 \times 10^{-6} \text{ m}^2/\text{s}$ that is initially at a uniform temperature of 200$^\circ\text{C}$. Heat is generated uniformly in the plate at a constant rate of $g = 5 \times 10^6 \text{ W/m}^3$. At time $t = 0$, one side of the plate is brought into contact with iced water and is maintained at 0$^\circ\text{C}$ at all times, while the other side is subjected to convection to an environment at $T_e = 30^\circ\text{C}$ with a heat transfer coefficient of $h = 45 \text{ W/m}^2 \cdot ^\circ\text{C}$, as shown in Figure 5–44. Considering a total of three equally spaced nodes in the medium, two at the boundaries and one at the middle, estimate the exposed surface temperature of the plate 2.5 min after the start of cooling using (a) the explicit method and (b) the implicit method.
SOLUTION  We have solved this problem in Example 5–1 for the steady case, and here we repeat it for the transient case to demonstrate the application of the transient finite difference methods. Again we assume one-dimensional heat transfer in rectangular coordinates and constant thermal conductivity. The number of nodes is specified to be \( M = 3 \), and they are chosen to be at the two surfaces of the plate and at the middle, as shown in the figure. Then the nodal spacing \( \Delta x \) becomes

\[
\Delta x = \frac{L}{M} = \frac{0.04 \text{ m}}{3 - 1} = 0.02 \text{ m}
\]

We number the nodes as 0, 1, and 2. The temperature at node 0 is given to be \( T_0 = 0^\circ \text{C} \) at all times, and the temperatures at nodes 1 and 2 are to be determined. This problem involves only two unknown nodal temperatures, and thus we need to have only two equations to determine them uniquely. These equations are obtained by applying the finite difference method to nodes 1 and 2.

(a) Node 1 is an interior node, and the explicit finite difference formulation at that node is obtained directly from Eq. 5–47 by setting \( m = 1 \):

\[
T_i^{n+1} = \tau (T_0 + T_j) + (1 - 2\tau) T_i^n + \frac{\dot{g}_i \Delta x^2}{k} \tag{1}
\]

Node 2 is a boundary node subjected to convection, and the finite difference formulation at that node is obtained by writing an energy balance on the volume element of thickness \( 2 \Delta x \) at that boundary by assuming heat transfer to be into the medium at all sides (Fig. 5–45):

\[
hA(T_a - T_2^n) + kA \frac{T_i^n - T_j^n}{2} A + \dot{g}_2 \frac{\Delta x}{2} = \rho A \frac{\Delta x}{2} C \frac{T_i^{n+1} - T_j^n}{\Delta x}
\]

Dividing by \( kA/2\Delta x \) and using the definitions of thermal diffusivity \( \alpha = k/\rho C \) and the dimensionless mesh Fourier number \( \tau = \alpha \Delta t/\Delta x^2 \) gives

\[
\frac{2h\Delta x}{k} (T_a - T_2^n) + 2(T_i^n - T_j^n) + \frac{\dot{g}_2 \Delta x^2}{k} = \frac{T_i^{n+1} - T_j^n}{\tau}
\]

which can be solved for \( T_2^{n+1} \) to give

\[
T_2^{n+1} = \left( 1 - 2\tau - 2\tau \frac{h\Delta x}{k} \right) T_2^n + \tau \left( 2T_i^n + 2 \frac{h\Delta x}{k} T_a + \frac{\dot{g}_2 \Delta x^2}{k} \right) \tag{2}
\]

Note that we did not use the superscript \( i \) for quantities that do not change with time. Next we need to determine the upper limit of the time step \( \Delta t \) from the stability criterion, which requires the coefficient of \( T_i^n \) in Equation 1 and the coefficient of \( T_j^n \) in the second equation to be greater than or equal to zero. The coefficient of \( T_j^n \) is smaller in this case, and thus the stability criterion for this problem can be expressed as

\[
1 - 2\tau - 2\tau \frac{h\Delta x}{k} \leq 0 \quad \Rightarrow \quad \tau \leq \frac{1}{2(1 + h\Delta x/k)} \quad \Rightarrow \quad \Delta t \leq \frac{\Delta x^2}{2\alpha(1 + h\Delta x/k)}
\]
since \( \tau = \alpha \Delta t (\Delta x)^2 \). Substituting the given quantities, the maximum allowable value of the time step is determined to be

\[
\Delta t \leq \frac{(0.02 \text{ m})^2}{2(12.5 \times 10^{-6} \text{ m}^2/\text{s})[1 + (45 \text{ W/m}^2 \cdot \degree\text{C})(0.02 \text{ m})/28 \text{ W/m} \cdot \degree\text{C}]} = 15.5 \text{ s}
\]

Therefore, any time step less than 15.5 s can be used to solve this problem. For convenience, let us choose the time step to be \( \Delta t = 15 \) s. Then the mesh Fourier number becomes

\[
\tau = \frac{\alpha \Delta t}{(\Delta x)^2} = \frac{(12.5 \times 10^{-6} \text{ m}^2/\text{s})(15 \text{ s})}{(0.02 \text{ m})^2} = 0.46875 \quad \text{(for } \Delta t = 15 \text{ s)}
\]

Substituting this value of \( \tau \) and other given quantities, the explicit finite difference equations (1) and (2) developed here reduce to

\[
T_{i}^{n+1} = 0.0625T_{i}^{n} + 0.46875T_{i+1}^{n} + 33.482
\]

\[
T_{2}^{n+1} = 0.9375T_{1}^{n} + 0.032366T_{2}^{n} + 34.386
\]

The initial temperature of the medium at \( t = 0 \) and \( i = 0 \) is given to be 200\degree\text{C} throughout, and thus \( T_{0}^{1} = T_{0}^{2} = 200\degree\text{C} \). Then the nodal temperatures at \( T_{1}^{1} \) and \( T_{2}^{1} \) at \( t = \Delta t = 15 \text{ s} \) are determined from these equations to be

\[
T_{1}^{1} = 0.0625T_{0}^{1} + 0.46875T_{1}^{0} + 33.482
\]

\[
= 0.0625 \times 200 + 0.46875 \times 200 + 33.482 = 139.7\degree\text{C}
\]

\[
T_{2}^{1} = 0.9375T_{1}^{0} + 0.032366T_{2}^{0} + 34.386
\]

\[
= 0.9375 \times 200 + 0.032366 \times 200 + 34.386 = 228.4\degree\text{C}
\]

Similarly, the nodal temperatures \( T_{1}^{2} \) and \( T_{2}^{2} \) at \( t = 2\Delta t = 2 \times 15 = 30 \) s are determined to be

\[
T_{1}^{2} = 0.0625T_{1}^{1} + 0.46875T_{2}^{1} + 33.482
\]

\[
= 0.0625 \times 139.7 + 0.46875 \times 228.4 + 33.482 = 149.3\degree\text{C}
\]

\[
T_{2}^{2} = 0.9375T_{1}^{1} + 0.032366T_{2}^{1} + 34.386
\]

\[
= 0.9375 \times 139.7 + 0.032366 \times 228.4 + 34.386 = 172.8\degree\text{C}
\]

Continuing in the same manner, the temperatures at nodes 1 and 2 are determined for \( i = 1, 2, 3, 4, 5, \ldots, 50 \) and are given in Table 5–2. Therefore, the temperature at the exposed boundary surface 2.5 min after the start of cooling is

\[
T_{2}^{5.25 \text{ min}} = T_{2}^{10} = 139.0\degree\text{C}
\]

(b) Node 1 is an interior node, and the implicit finite difference formulation at that node is obtained directly from Eq. 5–49 by setting \( m = 1 \):

\[
\tau T_{0} - (1 + 2\tau) T_{1}^{i+1} + \tau T_{2}^{i+1} + \tau \frac{g_{0} \Delta x^{2}}{k} + T_{i} = 0 \quad (3)
\]

Node 2 is a boundary node subjected to convection, and the implicit finite difference formulation at that node can be obtained from this formulation by expressing the left side of the equation at time step \( i + 1 \) instead of \( i \) as

**TABLE 5–2**

The variation of the nodal temperatures in Example 5–5 with time obtained by the explicit method

<table>
<thead>
<tr>
<th>Time Step, ( i )</th>
<th>Time, ( t ) (s)</th>
<th>Node 1 Temperature, ( T_{1} ) (\degree\text{C})</th>
<th>Node 2 Temperature, ( T_{2} ) (\degree\text{C})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>200.0</td>
<td>200.0</td>
</tr>
<tr>
<td>1</td>
<td>15</td>
<td>139.7</td>
<td>228.4</td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>149.3</td>
<td>172.8</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
<td>123.8</td>
<td>179.9</td>
</tr>
<tr>
<td>4</td>
<td>60</td>
<td>125.6</td>
<td>156.3</td>
</tr>
<tr>
<td>5</td>
<td>75</td>
<td>114.6</td>
<td>157.1</td>
</tr>
<tr>
<td>6</td>
<td>90</td>
<td>114.3</td>
<td>146.9</td>
</tr>
<tr>
<td>7</td>
<td>105</td>
<td>109.5</td>
<td>146.3</td>
</tr>
<tr>
<td>8</td>
<td>120</td>
<td>108.9</td>
<td>141.8</td>
</tr>
<tr>
<td>9</td>
<td>135</td>
<td>106.7</td>
<td>141.1</td>
</tr>
<tr>
<td>10</td>
<td>150</td>
<td>106.3</td>
<td>139.0</td>
</tr>
<tr>
<td>20</td>
<td>300</td>
<td>103.8</td>
<td>136.1</td>
</tr>
<tr>
<td>30</td>
<td>450</td>
<td>103.7</td>
<td>136.0</td>
</tr>
<tr>
<td>40</td>
<td>600</td>
<td>103.7</td>
<td>136.0</td>
</tr>
</tbody>
</table>
\[ \frac{2h \Delta x}{k} (T_i - T_{i+1}^j) + 2(T_i^{j+1} - T_{i+1}^{j+1}) + \frac{\dot{g}_i \Delta x^2}{k} = \frac{T_{i+1}^{j+1} - T_i^j}{\tau} \]

which can be rearranged as
\[ 2\tau T_i^{j+1} - \left( 1 + 2\tau + 2\tau \frac{h \Delta x}{k} \right) T_i^{j+1} + 2\tau \frac{h \Delta x}{k} T_i + \tau \frac{\dot{g}_i \Delta x^2}{k} + T_i^j = 0 \quad (4) \]

Again we did not use the superscript \( i \) or \( i + 1 \) for quantities that do not change with time. The implicit method imposes no limit on the time step, and thus we can choose any value we want. However, we will again choose \( \Delta t = 15 \) s, and thus \( \tau = 0.46875 \), to make a comparison with part (a) possible. Substituting this value of \( \tau \) and other given quantities, the two implicit finite difference equations developed here reduce to
\[ -1.9375T_i^{j+1} + 0.46875T_{i+1}^{j+1} + T_i + 33.482 = 0 \]
\[ 0.9375T_i^{j+1} - 1.9676T_{i+1}^{j+1} + T_i + 34.382 = 0 \]

Again \( T_0^i = T_1^i = 200^\circ C \) at \( t = 0 \) and \( i = 0 \) because of the initial condition, and for \( i = 0 \), these two equations reduce to
\[ -1.9375T_i^j + 0.46875T_{i+1}^j + 200 + 33.482 = 0 \]
\[ 0.9375T_i^j - 1.9676T_{i+1}^j + 200 + 34.382 = 0 \]

The unknown nodal temperatures \( T_1^j \) and \( T_2^j \) at \( t = \Delta t = 15 \) s are determined by solving these two equations simultaneously to be
\[ T_1^j = 168.8^\circ C \quad \text{and} \quad T_2^j = 199.6^\circ C \]

Similarly, for \( i = 1 \), these equations reduce to
\[ -1.9375T_i^j + 0.46875T_{i+1}^j + 168.8 + 33.482 = 0 \]
\[ 0.9375T_i^j - 1.9676T_{i+1}^j + 199.6 + 34.382 = 0 \]

The unknown nodal temperatures \( T_1^j \) and \( T_2^j \) at \( t = \Delta t = 2 \times 15 = 30 \) s are determined by solving these two equations simultaneously to be
\[ T_1^j = 150.5^\circ C \quad \text{and} \quad T_2^j = 190.6^\circ C \]

Continuing in this manner, the temperatures at nodes 1 and 2 are determined for \( i = 2, 3, 4, 5, \ldots, 40 \) and are listed in Table 5–3, and the temperature at the exposed boundary surface (node 2) 2.5 min after the start of cooling is obtained to be
\[ T_{2,5}^{2.5 \text{ min}} = T_2^{10} = 143.9^\circ C \]

which is close to the result obtained by the explicit method. Note that either method could be used to obtain satisfactory results to transient problems, except, perhaps, for the first few time steps. The implicit method is preferred when it is desirable to use large time steps, and the explicit method is preferred when one wishes to avoid the simultaneous solution of a system of algebraic equations.

### Table 5–3

<table>
<thead>
<tr>
<th>Time, s</th>
<th>Node Temperature, °C</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( T_1^i )</td>
</tr>
<tr>
<td>0</td>
<td>200.0</td>
</tr>
<tr>
<td>1</td>
<td>168.8</td>
</tr>
<tr>
<td>2</td>
<td>150.5</td>
</tr>
<tr>
<td>3</td>
<td>138.6</td>
</tr>
<tr>
<td>4</td>
<td>130.3</td>
</tr>
<tr>
<td>5</td>
<td>124.1</td>
</tr>
<tr>
<td>6</td>
<td>119.5</td>
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<td>7</td>
<td>115.9</td>
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<tr>
<td>8</td>
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<td>9</td>
<td>111.0</td>
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<tr>
<td>10</td>
<td>109.4</td>
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<td>104.2</td>
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<tr>
<td>30</td>
<td>103.8</td>
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<tr>
<td>40</td>
<td>103.8</td>
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</table>
EXAMPLE 5–6 Solar Energy Storage in Trombe Walls

Dark painted thick masonry walls called Trombe walls are commonly used on south sides of passive solar homes to absorb solar energy, store it during the day, and release it to the house during the night (Fig. 5–46). The idea was proposed by E. L. Morse of Massachusetts in 1881 and is named after Professor Felix Trombe of France, who used it extensively in his designs in the 1970s. Usually a single or double layer of glazing is placed outside the wall and transmits most of the solar energy while blocking heat losses from the exposed surface of the wall to the outside. Also, air vents are commonly installed at the bottom and top of the Trombe walls so that the house air enters the parallel flow channel between the Trombe wall and the glazing, rises as it is heated, and enters the room through the top vent.

Consider a house in Reno, Nevada, whose south wall consists of a 1-ft-thick Trombe wall whose thermal conductivity is $k = 0.40 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ \text{F}$ and whose thermal diffusivity is $\alpha = 4.78 \times 10^{-6} \text{ ft}^2/\text{s}$. The variation of the ambient temperature $T_{\text{out}}$ and the solar heat flux $q_{\text{solar}}$ incident on a south-facing vertical surface throughout the day for a typical day in January is given in Table 5–4 in 3-h intervals. The Trombe wall has single glazing with an absorptivity-transmissivity product of $\kappa = 0.77$ (that is, 77 percent of the solar energy incident is absorbed by the exposed surface of the Trombe wall), and the average combined heat transfer coefficient for heat loss from the Trombe wall to the ambient is determined to be $h_{\text{wall}} = 0.7 \text{ Btu/h} \cdot \text{ft}^2 \cdot ^\circ \text{F}$. The interior of the house is maintained at $T_{\text{in}} = 70^\circ \text{F}$ at all times, and the heat transfer coefficient at the interior surface of the Trombe wall is $h_{\text{in}} = 1.8 \text{ Btu/h} \cdot \text{ft}^2 \cdot ^\circ \text{F}$. Also, the vents on the Trombe wall are kept closed, and thus the only heat transfer between the air in the house and the Trombe wall is through the interior surface of the wall. Assuming the temperature of the Trombe wall to vary linearly between 70°F at the interior surface and 30°F at the exterior surface at 7 AM and using the explicit finite difference method with a uniform nodal spacing of $\Delta x = 0.2 \text{ ft}$, determine the temperature distribution along the thickness of the Trombe wall after 12, 24, 36, and 48 h. Also, determine the net amount of heat transferred to the house from the Trombe wall during the first day and the second day. Assume the wall is 10 ft high and 25 ft long.

### SOLUTION

The passive solar heating of a house through a Trombe wall is considered. The temperature distribution in the wall in 12-h intervals and the amount of heat transfer during the first and second days are to be determined.

#### Assumptions
1. Heat transfer is one-dimensional since the exposed surface of the wall is large relative to its thickness.
2. Thermal conductivity is constant.
3. The heat transfer coefficients are constant.

#### Properties

The wall properties are given to be $k = 0.40 \text{ Btu/h} \cdot \text{ft} \cdot ^\circ \text{F}$, $\alpha = 4.78 \times 10^{-6} \text{ ft}^2/\text{s}$, and $\kappa = 0.77$.

#### Analysis

The nodal spacing is given to be $\Delta x = 0.2 \text{ ft}$, and thus the total number of nodes along the Trombe wall is

$$M = \frac{L}{\Delta x} + 1 = \frac{1 \text{ ft}}{0.2 \text{ ft}} + 1 = 6$$

We number the nodes as 0, 1, 2, 3, 4, and 5, with node 0 on the interior surface of the Trombe wall and node 5 on the exterior surface, as shown in Figure 5–47. Nodes 1 through 4 are interior nodes, and the explicit finite difference formulations of these nodes are obtained directly from Eq. 5–47 to be...
The interior surface is subjected to convection, and thus the explicit formulation of node 0 can be obtained directly from Eq. 5–51 to be

$$T_0^{i+1} = (1 - 2\tau - 2\tau \Delta x/k) T_0^i + 2\tau T_1^i + 2\tau h_{in} \Delta x/k T_{in}$$  \hspace{1cm} (5)$$

Substituting the quantities $h_{in}$, $\Delta x$, $k$, and $T_{in}$ which do not change with time, into this equation gives

$$T_0^{i+1} = (1 - 3.80\tau) T_0^i + \tau (2T_1^i + 126.0)$$  \hspace{1cm} (5-53)$$

The exterior surface of the Trombe wall is subjected to convection as well as to heat flux. The explicit finite difference formulation at that boundary is obtained by writing an energy balance on the volume element represented by node 5,

$$h_{out} A(T_{out}^i - T_5^i) + \kappa A q_{solar}^i + k A \frac{T_5^i - T_4^i}{\Delta x} = \rho A \frac{\Delta x}{2} C T_5^{i+1} - T_5^i \frac{T_5^i - T_4^i}{\Delta t}$$  \hspace{1cm} (5-54)$$

which simplifies to

$$T_5^{i+1} = (1 - 2\tau - 2\tau h_{out} \Delta x/k) T_5^i + 2\tau T_4^i + 2\tau h_{out} \Delta x/k T_{out}^i + 2\tau \kappa q_{solar}^i \Delta x/k$$

where $\tau = \alpha \Delta t / \Delta x^2$ is the dimensionless mesh Fourier number. Note that we kept the superscript $i$ for quantities that vary with time. Substituting the quantities $h_{out}$, $\Delta x$, $k$, and $\kappa$, which do not change with time, into this equation gives

$$T_5^{i+1} = (1 - 2.70\tau) T_5^i + \tau (2T_4^i + 0.70T_{out}^i + 0.770 q_{solar}^i)$$  \hspace{1cm} (6)$$

where the unit of $q_{solar}$ is Btu/h · ft².

Next we need to determine the upper limit of the time step $\Delta t$ from the stability criterion since we are using the explicit method. This requires the identification of the smallest primary coefficient in the system. We know that the boundary nodes are more restrictive than the interior nodes, and thus we examine the formulations of the boundary nodes 0 and 5 only. The smallest and thus the most restrictive primary coefficient in this case is the coefficient of $T_0^i$ in the formulation of node 0 since $1 - 3.8\tau < 1 - 2.7\tau$, and thus the stability criterion for this problem can be expressed as

$$1 - 3.8\tau \geq 0 \rightarrow \tau = \frac{\alpha \Delta x}{\Delta t} \leq \frac{1}{3.80}$$

Substituting the given quantities, the maximum allowable value of the time step is determined to be

$$\Delta t \leq \frac{\Delta x^2}{3.80\alpha} = \frac{(0.2 \text{ ft})^2}{3.80 \times (4.78 \times 10^{-6} \text{ ft/s})^2} = 2202 \text{ s}$$

The nodal network for the Trombe wall discussed in Example 5–6.
Therefore, any time step less than 2202 s can be used to solve this problem. For convenience, let us choose the time step to be \( \Delta t = 900 \text{ s} = 15 \text{ min} \). Then the mesh Fourier number becomes

\[
\tau = \frac{\alpha \Delta t}{(\Delta x)^2} = \frac{(4.78 \times 10^{-3} \text{ ft}^2/\text{s})(900 \text{ s})}{(0.2 \text{ ft})^2} = 0.10755 \quad \text{(for } \Delta t = 15 \text{ min)}
\]

Initially (at 7 AM or \( t = 0 \)), the temperature of the wall is said to vary linearly between 70°F at node 0 and 30°F at node 5. Noting that there are five nodal spacings of equal length, the temperature change between two neighboring nodes is \( (70 - 30)\text{°F}/5 = 8\text{°F} \). Therefore, the initial nodal temperatures are

\[
T_0^0 = 70\text{°F}, \quad T_1^0 = 62\text{°F}, \quad T_2^0 = 54\text{°F},
\]

\[
T_3^0 = 46\text{°F}, \quad T_4^0 = 38\text{°F}, \quad T_5^0 = 30\text{°F}.
\]

Then the nodal temperatures at \( t = \Delta t = 15 \text{ min} \) (at 7:15 AM) are determined from these equations to be

\[
T_0^1 = (1 - 3.80\tau) T_0^0 + \tau(2T_1^0 + 126.0)
\]

\[
= (1 - 3.80 \times 0.10755) \times 70 + 0.10755(2 \times 62 + 126.0) = 68.3\text{°F}
\]

\[
T_1^1 = \tau(T_0^0 + T_1^0) + (1 - 2\tau) T_1^0
\]

\[
= 0.10755(70 + 54) + (1 - 2 \times 0.10755)62 = 62\text{°F}
\]

\[
T_2^1 = \tau(T_1^0 + T_2^0) + (1 - 2\tau) T_2^0
\]

\[
= 0.10755(62 + 46) + (1 - 2 \times 0.10755)54 = 54\text{°F}
\]

\[
T_3^1 = \tau(T_2^0 + T_3^0) + (1 - 2\tau) T_3^0
\]

\[
= 0.10755(54 + 38) + (1 - 2 \times 0.10755)46 = 46\text{°F}
\]

\[
T_4^1 = \tau(T_3^0 + T_4^0) + (1 - 2\tau) T_4^0
\]

\[
= 0.10755(46 + 30) + (1 - 2 \times 0.10755)38 = 38\text{°F}
\]

\[
T_5^1 = (1 - 2.70\tau) T_5^0 + \tau(2T_4^0 + 0.70T_{\text{out}}^0 + 0.770q_{\text{loss}}^0)
\]

\[
= (1 - 2.70 \times 0.10755)30 + 0.10755(2 \times 38 + 0.70 \times 33 + 0.770 \times 114)
\]

\[
= 41\text{°F}
\]

Note that the inner surface temperature of the Trombe wall dropped by 1.7°F and the outer surface temperature rose by 11.4°F during the first time step while the temperatures at the interior nodes remained the same. This is typical of transient problems in mediums that involve no heat generation. The nodal temperatures at the following time steps are determined similarly with the help of a computer. Note that the data for ambient temperature and the incident solar radiation change every 3 hours, which corresponds to 12 time steps, and this must be reflected in the computer program. For example, the value of \( q_{\text{solar}} \) must be taken to be \( q_{\text{solar}}^i = 75 \) for \( i = 1-12 \), \( q_{\text{solar}}^i = 242 \) for \( i = 13-24 \), \( q_{\text{solar}}^i = 178 \) for \( i = 25-36 \), and \( q_{\text{solar}}^i = 0 \) for \( i = 37-96 \).

The results after 6, 12, 18, 24, 30, 36, 42, and 48 h are given in Table 5–5 and are plotted in Figure 5–48 for the first day. Note that the interior temperature of the Trombe wall drops in early morning hours, but then rises as the solar energy absorbed by the exterior surface diffuses through the wall. The exterior surface temperature of the Trombe wall rises from 30 to 142°F in just 6 h because of the solar energy absorbed, but then drops to 53°F by next morning as a result of heat loss at night. Therefore, it may be worthwhile to cover the outer surface at night to minimize the heat losses.
The rate of heat transfer from the Trombe wall to the interior of the house during each time step is determined from Newton’s law using the average temperature at the inner surface of the wall (node 0) as

\[ Q_{\text{Trombe wall}} = \dot{Q}_{\text{Trombe wall}} = h_{\text{in}} A (T_0^i - T_{\text{in}}) \Delta t = h_{\text{in}} A [(T_0^i + T_{\text{in}}^{-1})/2 - T_{\text{in}}] \Delta t \]

Therefore, the amount of heat transfer during the first time step \((i = 1)\) or during the first 15-min period is

\[ Q_{\text{Trombe wall}} = h_{\text{in}} A [(T_0^1 + T_{\text{in}})^{-1}/2 - T_{\text{in}}] \Delta t \]

\[ = (1.8 \text{ Btu/h} \cdot \text{ft}^2 \cdot \circF)(10 \times 25 \text{ ft}^2)[(68.3 + 70)/2 - 70\circF](0.25 \text{ h}) \]

\[ = -95.6 \text{ Btu} \]

The negative sign indicates that heat is transferred to the Trombe wall from the air in the house, which represents a heat loss. Then the total heat transfer during a specified time period is determined by adding the heat transfer amounts for each time step as

\[ Q_{\text{Trombe wall}} = \sum_{i=1}^{I} \dot{Q}_{\text{Trombe wall}} = \sum_{i=1}^{I} h_{\text{in}} A [(T_0^i + T_{\text{in}}^{-1})/2 - T_{\text{in}}] \Delta t \]  \hspace{1cm} (5-55)

where \(I\) is the total number of time intervals in the specified time period. In this case \(I = 48\) for 12 h, 96 for 24 h, and so on. Following the approach described here using a computer, the amount of heat transfer between the Trombe wall and the interior of the house is determined to be

\[ Q_{\text{Trombe wall}} = -17,048 \text{ Btu after 12 h} \]
\[ Q_{\text{Trombe wall}} = -2483 \text{ Btu after 24 h} \]
\[ Q_{\text{Trombe wall}} = 5610 \text{ Btu after 36 h} \]
\[ Q_{\text{Trombe wall}} = 34,400 \text{ Btu after 48 h} \]

Therefore, the house loses 2483 Btu through the Trombe wall the first day as a result of the low start-up temperature but delivers a total of 36,883 Btu of heat to the house the second day. It can be shown that the Trombe wall will deliver even more heat to the house during the third day since it will start the day at a higher average temperature.
Two-Dimensional Transient Heat Conduction

Consider a rectangular region in which heat conduction is significant in the \( x \)- and \( y \)-directions, and consider a unit depth of \( \Delta z = 1 \) in the \( z \)-direction. Heat may be generated in the medium at a rate of \( q(x, y, t) \), which may vary with time and position, with the thermal conductivity \( k \) of the medium assumed to be constant. Now divide the \( x-y \)-plane of the region into a rectangular mesh of nodal points spaced \( \Delta x \) and \( \Delta y \) apart in the \( x \)- and \( y \)-directions, respectively, and consider a general interior node \((m, n)\) whose coordinates are \( x = m\Delta x \) and \( y = n\Delta y \), as shown in Figure 5–49. Noting that the volume element centered about the general interior node \((m, n)\) involves heat conduction from four sides (right, left, top, and bottom) and the volume of the element is 

\[
V_{\text{element}} = \Delta x \times \Delta y \times 1 = \Delta x \Delta y,
\]

the transient finite difference formulation for a general interior node can be expressed on the basis of Eq. 5–39 as

\[
k\Delta y \left( \frac{T_{m-1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n+1} - T_{m,n}}{\Delta y} + k\Delta y \frac{T_{m+1,n} - T_{m,n}}{\Delta x} + k\Delta x \frac{T_{m,n-1} - T_{m,n}}{\Delta y} + \frac{q_{m,n} \Delta x \Delta y}{\rho c} \right) = \left( \frac{T_{m,n}^{i+1} - T_{m,n}^{i}}{\Delta t} \right)
\]

(5-56)

Taking a square mesh \((\Delta x = \Delta y = 1)\) and dividing each term by \(k\) gives after simplifying,

\[
T_{m-1,n} + T_{m+1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} + \frac{q_{m,n} \Delta x \Delta y}{\rho c} \Delta x \Delta y = \left( \frac{T_{m,n}^{i+1} - T_{m,n}^{i}}{\Delta t} \right)
\]

(5-57)

where again \( \alpha = k/\rho c \) is the thermal diffusivity of the material and \( \tau = \alpha \Delta t/\Delta x^2 \) is the dimensionless mesh Fourier number. It can also be expressed in terms of the temperatures at the neighboring nodes in the following easy-to-remember form:

\[
T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{q_{\text{node}} \Delta x \Delta y}{\rho c} = \left( \frac{T_{\text{node}}^{i+1} - T_{\text{node}}^{i}}{\tau} \right)
\]

(5-58)

Again the left side of this equation is simply the finite difference formulation of the problem for the steady case, as expected. Also, we are still not committed to explicit or implicit formulation since we did not indicate the time step on the left side of the equation. We now obtain the explicit finite difference formulation by expressing the left side at time step \( i \) as

\[
T_{\text{left}}^{i} + T_{\text{top}}^{i} + T_{\text{right}}^{i} + T_{\text{bottom}}^{i} - 4T_{\text{node}}^{i} + \frac{q_{\text{node}} \Delta x \Delta y}{\rho c} = \left( \frac{T_{\text{node}}^{i+1} - T_{\text{node}}^{i}}{\tau} \right)
\]

(5-59)

Expressing the left side at time step \( i + 1 \) instead of \( i \) would give the implicit formulation. This equation can be solved explicitly for the new temperature \( T_{\text{node}}^{i+1} \) to give

\[
T_{\text{node}}^{i+1} = \tau(T_{\text{left}}^{i} + T_{\text{top}}^{i} + T_{\text{right}}^{i} + T_{\text{bottom}}^{i}) + (1 - 4\tau) T_{\text{node}}^{i} + \frac{\tau q_{\text{node}} \Delta x \Delta y}{\rho c}
\]

(5-60)

for all interior nodes \((m, n)\) where \( m = 1, 2, 3, \ldots, M - 1 \) and \( n = 1, 2, 3, \ldots, N - 1 \) in the medium. In the case of no heat generation and \( \tau = \frac{1}{4} \), the
explicit finite difference formulation for a general interior node reduces to
\[ T_{i}^{m+1} = \frac{T_{i}^{m} + T_{i}^{t} + T_{i}^{t+1} + T_{i}^{b}}{4} \]
which has the interpretation that the temperature of an interior node at the new time step is simply the average of the temperatures of its neighboring nodes at the previous time step (Fig. 5–50).

The stability criterion that requires the coefficient of \( T_{i}^{m} \) in the \( T_{i}^{m+1} \) expression to be greater than or equal to zero for all nodes is equally valid for two- or three-dimensional cases and severely limits the size of the time step \( \Delta t \) that can be used with the explicit method. In the case of transient two-dimensional heat transfer in rectangular coordinates, the coefficient of \( T_{i}^{m} \) in the \( T_{i}^{m+1} \) expression is \( 1 - 4\tau \), and thus the stability criterion for all interior nodes in this case is \( 1 - 4\tau > 0 \),

\[ \tau = \frac{\alpha \Delta t}{\Delta x^2} \leq \frac{1}{4} \quad \text{(interior nodes, two-dimensional heat transfer in rectangular coordinates)} \]  

(5-61)

where \( \Delta x = \Delta y = l \). When the material of the medium and thus its thermal diffusivity \( \alpha \) are known and the value of the mesh size \( l \) is specified, the largest allowable value of the time step \( \Delta t \) can be determined from the relation above. Again the boundary nodes involving convection and/or radiation are more restrictive than the interior nodes and thus require smaller time steps. Therefore, the most restrictive boundary node should be used in the determination of the maximum allowable time step \( \Delta t \) when a transient problem is solved with the explicit method.

The application of Eq. 5–60 to each of the \((M - 1) \times (N - 1)\) interior nodes gives \((M - 1) \times (N - 1)\) equations. The remaining equations are obtained by applying the method to the boundary nodes unless, of course, the boundary temperatures are specified as being constant. The development of the transient finite difference formulation of boundary nodes in two- (or three-) dimensional problems is similar to the development in the one-dimensional case discussed earlier. Again the region is partitioned between the nodes by forming volume elements around the nodes, and an energy balance is written for each boundary node on the basis of Eq. 5–39. This is illustrated in Example 5–7.

**EXAMPLE 5–7 Transient Two-Dimensional Heat Conduction in L-Bars**

Consider two-dimensional transient heat transfer in an L-shaped solid body that is initially at a uniform temperature of 90°C and whose cross section is given in Figure 5–51. The thermal conductivity and diffusivity of the body are \( k = 15 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 3.2 \times 10^{-6} \text{ m}^2/\text{s} \), respectively, and heat is generated in the body at a rate of \( q = 2 \times 10^6 \text{ W/m}^3 \). The left surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of 90°C at all times. At time \( t = 0 \), the entire top surface is subjected to convection to ambient air at \( T_\infty = 25°C \) with a convection coefficient of \( h = 80 \text{ W/m}^2 \cdot \text{°C} \), and the right surface is subjected to heat flux at a uniform rate of \( q_R = 5000 \text{ W/m}^2 \). The nodal network of the problem consists of 15 equally spaced nodes with \( \Delta x = \Delta y = 1.2 \text{ cm} \), as shown in the figure. Five of the nodes are at the bottom surface, and thus their temperatures are known. Using the explicit method, determine the temperature at the top corner (node 3) of the body after 1, 3, 5, 10, and 60 min.
SOLUTION  This is a transient two-dimensional heat transfer problem in rectangular coordinates, and it was solved in Example 5–3 for the steady case. Therefore, the solution of this transient problem should approach the solution for the steady case when the time is sufficiently large. The thermal conductivity and heat generation rate are given to be constants. We observe that all nodes are boundary nodes except node 5, which is an interior node. Therefore, we will have to rely on energy balances to obtain the finite difference equations. The region is partitioned among the nodes equitably as shown in the figure, and the explicit finite difference equations are determined on the basis of the energy balance for the transient case expressed as

$$\sum_{\text{All sides}} Q_i^t + G_{i,\text{element}} = \rho V_{\text{element}} C \frac{T_{i,m}^{t+1} - T_{i,m}^t}{\Delta t}$$

The quantities $h$, $T_m$, $g_1$, and $g_2$ do not change with time, and thus we do not need to use the superscript $i$ for them. Also, the energy balance expressions are simplified using the definitions of thermal diffusivity $\alpha = \frac{k}{\rho C}$ and the dimensionless mesh Fourier number $\tau = \alpha \Delta t / l^2$, where $\Delta x = \Delta y = l$.

(a) Node 1. (Boundary node subjected to convection and insulation, Fig. 5–52a)

$$h \frac{\Delta x}{2} (T_m - T_1^i) + k \left( \frac{\Delta y}{2} \right) T_2^i - T_1^i \frac{\Delta x}{\Delta x} + k \frac{\Delta x}{\Delta y} T_1^i - T_1^i \frac{\Delta y}{\Delta y} + g_1 \frac{\Delta x}{2} \frac{\Delta y}{2} = \rho \frac{\Delta x}{2} \frac{\Delta y}{2} C \frac{T_1^{i+1} - T_1^i}{\Delta t}$$

Dividing by $k/4$ and simplifying,

$$\frac{2hl}{k} (T_m - T_1^i) + 2(T_2^i - T_1^i) + 2(T_1^i - T_1^i) + \frac{g_1 l^2}{k} = \frac{T_1^{i+1} - T_1^i}{\tau}$$

which can be solved for $T_1^{i+1}$ to give

$$T_1^{i+1} = \left( 1 - 4\tau - 2\tau \frac{hl}{k} \right) T_1^i + 2\tau \left( T_2^i + T_1^i + \frac{hl}{k} T_m + \frac{g_1 l^2}{2k} \right)$$

(b) Node 2. (Boundary node subjected to convection, Fig. 5–52b)

$$h \Delta x (T_m - T_2^i) + k \left( \frac{\Delta y}{2} \right) T_1^i - T_2^i \frac{\Delta x}{\Delta x} + k \frac{\Delta x}{\Delta y} T_2^i - T_2^i \frac{\Delta y}{\Delta y} + g_2 \frac{\Delta x}{2} \frac{\Delta y}{2} = \rho \frac{\Delta x}{2} \frac{\Delta y}{2} C \frac{T_2^{i+1} - T_2^i}{\Delta t}$$

Dividing by $k/2$, simplifying, and solving for $T_2^{i+1}$ gives

$$T_2^{i+1} = \left( 1 - 4\tau - 2\tau \frac{hl}{k} \right) T_2^i + \tau \left( T_1^i + T_2^i + 2T_3^i + \frac{2hl}{k} T_m + \frac{g_1 l^2}{k} \right)$$
(c) Node 3. (Boundary node subjected to convection on two sides, Fig. 5–53a)

\[
\begin{align*}
\frac{h}{2} \left( \frac{\Delta x + \Delta y}{2} \right) (T_x - T_i) + k \frac{\Delta x}{2} \frac{T_{i} - T_{i}^{0}}{\Delta y} + k \frac{\Delta y}{2} \frac{T_{j} - T_{j}^{0}}{\Delta x} + k \Delta x \frac{T_{i} - T_{i}^{0}}{\Delta y} + k \Delta y \frac{T_{j} - T_{j}^{0}}{\Delta x} + \frac{\Delta x \Delta y}{2} = \rho \frac{\Delta x \Delta y}{2} \frac{T_{i}^{+1} - T_{i}}{\Delta t} + \frac{\Delta x \Delta y}{2} = \rho \frac{\Delta x \Delta y}{2} \frac{T_{j}^{+1} - T_{j}}{\Delta t}.
\end{align*}
\]

Dividing by \(k/4\), simplifying, and solving for \(T_{j}^{+1}\) gives

\[
T_{j}^{+1} = \left(1 - 4\tau - 4\frac{hl}{k}\right) T_{j} + \tau \left( T_{j} + T_{j}^{0} + 2 \frac{hl}{k} T_{x} + \frac{ghl^{2}}{2k} \right)
\]

(d) Node 4. (On the insulated boundary, and can be treated as an interior node, Fig. 5–53b). Noting that \(T_{10} = 90^\circ\text{C}\), Eq. 5–60 gives

\[
T_{4}^{+1} = \left(1 - 4\tau\right) T_{4} + \tau \left( T_{4} + T_{4}^{0} + 90 + \frac{ghl^{2}}{k} \right)
\]

(e) Node 5. (Interior node, Fig. 5–54a). Noting that \(T_{11} = 90^\circ\text{C}\), Eq. 5–60 gives

\[
T_{5}^{+1} = \left(1 - 4\tau\right) T_{5} + \tau \left( T_{5} + T_{5}^{0} + T_{6}^{0} + 90 + \frac{ghl^{2}}{k} \right)
\]

(f) Node 6. (Boundary node subjected to convection on two sides, Fig. 5–54b)

\[
\begin{align*}
\frac{h}{2} \left( \frac{\Delta x + \Delta y}{2} \right) (T_x - T_i) + k \frac{\Delta x}{2} \frac{T_{i} - T_{i}^{0}}{\Delta y} + k \frac{\Delta y}{2} \frac{T_{j} - T_{j}^{0}}{\Delta x} + k \Delta x \frac{T_{i} - T_{i}^{0}}{\Delta y} + k \Delta y \frac{T_{j} - T_{j}^{0}}{\Delta x} + \frac{\Delta x \Delta y}{4} = \rho \frac{\Delta x \Delta y}{4} \frac{T_{i}^{+1} - T_{i}}{\Delta t} + \frac{\Delta x \Delta y}{4} = \rho \frac{\Delta x \Delta y}{4} \frac{T_{j}^{+1} - T_{j}}{\Delta t}.
\end{align*}
\]

Dividing by \(3k/4\), simplifying, and solving for \(T_{j}^{+1}\) gives

\[
T_{j}^{+1} = \left(1 - 4\tau - 4\frac{hl}{3k}\right) T_{j} + \tau \left[ 2T_{j} + 4T_{j}^{0} + 2T_{j}^{0} + 4 \times 90 + 4 \frac{hl}{k} T_{x} + 3 \frac{ghl^{2}}{k} \right]
\]

(g) Node 7. (Boundary node subjected to convection, Fig. 5–55)

\[
\begin{align*}
h \Delta x (T_x - T_i) + k \frac{\Delta y T_{j} - T_{j}^{0}}{\Delta x} + k \Delta x \frac{T_{i} - T_{i}^{0}}{\Delta y} + k \frac{\Delta y T_{j} - T_{j}^{0}}{\Delta x} + \frac{\Delta y}{2} \frac{\Delta x \Delta y}{2} = \rho \frac{\Delta x \Delta y}{2} \frac{T_{i}^{+1} - T_{i}}{\Delta t} + \frac{\Delta y}{2} \frac{\Delta x \Delta y}{2} = \rho \frac{\Delta x \Delta y}{2} \frac{T_{j}^{+1} - T_{j}}{\Delta t}.
\end{align*}
\]

Dividing by \(k/2\), simplifying, and solving for \(T_{j}^{+1}\) gives

\[
T_{j}^{+1} = \left(1 - 4\tau - 2\frac{hl}{k}\right) T_{j} + \tau \left[ T_{j} + T_{j}^{0} + 2 \times 90 + 2 \frac{hl}{k} T_{x} + \frac{ghl^{2}}{k} \right]
\]
(h) Node 8. This node is identical to node 7, and the finite difference formulation of this node can be obtained from that of node 7 by shifting the node numbers by 1 (i.e., replacing subscript \( m \) by subscript \( m + 1 \)). It gives

\[
T_i^{j+1} = \left( 1 - 4\tau - 2\tau \frac{hl}{k} \right) T_i^j + \tau \left[ T_j^j + T_i^j + 2 \times 90 + \frac{2hl}{k} T_m + \frac{\delta T^2}{k} \right]
\]

(i) Node 9. (Boundary node subjected to convection on two sides, Fig. 5–55)

\[
h \frac{\Delta x}{2} (T_m - T_i) + \frac{\Delta y}{2} h \frac{\Delta x}{2} T_i^j - T_i^j
\]

\[
+ \frac{k \Delta y}{\Delta x} T_i^j - T_i^j + \frac{\Delta x}{2} \frac{\Delta y}{2} = \rho \frac{\Delta x}{2} \frac{\Delta y}{2} C \frac{T_i^{j+1} - T_i^j}{\Delta t}
\]

Dividing by \( k/4 \), simplifying, and solving for \( T_i^{j+1} \) gives

\[
T_i^{j+1} = \left( 1 - 4\tau - 2\tau \frac{hl}{k} \right) T_i^j + \frac{\bar{q}}{k} \left( T_i^j + 90 + \frac{hl}{k} T_m + \frac{\delta T^2}{2k} \right)
\]

This completes the finite difference formulation of the problem. Next we need to determine the upper limit of the time step \( \Delta t \) from the stability criterion, which requires the coefficient of \( T_m \) in the \( T_i^{j+1} \) expression (the primary coefficient) to be greater than or equal to zero for all nodes. The smallest primary coefficient in the nine equations here is the coefficient of \( T_i^j \) in the expression, and thus the stability criterion for this problem can be expressed as

\[
1 - 4\tau - 4\tau \frac{hl}{k} \geq 0 \quad \Rightarrow \quad \tau \leq \frac{1}{4(1 + hl/k)} \quad \Rightarrow \quad \Delta t \leq \frac{l^2}{4\alpha(1 + hl/k)}
\]

since \( \tau = \alpha \Delta t/l^2 \). Substituting the given quantities, the maximum allowable value of the time step is determined to be

\[
\Delta t \leq \frac{0.012 \text{ m}^2}{4(3.2 \times 10^{-6} \text{ m}^2/\text{s})(1 + (80 \text{ W/m}^2 \cdot ^\circ\text{C})(0.012 \text{ m})/(15 \text{ W/m} \cdot ^\circ\text{C})]} = 10.6 \text{ s}
\]

Therefore, any time step less than 10.6 s can be used to solve this problem. For convenience, let us choose the time step to be \( \Delta t = 10 \text{ s} \). Then the mesh Fourier number becomes

\[
\tau = \frac{4\Delta t}{l^2} = \frac{4(3.2 \times 10^{-6} \text{ m}^2/\text{s})(10 \text{ s})}{(0.012 \text{ m})^2} = 0.222 \quad \text{for } \Delta t = 10 \text{ s}
\]

Substituting this value of \( \tau \) and other given quantities, the developed transient finite difference equations simplify to

\[
T_i^{j+1} = 0.0836 T_i^j + 0.444(T_i^j + T_j^j + 11.2)
\]
\[
T_j^{j+1} = 0.0836 T_j^j + 0.222(T_i^j + T_j^j + 2T_j^j + 22.4)
\]
\[
T_l^{j+1} = 0.0552 T_l^j + 0.444(T_i^j + T_m^j + 12.8)
\]
\[
T_m^{j+1} = 0.112 T_m^j + 0.222(T_i^j + 2T_j^j + 109.2)
\]
\[
T_0^{j+1} = 0.112 T_0^j + 0.222(T_i^j + T_j^j + 109.2)
\]
Having specified initial conditions as the solution at time $t = 0$ (for $i = 0$), sweeping through these nine equations will give the solution at intervals of 10 s. The solution at the upper corner node (node 3) is determined to be 100.2, 105.9, 106.5, 106.6, and 106.6 °C at 1, 3, 5, 10, and 60 min, respectively. Note that the last three solutions are practically identical to the solution for the steady case obtained in Example 5–3. This indicates that steady conditions are reached in the medium after about 5 min.

**TOPIC OF SPECIAL INTEREST**

*Controlling the Numerical Error*

A comparison of the numerical results with the exact results for temperature distribution in a cylinder would show that the results obtained by a numerical method are approximate, and they may or may not be sufficiently close to the exact (true) solution values. The difference between a numerical solution and the exact solution is the error involved in the numerical solution, and it is primarily due to two sources:

- **The discretization error** (also called the truncation or formulation error), which is caused by the approximations used in the formulation of the numerical method.
- **The round-off error**, which is caused by the computer’s use of a limited number of significant digits and continuously rounding (or chopping) off the digits it cannot retain.

Below we discuss both types of errors.

**Discretization Error**

The discretization error involved in numerical methods is due to replacing the derivatives by differences in each step, or the actual temperature distribution between two adjacent nodes by a straight line segment.

Consider the variation of the solution of a transient heat transfer problem with time at a specified nodal point. Both the numerical and actual (exact) solutions coincide at the beginning of the first time step, as expected, but the numerical solution deviates from the exact solution as the time $t$ increases. The difference between the two solutions at $t = \Delta t$ is due to the approximation at the first time step only and is called the local discretization error. One would expect the situation to get worse with each step since the second step uses the erroneous result of the first step as its starting point and adds a second local discretization error on top of it, as shown in Figure 5–56. The accumulation of the local discretization errors continues with the increasing number of time steps, and the total discretization error at any
step is called the **global or accumulated discretization error**. Note that the local and global discretization errors are identical for the first time step. The global discretization error usually increases with the increasing number of steps, but the opposite may occur when the solution function changes direction frequently, giving rise to local discretization errors of opposite signs, which tend to cancel each other.

To have an idea about the magnitude of the local discretization error, consider the Taylor series expansion of the temperature at a specified nodal point \( m \) about time \( t_i \),

\[
T(x_m, t_i + \Delta t) = T(x_m, t_i) + \Delta t \frac{dT(x_m, t_i)}{dt} + \frac{1}{2} \Delta t^2 \frac{d^2T(x_m, t_i)}{dt^2} + \cdots \tag{5-62}
\]

The finite difference formulation of the time derivative at the same nodal point is expressed as

\[
\frac{\partial T(x_m, t_i)}{\partial t} \equiv \frac{T(x_m, t_i + \Delta t) - T(x_m, t_i)}{\Delta t} = \frac{T_{m+1}^i - T_{m}^i}{\Delta t} \tag{5-63}
\]

or

\[
T(x_m, t_i + \Delta t) \equiv T(x_m, t_i) + \Delta t \frac{\partial T(x_m, t_i)}{\partial t} \tag{5-64}
\]

which resembles the **Taylor series expansion** terminated after the first two terms. Therefore, the third and later terms in the Taylor series expansion represent the error involved in the finite difference approximation. For a sufficiently small time step, these terms decay rapidly as the order of derivative increases, and their contributions become smaller and smaller. The first term neglected in the Taylor series expansion is proportional to \( \Delta t^2 \), and thus the local discretization error of this approximation, which is the error involved in each step, is also proportional to \( \Delta t^2 \).

The local discretization error is the formulation error associated with a single step and gives an idea about the accuracy of the method used. However, the solution results obtained at every step except the first one involve the **accumulated error** up to that point, and the local error alone does not have much significance. What we really need to know is the global discretization error. At the worst case, the accumulated discretization error after \( I \) time steps during a time period \( t_0 \) is \( I(\Delta t)^2 = (t_0/\Delta t)(\Delta t)^2 = t_0\Delta t \), which is proportional to \( \Delta t \). Thus, we conclude that the local discretization error is proportional to the square of the step size \( \Delta t^2 \) while the global discretization error is proportional to the step size \( \Delta t \) itself. Therefore, the smaller the mesh size (or the size of the time step in transient problems), the smaller the error, and thus the more accurate is the approximation. For example, halving the step size will reduce the global discretization error by half. It should be clear from the discussions above that the discretization error can be minimized by decreasing the step size in space or time as much as possible. The discretization error approaches zero as the difference quantities such as \( \Delta x \) and \( \Delta t \) approach the differential quantities such as \( dx \) and \( dt \).
Round-off Error

If we had a computer that could retain an infinite number of digits for all numbers, the difference between the exact solution and the approximate (numerical) solution at any point would entirely be due to discretization error. But we know that every computer (or calculator) represents numbers using a finite number of significant digits. The default value of the number of significant digits for many computers is 7, which is referred to as single precision. But the user may perform the calculations using 15 significant digits for the numbers, if he or she wishes, which is referred to as double precision. Of course, performing calculations in double precision will require more computer memory and a longer execution time.

In single precision mode with seven significant digits, a computer will register the number 44444.66666 as 44444.67 or 44444.66, depending on the method of rounding the computer uses. In the first case, the excess digits are said to be rounded to the closest integer, whereas in the second case they are said to be chopped off. Therefore, the numbers \( a = 44444.12345 \) and \( b = 44444.12032 \) are equivalent for a computer that performs calculations using seven significant digits. Such a computer would give \( a - b = 0 \) instead of the true value 0.00313.

The error due to retaining a limited number of digits during calculations is called the round-off error. This error is random in nature and there is no easy and systematic way of predicting it. It depends on the number of calculations, the method of rounding off, the type of computer, and even the sequence of calculations.

In algebra you learned that \( a + b + c = a + c + b \), which seems quite reasonable. But this is not necessarily true for calculations performed with a computer, as demonstrated in Figure 5–57. Note that changing the sequence of calculations results in an error of 30.8 percent in just two operations. Considering that any significant problem involves thousands or even millions of such operations performed in sequence, we realize that the accumulated round-off error has the potential to cause serious error without giving any warning signs. Experienced programmers are very much aware of this danger, and they structure their programs to prevent any buildup of the round-off error. For example, it is much safer to multiply a number by 10 than to add it 10 times. Also, it is much safer to start any addition process with the smallest numbers and continue with larger numbers. This rule is particularly important when evaluating series with a large number of terms with alternating signs.

The round-off error is proportional to the number of computations performed during the solution. In the finite difference method, the number of calculations increases as the mesh size or the time step size decreases. Halving the mesh or time step size, for example, will double the number of calculations and thus the accumulated round-off error.

Controlling the Error in Numerical Methods

The total error in any result obtained by a numerical method is the sum of the discretization error, which decreases with decreasing step size, and the round-off error, which increases with decreasing step size, as shown in Figure 5–58. Therefore, decreasing the step size too much in order to get more
accurate results may actually backfire and give less accurate results because of a faster increase in the round-off error. We should be careful not to let round-off error get out of control by avoiding a large number of computations with very small numbers.

In practice, we will not know the exact solution of the problem, and thus we will not be able to determine the magnitude of the error involved in the numerical method. Knowing that the global discretization error is proportional to the step size is not much help either since there is no easy way of determining the value of the proportionality constant. Besides, the global discretization error alone is meaningless without a true estimate of the round-off error. Therefore, we recommend the following practical procedures to assess the accuracy of the results obtained by a numerical method.

- Start the calculations with a reasonable mesh size $\Delta x$ (and time step size $\Delta t$ for transient problems) based on experience. Then repeat the calculations using a mesh size of $\Delta x/2$. If the results obtained by halving the mesh size do not differ significantly from the results obtained with the full mesh size, we conclude that the discretization error is at an acceptable level. But if the difference is larger than we can accept, then we have to repeat the calculations using a mesh size $\Delta x/4$ or even a smaller one at regions of high temperature gradients. We continue in this manner until halving the mesh size does not cause any significant change in the results, which indicates that the discretization error is reduced to an acceptable level.

- Repeat the calculations using double precision holding the mesh size (and the size of the time step in transient problems) constant. If the changes are not significant, we conclude that the round-off error is not a problem. But if the changes are too large to accept, then we may try reducing the total number of calculations by increasing the mesh size or changing the order of computations. But if the increased mesh size gives unacceptable discretization errors, then we may have to find a reasonable compromise.

It should always be kept in mind that the results obtained by any numerical method may not reflect any trouble spots in certain problems that require special consideration such as hot spots or areas of high temperature gradients. The results that seem quite reasonable overall may be in considerable error at certain locations. This is another reason for always repeating the calculations at least twice with different mesh sizes before accepting them as the solution of the problem. Most commercial software packages have built-in routines that vary the mesh size as necessary to obtain highly accurate solutions. But it is a good engineering practice to be aware of any potential pitfalls of numerical methods and to examine the results obtained with a critical eye.

**SUMMARY**

Analytical solution methods are limited to highly simplified problems in simple geometries, and it is often necessary to use a numerical method to solve real world problems with complicated geometries or nonuniform thermal conditions. The
Numerical finite difference method is based on replacing derivatives by differences, and the finite difference formulation of a heat transfer problem is obtained by selecting a sufficient number of points in the region, called the nodal points or nodes, and writing energy balances on the volume elements centered about the nodes.

For steady heat transfer, the energy balance on a volume element can be expressed in general as
\[ \sum_{\text{All sides}} \dot{Q} + gV_{\text{element}} = 0 \]

whether the problem is one-, two-, or three-dimensional. For convenience in formulation, we always assume all heat transfer to be into the volume element from all surfaces toward the node under consideration, except for specified heat flux whose direction is already specified. The finite difference formulation for a general interior node under steady conditions are expressed for some geometries as follows:

**One-dimensional steady conduction in a plane wall:**
\[ T_{m-1} - 2T_m + T_{m+1} = \frac{g_n}{k} = 0 \]

where \( \Delta x \) is the nodal spacing for the plane wall and \( \Delta y = l \) is the nodal spacing for the two-dimensional case. Insulated boundaries can be viewed as mirrors in formulation, and thus the nodes on insulated boundaries can be treated as interior nodes by using mirror images.

The finite difference formulation at node 0 at the left boundary of a plane wall for steady one-dimensional heat conduction can be expressed as
\[ \dot{Q}_{\text{left surface}} + kA \frac{T_1 - T_0}{\Delta x} + \frac{g_0(A \Delta x/2)}{k} = 0 \]

where \( A \Delta x/2 \) is the volume of the volume, \( g_0 \) is the rate of heat generation per unit volume at \( x = 0 \), and \( A \) is the heat transfer area. The form of the first term depends on the boundary condition at \( x = 0 \) (convection, radiation, specified heat flux, etc.).

The finite difference formulation of heat conduction problems usually results in a system of \( N \) algebraic equations in \( N \) unknown nodal temperatures that need to be solved simultaneously. There are numerous systematic approaches available in the literature. Several widely available equation solvers can also be used to solve a system of equations simultaneously at the press of a button.

The finite difference formulation of transient heat conduction problems is based on an energy balance that also accounts for the variation of the energy content of the volume element during a time interval \( \Delta t \). The heat transfer and heat generation terms are expressed at the previous time step \( i \) in the explicit method, and at the new time step \( i + 1 \) in the implicit method. For a general node \( m \), the finite difference formulations are expressed as

**Explicit method:**
\[ \sum_{\text{All sides}} \dot{Q} + gV_{\text{element}} C = \frac{T_m^{i+1} - T_m^i}{\Delta t} \]

**Implicit method:**
\[ \sum_{\text{All sides}} \dot{Q} + gV_{\text{element}} C = \frac{T_m^{i+1} - T_m^i}{\Delta t} \]

where \( T_m^i \) and \( T_m^{i+1} \) are the temperatures of node \( m \) at times \( t_i = i\Delta t \) and \( t_{i+1} = (i+1)\Delta t \), respectively, and \( T_m^{i+1} - T_m^i \) represents the temperature change of the node during the time interval \( \Delta t \) between the time steps \( i \) and \( i + 1 \). The explicit and implicit formulations given here are quite general and can be used in any coordinate system regardless of heat transfer being one-, two-, or three-dimensional.

The explicit formulation of a general interior node for one- and two-dimensional heat transfer in rectangular coordinates can be expressed as

**One-dimensional case:**
\[ T_m^{i+1} = \tau(T_m^{i-1} + T_m^{i+1}) + (1 - 2\tau) T_m^i + \frac{g_i \Delta x^2}{k} \]

**Two-dimensional case:**
\[ T_m^{i+1} = \tau(T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}}) + (1 - 4\tau) T_{\text{node}} + \frac{g_{\text{node}} \Delta x^2}{k} \]

where
\[ \tau = \frac{\alpha \Delta t}{k C} \]

is the dimensionless mesh Fourier number and \( \alpha = k/(\rho C) \) is the thermal diffusivity of the medium.

The implicit method is inherently stable, and any value of \( \Delta t \) can be used with that method as the time step. The largest value of the time step \( \Delta t \) in the explicit method is limited by the stability criterion, expressed as: the coefficients of all \( T_m^i \) in the \( T_m^{i+1} \) expressions (called the primary coefficients) must be greater than or equal to zero for all nodes \( m \). The maximum value of \( \Delta t \) is determined by applying the stability criterion to the equation with the smallest primary coefficient since it is the
most restrictive. For problems with specified temperatures or heat fluxes at all the boundaries, the stability criterion can be expressed as \( \tau \leq \frac{1}{4} \) for one-dimensional problems and \( \tau \leq \frac{1}{4} \) for the two-dimensional problems in rectangular coordinates.

REFERENCES AND SUGGESTED READING


PROBLEMS*

Why Numerical Methods?

5–1C What are the limitations of the analytical solution methods?

5–2C How do numerical solution methods differ from analytical ones? What are the advantages and disadvantages of numerical and analytical methods?

5–3C What is the basis of the energy balance method? How does it differ from the formal finite difference method? For a specified nodal network, will these two methods result in the same or a different set of equations?

5–4C Consider a heat conduction problem that can be solved both analytically, by solving the governing differential equation and applying the boundary conditions, and numerically, by a software package available on your computer. Which approach would you use to solve this problem? Explain your reasoning.

5–5C Two engineers are to solve an actual heat transfer problem in a manufacturing facility. Engineer A makes the necessary simplifying assumptions and solves the problem analytically, while engineer B solves it numerically using a powerful software package. Engineer A claims he solved the problem exactly and thus his results are better, while engineer B claims that he used a more realistic model and thus his results are better. To resolve the dispute, you are asked to solve the problem experimentally in a lab. Which engineer do you think the experiments will prove right? Explain.

Finite Difference Formulation of Differential Equations

5–6C Define these terms used in the finite difference formulation: node, nodal network, volume element, nodal spacing, and difference equation.

5–7 Consider three consecutive nodes \( n-1, n, \) and \( n+1 \) in a plane wall. Using the finite difference form of the first derivative at the midpoints, show that the finite difference form of the second derivative can be expressed as

\[
\frac{T_{n-1} - 2T_n + T_{n+1}}{\Delta x^2} = 0
\]
The finite difference formulation of steady two-dimensional heat conduction in a medium with heat generation and constant thermal conductivity is given by

\[ T_{m-1,n} - \frac{2T_{m,n} + T_{m+1,n}}{\Delta x^2} + T_{m,n-1} - \frac{2T_{m,n} + T_{m,n+1}}{\Delta y^2} + \frac{g_{m,n}}{k} = 0 \]

in rectangular coordinates. Modify this relation for the three-dimensional case.

Consider steady one-dimensional heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, 3, and 4 with a uniform nodal spacing of \( \Delta x \). Using the finite difference form of the first derivative (not the energy balance approach), obtain the finite difference formulation of the boundary nodes for the case of uniform heat flux \( q \) at the left boundary (node 0) and convection at the right boundary (node 4) with a convection coefficient of \( \varepsilon \) and an ambient temperature of \( T_{\text{sur}} \).

Consider steady one-dimensional heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, 3, 4, and 5 with a uniform nodal spacing of \( \Delta x \). Using the finite difference form of the first derivative (not the energy balance approach), obtain the finite difference formulation of the boundary nodes for the case of insulation at the left boundary (node 0) and radiation at the right boundary (node 5) with an emissivity of \( \varepsilon \) and surrounding temperature of \( T_{\text{sur}} \).

One-Dimensional Steady Heat Conduction

Explain how the finite difference form of a heat conduction problem is obtained by the energy balance method.

In the energy balance formulation of the finite difference method, it is recommended that all heat transfer at the boundaries of the volume element be assumed to be into the volume element even for steady heat conduction. Is this a valid recommendation even though it seems to violate the conservation of energy principle?

How is an insulated boundary handled in the finite difference formulation of a problem? How does a symmetry line differ from an insulated boundary in the finite difference formulation?

How can a node on an insulated boundary be treated as an interior node in the finite difference formulation of a plane wall? Explain.

Consider a medium in which the finite difference formulation of a general interior node is given in its simplest form as

\[ T_{m-1} - \frac{2T_m + T_{m+1}}{\Delta x^2} + \frac{g_m}{k} = 0 \]

(a) Is heat transfer in this medium steady or transient?
(b) Is heat transfer one-, two-, or three-dimensional?
(c) Is there heat generation in the medium?
(d) Is the nodal spacing constant or variable?
(e) Is the thermal conductivity of the medium constant or variable?

Consider steady heat conduction in a plane wall whose left surface (node 0) is maintained at 30°C while the right surface (node 8) is subjected to a heat flux of 800 W/m². Express the finite difference formulation of the boundary nodes 0 and 8.
for the case of no heat generation. Also obtain the finite difference formulation for the rate of heat transfer at the left boundary.

5–17 Consider steady heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, 3, and 4 with a uniform nodal spacing of $\Delta x$. Using the energy balance approach, obtain the finite difference formulation of the boundary nodes for the case of uniform heat flux $\dot{q}_0$ at the left boundary (node 0) and convection at the right boundary (node 4) with a convection coefficient of $h$ and an ambient temperature of $T_{\text{surr}}$.

5–18 Consider steady one-dimensional heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, 3, 4, and 5 with a uniform nodal spacing of $\Delta x$. Using the energy balance approach, obtain the finite difference formulation of the boundary nodes for the case of insulation at the left boundary (node 0) and radiation at the right boundary (node 5) with an emissivity of $\varepsilon$ and surrounding temperature of $T_{\text{surr}}$.

5–19 Consider steady one-dimensional heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, 3, 4, and 5 with a uniform nodal spacing of $\Delta x$. The temperature at the right boundary (node 5) is specified. Using the energy balance approach, obtain the finite difference formulation of the boundary node 0 on the left boundary for the case of combined convection, radiation, and heat flux at the left boundary with an emissivity of $\varepsilon$, convection coefficient of $h$, ambient temperature of $T_{\text{a}}$, surrounding temperature of $T_{\text{surr}}$, and uniform heat flux of $\dot{q}_0$. Also, obtain the finite difference formulation for the rate of heat transfer at the right boundary.

5–20 Consider steady one-dimensional heat conduction in a composite plane wall consisting of two layers A and B in perfect contact at the interface. The wall involves no heat generation. The nodal network of the medium consists of nodes 0, 1 (at the interface), and 2 with a uniform nodal spacing of $\Delta x$. Using the energy balance approach, obtain the finite difference formulation of this problem for the case of insulation at the left boundary (node 0) and radiation at the right boundary (node 2) with an emissivity of $\varepsilon$ and surrounding temperature of $T_{\text{surr}}$.

5–21 Consider steady one-dimensional heat conduction in a plane wall with variable heat generation and variable thermal conductivity. The nodal network of the medium consists of nodes 0, 1, and 2 with a uniform nodal spacing of $\Delta x$. Using the energy balance approach, obtain the finite difference formulation of this problem for the case of specified heat flux $\dot{q}_0$ to the wall and convection at the left boundary (node 0) with a convection coefficient of $h$ and ambient temperature of $T_{\text{a}}$, and radiation at the right boundary (node 2) with an emissivity of $\varepsilon$ and surrounding surface temperature of $T_{\text{surr}}$.

5–22 Consider steady one-dimensional heat conduction in a pin fin of constant diameter $D$ with constant thermal conductivity. The fin is losing heat by convection to the ambient air at $T_{\text{a}}$ with a heat transfer coefficient of $h$. The nodal network of the fin consists of nodes 0 (at the base), 1 (in the middle), and 2 (at the fin tip) with a uniform nodal spacing of $\Delta x$. Using the energy balance approach, obtain the finite difference formulation of this problem to determine $T_1$ and $T_2$ for the case of specified temperature at the fin base and negligible heat transfer at the fin tip. All temperatures are in °C.

5–23 Consider steady one-dimensional heat conduction in a pin fin of constant diameter $D$ with constant thermal conductivity. The fin is losing heat by convection to the ambient air at $T_{\text{a}}$ with a convection coefficient of $h$, and by radiation to the surrounding surfaces at an average temperature of $T_{\text{surr}}$. 

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**FIGURE P5–19**

**FIGURE P5–20**

**FIGURE P5–21**

**FIGURE P5–22**

**FIGURE P5–23**
Consider an aluminum alloy fin (5–25 equations). Nodal temperatures under steady conditions by solving those difference formulation of this problem and \( T_1 \) and \( T_2 \) for the case of specified temperature at the fin base and negligible heat transfer at the fin tip. All temperatures are in °C.

5–24 Consider a large uranium plate of thickness 5 cm and thermal conductivity \( k = 28 \text{ W/m} \cdot \text{°C} \) in which heat is generated uniformly at a constant rate of \( g = 6 \times 10^5 \text{ W/m}^2 \). One side of the plate is insulated while the other side is subjected to convection to an environment at 30°C with a heat transfer coefficient of \( h = 60 \text{ W/m}^2 \cdot \text{°C} \). Considering six equally spaced nodes with a nodal spacing of 1 cm, (a) obtain the finite difference formulation of this problem and (b) determine the nodal temperatures under steady conditions by solving those equations.

5–25 Consider an aluminum alloy fin (5–25 equations) of triangular cross section whose length is \( L = 5 \text{ cm} \), base thickness is \( b = 1 \text{ cm} \), and width \( w \) in the direction normal to the plane of paper is very large. The base of the fin is maintained at a temperature of \( T_0 = 180^\circ \text{C} \). The fin is losing heat by convection to the ambient air at \( T_\infty = 25^\circ \text{C} \) with a heat transfer coefficient of \( h = 25 \text{ W/m}^2 \cdot \text{°C} \) and by radiation to the surrounding surfaces at an average temperature of \( T_\text{surf} = 290 \text{ K} \). Using the finite difference method with six equally spaced nodes along the fin in the \( x \)-direction, determine (a) the temperatures at the nodes and (b) the rate of heat transfer from the fin for \( w = 1 \text{ m} \). Take the emissivity of the fin surface to be 0.9 and assume steady one-dimensional heat transfer in the fin.

5–26 Reconsider Problem 5–25. Using EES (or other) software, investigate the effect of the fin base temperature on the fin tip temperature and the rate of heat transfer from the fin. Let the temperature at the fin base vary from 100°C to 200°C. Plot the fin tip temperature and the rate of heat transfer as a function of the fin base temperature, and discuss the results.

5–27 Consider a large plane wall of thickness \( L = 0.4 \text{ m} \), thermal conductivity \( k = 2.3 \text{ W/m} \cdot \text{°C} \), and surface area \( A = 20 \text{ m}^2 \). The left side of the wall is maintained at a constant temperature of 80°C, while the right side loses heat by convection to the surrounding air at \( T_\infty = 15^\circ \text{C} \) with a heat transfer coefficient of \( h = 24 \text{ W/m}^2 \cdot \text{°C} \). Assuming steady one-dimensional heat transfer and taking the nodal spacing to be \( 10 \text{ cm} \), (a) obtain the finite difference formulation for all nodes, (b) determine the nodal temperatures by solving those equations, and (c) evaluate the rate of heat transfer through the wall.

5–28 Consider the base plate of a 800-W household iron having a thickness of \( A = 0.6 \text{ cm} \), base area of \( A = 160 \text{ cm}^2 \), and thermal conductivity of \( k = 20 \text{ W/m} \cdot \text{°C} \). The inner surface of the base plate is subjected to uniform heat flux generated by the resistance heaters inside. When steady operating conditions are reached, the outer surface temperature of the plate is measured to be \( 85^\circ \text{C} \). Disregarding any heat loss through the upper part of the iron and taking the nodal spacing to be \( 0.2 \text{ cm} \), (a) obtain the finite difference formulation for the nodes and (b) determine the inner surface temperature of the plate by solving those equations. Answer: (b) 100°C

5–29 Consider a large plane wall of thickness \( L = 0.3 \text{ m} \), thermal conductivity \( k = 2.5 \text{ W/m} \cdot \text{°C} \), and surface area \( A = 12 \text{ m}^2 \). The left side of the wall is subjected to a heat flux of \( \dot{q}_0 = 700 \text{ W/m}^2 \) while the temperature at that surface is measured to be \( T_0 = 60^\circ \text{C} \). Assuming steady one-dimensional heat transfer and taking the nodal spacing to be \( 6 \text{ cm} \), (a) obtain the finite difference formulation for the six nodes and (b) determine the temperature of the other surface of the wall by solving those equations.

5–30E A large steel plate having a thickness of \( L = 5 \text{ in} \), thermal conductivity of \( k = 7.2 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F} \), and an emissivity of \( \varepsilon = 0.6 \) is lying on the ground. The exposed surface of
the plate exchanges heat by convection with the ambient air at \( T_w = 80°F \) with an average heat transfer coefficient of \( h = 3.5 \) Btu/h \( \cdot \) ft\(^2\) \( \cdot \) °F as well as by radiation with the open sky at an equivalent sky temperature of \( T_{sky} = 510 \) R. The ground temperature below a certain depth (say, 3 ft) is not affected by the weather conditions outside and remains fairly constant at \( 50°F \) at that location. The thermal conductivity of the soil can be taken to be \( k_{soil} = 0.49 \) Btu/h \( \cdot \) ft \( \cdot \) °F, and the steel plate can be assumed to be in perfect contact with the ground. Assuming steady one-dimensional heat transfer and taking the nodal spacings to be 1 in. in the plate and 0.6 ft in the ground, (a) obtain the finite difference formulation for all 11 nodes shown in Figure P5–30E and (b) determine the top and bottom surface temperatures of the plate by solving those equations.

5–31E Repeat Problem 5–30E by disregarding radiation heat transfer from the upper surface. 

5–32 Consider a stainless steel spoon \( (k = 15.1 \) W/m \( \cdot \) °C, \( \varepsilon = 0.6) \) that is partially immersed in boiling water at 95°C in a kitchen at 25°C. The handle of the spoon has a cross section of about 0.2 cm \( \times \) 1 cm and extends 18 cm in the air from the free surface of the water. The spoon loses heat by convection to the ambient air with an average heat transfer coefficient of \( h = 13 \) W/m\(^2\) \( \cdot \) °C as well as by radiation to the surrounding surfaces at an average temperature of \( T_{sur} = 295 \) K. Assuming steady one-dimensional heat transfer along the spoon and taking the nodal spacing to be 3 cm, (a) obtain the finite difference formulation for all nodes, (b) determine the temperature of the tip of the spoon by solving those equations, and (c) determine the rate of heat transfer from the exposed surfaces of the spoon.

5–33 Repeat Problem 5–32 using a nodal spacing of 1.5 cm.

5–34 Reconsider Problem 5–33. Using EES (or other) software, investigate the effects of the thermal conductivity and the emissivity of the spoon material on the temperature at the spoon tip and the rate of heat transfer from the exposed surfaces of the spoon. Let the thermal conductivity vary from 10 W/m \( \cdot \) °C to 400 W/m \( \cdot \) °C, and the emissivity from 0.1 to 1.0. Plot the spoon tip temperature and the heat transfer rate as functions of thermal conductivity and emissivity, and discuss the results.

5–35 One side of a 2-m-high and 3-m-wide vertical plate at 130°C is to be cooled by attaching aluminum fins \( (k = 237 \) W/m \( \cdot \) °C) of rectangular profile in an environment at 35°C. The fins are 2 cm long, 0.3 cm thick, and 0.4 cm apart. The heat transfer coefficient between the fins and the surrounding air for combined convection and radiation is estimated to be 30 W/m\(^2\) \( \cdot \) °C. Assuming steady one-dimensional heat transfer along the fin and taking the nodal spacing to be 0.5 cm, determine (a) the finite difference formulation of this problem, (b) the nodal temperatures along the fin by solving these equations, (c) the rate of heat transfer from a single fin,
and (d) the rate of heat transfer from the entire finned surface of the plate.

5–36 A hot surface at 100°C is to be cooled by attaching 3-cm-long, 0.25-cm-diameter aluminum pin fins (\( k = 237 \text{ W/m} \cdot \text{°C} \)) with a center-to-center distance of 0.6 cm. The temperature of the surrounding medium is 30°C, and the combined heat transfer coefficient on the surfaces is 35 W/m² · °C. Assuming steady one-dimensional heat transfer along the fin and taking the nodal spacing to be 0.5 cm, determine (a) the finite difference formulation of this problem, (b) the nodal temperatures along the fin by solving these equations, (c) the rate of heat transfer from a single fin, and (d) the rate of heat transfer from a 1-m × 1-m section of the plate.

5–37 Repeat Problem 5–36 using copper fins (\( k = 386 \text{ W/m} \cdot \text{°C} \)) instead of aluminum ones.

Answers: (b) 98.6°C, 97.5°C, 96.7°C, 96.0°C, 95.7°C, 95.5°C

5–38 Two 3-m-long and 0.4-cm-thick cast iron (\( k = 52 \text{ W/m} \cdot \text{°C} \), \( \varepsilon = 0.8 \)) steam pipes of outer diameter 10 cm are connected to each other through two 1-cm-thick flanges of outer diameter 20 cm, as shown in the figure. The steam flows inside the pipe at an average temperature of 200°C with a heat transfer coefficient of 180 W/m² · °C. The outer surface of the pipe is exposed to convection with ambient air at 8°C with a heat transfer coefficient of 25 W/m² · °C as well as radiation with the surrounding surfaces at an average temperature of \( T_{\text{sur}} = 290 \text{ K} \). Assuming steady one-dimensional heat conduction along the flanges and taking the nodal spacing to be 1 cm along the flange (a) obtain the finite difference formulation for all nodes, (b) determine the temperature at the tip of the flange by solving those equations, and (c) determine the rate of heat transfer from the exposed surfaces of the flange.

5–39 Reconsider Problem 5–38. Using EES (or other) software, investigate the effects of the steam temperature and the outer heat transfer coefficient on the flange tip temperature and the rate of heat transfer from the exposed surfaces of the flange. Let the steam temperature vary from 150°C to 300°C and the heat transfer coefficient from 15 W/m² · °C to 60 W/m² · °C. Plot the flange tip temperature and the heat transfer rate as functions of steam temperature and heat transfer coefficient, and discuss the results.

5–40 Using EES (or other) software, solve these systems of algebraic equations.

(a) \[
3x_1 - x_2 + 3x_3 = 0 \\
-x_1 + 2x_2 + x_3 = 3 \\
2x_1 - x_2 - x_3 = 2
\]

(b) \[
4x_1 - 2x_2^2 + 0.5x_3 = -2 \\
x_1^2 - x_2 + x_3 = 11.964 \\
x_1 + x_2 + x_3 = 3
\]

Answers: (a) \( x_1 = 2, x_2 = 3, x_3 = -1 \), (b) \( x_1 = 2.33, x_2 = 2.29, x_3 = -1.62 \)

5–41 Using EES (or other) software, solve these systems of algebraic equations.

(a) \[
3x_1 + 2x_2 - x_3 + x_4 = 6 \\
x_1 + 2x_2 - x_4 = -3 \\
-2x_1 + x_2 + 3x_3 + x_4 = 2 \\
3x_2 + x_3 - 4x_4 = -6
\]

(b) \[
3x_1 + x_2^2 + 2x_3 = 8 \\
x_1^2 + 3x_2 + 2x_3 = -6.293 \\
2x_1 - x_2^3 + 4x_3 = -12
\]

FIGURE P5–38

FIGURE P5–36

FIGURE P5–37

FIGURE P5–38
5–42 Using EES (or other) software, solve these systems of algebraic equations.

\[(a) \quad 4x_1 - x_2 + 2x_3 + x_4 = -6 \]
\[x_1 + 3x_2 - x_3 + 4x_4 = -1 \]
\[-x_1 + 2x_2 + 5x_4 = 5 \]
\[2x_2 - 4x_3 - 3x_4 = -5 \]

\[(b) \quad 2x_1 + x_2^2 - 2x_3 + x_4 = 1 \]
\[x_1^2 + 4x_2 + 2x_3^2 - 2x_4 = -3 \]
\[-x_1 + x_2^2 + 5x_3 = 10 \]
\[3x_1 - x_3^2 + 8x_4 = 15 \]

Two-Dimensional Steady Heat Conduction

5–43C Consider a medium in which the finite difference formulation of a general interior node is given in its simplest form as

\[T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}} - 4T_{\text{node}} + \frac{\theta_{\text{node}}}{k} = 0 \]

(a) Is heat transfer in this medium steady or transient?
(b) Is heat transfer one-, two-, or three-dimensional?
(c) Is there heat generation in the medium?
(d) Is the nodal spacing constant or variable?
(e) Is the thermal conductivity of the medium constant or variable?

5–44C Consider a medium in which the finite difference formulation of a general interior node is given in its simplest form as

\[T_{\text{node}} = \frac{T_{\text{left}} + T_{\text{top}} + T_{\text{right}} + T_{\text{bottom}}}{4} \]

(a) Is heat transfer in this medium steady or transient?
(b) Is heat transfer one-, two-, or three-dimensional?
(c) Is there heat generation in the medium?
(d) Is the nodal spacing constant or variable?
(e) Is the thermal conductivity of the medium constant or variable?

5–45C What is an irregular boundary? What is a practical way of handling irregular boundary surfaces with the finite difference method?

5–46 Consider steady two-dimensional heat transfer in a long solid body whose cross section is given in the figure. The temperatures at the selected nodes and the thermal conditions at the boundaries are as shown. The thermal conductivity of the body is \(k = 45 \text{ W/m} \cdot \text{°C}\), and there is no heat generation. Using the finite difference method with a mesh size of \(\Delta x = \Delta y = 2.0 \text{ cm}\), determine the temperatures at the indicated points in the medium. 

**FIGURE P5–46**

5–47 Consider steady two-dimensional heat transfer in a long solid bar whose cross section is given in the figure. The measured temperatures at selected points of the outer surfaces are as shown. The thermal conductivity of the body is \(k = 45 \text{ W/m} \cdot \text{°C}\), and there is no heat generation. Using the finite difference method with a mesh size of \(\Delta x = \Delta y = 2.0 \text{ cm}\), determine the temperatures at the indicated points in the medium.

**FIGURE P5–47**

5–48 Consider steady two-dimensional heat transfer in a long solid bar whose cross section is given in the figure. The measured temperatures at selected points of the outer surfaces are as shown. The thermal conductivity of the body is \(k = 45 \text{ W/m} \cdot \text{°C}\), and there is no heat generation. Using the finite difference method with a mesh size of \(\Delta x = \Delta y = 2.0 \text{ cm}\), determine the temperatures at the indicated points in the medium. 

**Answers:** \(T_1 = 185^\circ \text{C}, T_2 = T_3 = T_4 = 190^\circ \text{C}\)

5–49 Starting with an energy balance on a volume element, obtain the steady two-dimensional finite difference equation for a general interior node in rectangular coordinates for \(T(x, y)\) for the case of variable thermal conductivity and uniform heat generation.
5–50 Consider steady two-dimensional heat transfer in a long solid body whose cross section is given in the figure. The temperatures at the selected nodes and the thermal conditions on the boundaries are as shown. The thermal conductivity of the body is $k = 180 \text{ W/m} \cdot \text{°C}$, and heat is generated in the body uniformly at a rate of $g = 10^7 \text{ W/m}^3$. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 10 \text{ cm}$, determine (a) the temperatures at nodes 1, 2, 3, and 4 and (b) the rate of heat loss from the top surface through a 1-m-long section of the body.

5–51 Reconsider Problem 5–50. Using EES (or other) software, investigate the effects of the thermal conductivity and the heat generation rate on the temperatures at nodes 1 and 3, and the rate of heat loss from the top surface. Let the thermal conductivity vary from $10 \text{ W/m} \cdot \text{°C}$ to $400 \text{ W/m} \cdot \text{°C}$ and the heat generation rate from $10^5 \text{ W/m}^3$ to $10^8 \text{ W/m}^3$. Plot the temperatures at nodes 1 and 3, and the rate of heat loss as functions of thermal conductivity and heat generation rate, and discuss the results.

5–52 Consider steady two-dimensional heat transfer in a long solid bar whose cross section is given in the figure. The measured temperatures at selected points on the outer surfaces are as shown. The thermal conductivity of the body is $k = 20 \text{ W/m} \cdot \text{°C}$, and there is no heat generation. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 1.0 \text{ cm}$, determine the temperatures at the indicated points in the medium. 

Hint: Take advantage of symmetry.

Answers: (b) $T_1 = T_4 = 143 \text{ °C}$, $T_2 = T_3 = 136 \text{ °C}$

5–53 Consider steady two-dimensional heat transfer in an L-shaped solid body whose cross section is given in the figure. The thermal conductivity of the body is $k = 45 \text{ W/m} \cdot \text{°C}$, and heat is generated in the body at a rate of $g = 5 \times 10^6 \text{ W/m}^3$. 

5–54 Consider steady two-dimensional heat transfer in an L-shaped solid body whose cross section is given in the figure. The thermal conductivity of the body is $k = 45 \text{ W/m} \cdot \text{°C}$, and heat is generated in the body at a rate of $g = 5 \times 10^6 \text{ W/m}^3$. 

5–55 Reconsider Problem 5–54. Using EES (or other) software, investigate the effects of the thermal conductivity and the heat generation rate on the temperatures at nodes 1 and 3, and the rate of heat loss from the top surface. Let the thermal conductivity vary from $10 \text{ W/m} \cdot \text{°C}$ to $400 \text{ W/m} \cdot \text{°C}$ and the heat generation rate from $10^5 \text{ W/m}^3$ to $10^8 \text{ W/m}^3$. Plot the temperatures at nodes 1 and 3, and the rate of heat loss as functions of thermal conductivity and heat generation rate, and discuss the results.
The right surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of 120°C. The entire top surface is subjected to convection with ambient air at $T_a = 30$°C with a heat transfer coefficient of $h = 55$ W/m² · °C, and the left surface is subjected to heat flux at a uniform rate of $q_s = 8000$ W/m². The nodal network of the problem consists of 13 equally spaced nodes with $\Delta x = \Delta y = 1.5$ cm. Five of the nodes are at the bottom surface and thus their temperatures are known. (a) Obtain the finite difference equations at the remaining eight nodes and (b) determine the nodal temperatures by solving those equations.

Consider steady two-dimensional heat transfer in a long solid bar of square cross section in which heat is generated uniformly at a rate of $g = 0.19 \times 10^9$ Btu · ft⁻³. The cross section of the bar is 0.4 ft × 0.4 ft in size, and its thermal conductivity is $k = 16$ Btu/h · ft · °F. All four sides of the bar are subjected to convection with the ambient air at $T_a = 70$°F with a heat transfer coefficient of $h = 7.9$ Btu/h · ft² · °F. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 0.2$ ft, determine (a) the temperatures at the nine nodes and (b) the rate of heat loss from the bar through a 1-ft-long section.

**Answer:** (b) 3040 Btu/h

Consider steady two-dimensional heat transfer in a long concrete dam ($k = 0.6$ W/m · °C, $\alpha_s = 0.7$ m²/s) of triangular cross section whose exposed surface is subjected to solar heat flux of $q_s = 800$ W/m² and to convection and radiation to the environment at 25°C with a combined heat transfer coefficient of 30 W/m² · °C. The 2-m-high vertical section of the dam is subjected to convection by water at 15°C with a heat transfer coefficient of 150 W/m² · °C, and heat transfer through the 2-m-long base is considered to be negligible. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 1$ m and assuming steady two-dimensional heat transfer, determine the temperature of the top, middle, and bottom of the exposed surface of the dam.

**Answers:** 21.3°C, 43.2°C, 43.6°C
5–59E Consider steady two-dimensional heat transfer in a V-grooved solid body whose cross section is given in the figure. The top surfaces of the groove are maintained at 32°F while the bottom surface is maintained at 212°F. The side surfaces of the groove are insulated. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 1$ ft and taking advantage of symmetry, determine the temperatures at the middle of the insulated surfaces.

![Figure P5–59E](image)

5–60 Reconsider Problem 5–59E. Using EES (or other) software, investigate the effects of the temperatures at the top and bottom surfaces on the temperature in the middle of the insulated surface. Let the temperatures at the top and bottom surfaces vary from 32°F to 212°F. Plot the temperature in the middle of the insulated surface as functions of the temperatures at the top and bottom surfaces, and discuss the results.

5–61 Consider a long solid bar whose thermal conductivity is $k = 12$ W/m·°C and whose cross section is given in the figure. The top surface of the bar is maintained at 50°C while the bottom surface is maintained at 120°C. The left surface is insulated and the remaining three surfaces are subjected to convection with ambient air at $T_a = 25°C$ with a heat transfer coefficient of $h = 30$ W/m²·°C. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 10$ cm, (a) obtain the finite difference formulation of this problem for steady two-dimensional heat transfer and (b) determine the unknown nodal temperatures by solving those equations.

Answers: (b) 85.7°C, 86.4°C, 87.6°C

5–62 Consider a 5-m-long constantan block ($k = 23$ W/m·°C) 30 cm high and 50 cm wide. The block is completely submerged in iced water at 0°C that is well stirred, and the heat transfer coefficient is so high that the temperatures on both sides of the block can be taken to be 0°C. The bottom surface of the bar is covered with a low-conductivity material so that heat transfer through the bottom surface is negligible. The top surface of the block is heated uniformly by a 6-kW resistance heater. Using the finite difference method with a mesh size of $\Delta x = \Delta y = 10$ cm and taking advantage of symmetry, (a) obtain the finite difference formulation of this problem for steady two-dimensional heat transfer, (b) determine the unknown nodal temperatures by solving those equations, and (c) determine the rate of heat transfer from the block to the iced water.

Transient Heat Conduction

5–63C How does the finite difference formulation of a transient heat conduction problem differ from that of a steady heat conduction problem? What does the term $\rho A \Delta x C (T_{m+1}^i - T_m^i) / \Delta t$ represent in the transient finite difference formulation?

5–64C What are the two basic methods of solution of transient problems based on finite differencing? How do heat transfer terms in the energy balance formulation differ in the two methods?

5–65C The explicit finite difference formulation of a general interior node for transient heat conduction in a plane wall is given by

$$T_m^i - 2T_m^i + T_{m+1}^i + \frac{h_i \Delta x^2}{k} = \frac{T_{m+1}^{i+1} - T_m^i}{\tau}$$

Obtain the finite difference formulation for the steady case by simplifying the relation above.
5–66C The explicit finite difference formulation of a general interior node for transient two-dimensional heat conduction is given by

\[
T_{\text{node}}^{i+1} = \tau(T_{\text{left}}^i + T_{\text{top}}^i + T_{\text{right}}^i + T_{\text{bottom}}^i) + (1 - 4\tau)T_{\text{node}}^i + \frac{\tau g_{\text{node}}}{k}
\]

Obtain the finite difference formulation for the steady case by simplifying the relation above.

5–67C Is there any limitation on the size of the time step \(\Delta t\) in the solution of transient heat conduction problems using (a) the explicit method and (b) the implicit method?

5–68C Express the general stability criterion for the explicit method of solution of transient heat conduction problems.

5–69C Consider transient one-dimensional heat conduction in a plane wall that is to be solved by the explicit method. If both sides of the wall are at specified temperatures, express the stability criterion for this problem in its simplest form.

5–70C Consider transient one-dimensional heat conduction in a plane wall that is to be solved by the explicit method. If both sides of the wall are subjected to specified heat flux, express the stability criterion for this problem in its simplest form.

5–71C Consider transient two-dimensional heat conduction in a rectangular region that is to be solved by the explicit method. If all boundaries of the region are either insulated or at specified temperatures, express the stability criterion for this problem in its simplest form.

5–72C The implicit method is unconditionally stable and thus any value of time step \(\Delta t\) can be used in the solution of transient heat conduction problems. To minimize the computation time, someone suggests using a very large value of \(\Delta t\) since there is no danger of instability. Do you agree with this suggestion? Explain.

5–73 Consider transient heat conduction in a plane wall whose left surface (node 0) is maintained at 50°C while the right surface (node 6) is subjected to a solar heat flux of 600 W/m². The wall is initially at a uniform temperature of 50°C. Express the explicit finite difference formulation of the boundary nodes 0 and 6 for the case of no heat generation. Also, obtain the finite difference formulation for the total amount of heat transfer at the left boundary during the first three time steps.

5–74 Consider transient heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, 3, and 4 with a uniform nodal spacing of \(\Delta x\). The wall is initially at a specified temperature. Using the energy balance approach, obtain the explicit finite difference formulation of the boundary nodes for the case of uniform heat flux \(q_0\) at the left boundary (node 0) and convection at the right boundary (node 4) with a convection coefficient of \(h\) and an ambient temperature of \(T_\infty\). Do not simplify.

5–75 Repeat Problem 5–74 for the case of implicit formulation.

5–76 Consider transient heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, 3, 4, and 5 with a uniform nodal spacing of \(\Delta x\). The wall is initially at a specified temperature. Using the energy balance approach, obtain the explicit finite difference formulation of the boundary nodes for the case of insulation at the left boundary (node 0) and radiation at the right boundary (node 5) with an emissivity of \(\varepsilon\) and surrounding temperature of \(T_{\text{surr}}\).

5–77 Consider transient heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, 3, and 4 with a uniform nodal spacing of \(\Delta x\). The wall is initially at a specified temperature. The temperature at the right boundary (node 4) is specified. Using the energy balance approach, obtain the explicit finite difference formulation of the boundary node 0 for the case of combined convection, radiation, and heat flux at the left boundary with an emissivity of \(\varepsilon\), convection coefficient of \(h\), ambient temperature of \(T_\infty\), surrounding temperature of \(T_{\text{surr}}\), and uniform heat flux of \(q_0\) toward the wall. Also, obtain the finite difference formulation for the total amount of heat transfer at the right boundary for the first 20 time steps.
5–78 Starting with an energy balance on a volume element, obtain the two-dimensional transient explicit finite difference equation for a general interior node in rectangular coordinates for \( T(x, y, t) \) for the case of constant thermal conductivity and no heat generation.

5–79 Starting with an energy balance on a volume element, obtain the two-dimensional transient implicit finite difference equation for a general interior node in rectangular coordinates for \( T(x, y, t) \) for the case of constant thermal conductivity and no heat generation.

5–80 Starting with an energy balance on a disk volume element, derive the one-dimensional transient explicit finite difference equation for a general interior node for \( T(z, t) \) in a cylinder whose side surface is insulated for the case of constant thermal conductivity with uniform heat generation.

5–81 Consider one-dimensional transient heat conduction in a composite plane wall that consists of two layers \( A \) and \( B \) with perfect contact at the interface. The wall involves no heat generation and initially is at a specified temperature. The nodal network of the medium consists of nodes 0, 1 (at the interface), and 2 with a uniform nodal spacing of \( \Delta x \). Using the energy balance approach, obtain the explicit finite difference formulation of this problem for the case of insulation at the left boundary (node 0) and radiation at the right boundary (node 2) with an emissivity of \( \varepsilon \) and surrounding temperature of \( T_{\text{surr}} \).

5–82 Consider transient one-dimensional heat conduction in a pin fin of constant diameter \( D \) with constant thermal conductivity. The fin is losing heat by convection to the ambient air at \( T_{a} \), with a heat transfer coefficient of \( h \) and by radiation to the surrounding surfaces at an average temperature of \( T_{\text{surr}} \). The nodal network of the fin consists of nodes 0 (at the base), 1 (in the middle), and 2 (at the fin tip) with a uniform nodal spacing of \( \Delta x \). Using the energy balance approach, obtain the explicit finite difference formulation of this problem for the case of a specified temperature at the fin base and negligible heat transfer at the fin tip.

5–83 Repeat Problem 5–82 for the case of implicit formulation.

5–84 Consider a large uranium plate of thickness \( L = 8 \text{ cm} \), thermal conductivity \( k = 28 \text{ W/m} \cdot \text{°C} \), and thermal diffusivity \( \alpha = 12.5 \times 10^{-6} \text{ m}^2/\text{s} \) that is initially at a uniform temperature of \( 100 \text{ °C} \). Heat is generated uniformly in the plate at a constant rate of \( g = 10^6 \text{ W/m}^3 \). At time \( t = 0 \), the left side of the plate is insulated while the other side is subjected to convection with an environment at \( T_{\text{surr}} = 20 \text{ °C} \) with a heat transfer coefficient of \( h = 35 \text{ W/m}^2 \cdot \text{°C} \). Using the explicit finite difference approach with a uniform nodal spacing of \( \Delta x = 2 \text{ cm} \), determine (a) the temperature distribution in the plate after 5 min and (b) how long it will take for steady conditions to be reached in the plate.

5–85 Reconsider Problem 5–84. Using EES (or other) software, investigate the effect of the cooling time on the temperatures of the left and right sides of the plate. Let the time vary from 5 min to 60 min. Plot the temperatures at the left and right surfaces as a function of time, and discuss the results.

5–86 Consider a house whose south wall consists of a 30-cm-thick Trombe wall whose thermal conductivity is \( k = 0.70 \text{ W/m} \cdot \text{°C} \) and whose thermal diffusivity is \( \alpha = 0.44 \times 10^{-6} \text{ m}^2/\text{s} \). The variations of the ambient temperature \( T_{\text{out}} \) and the solar heat flux \( q_{\text{solar}} \) incident on a south-facing vertical surface throughout the day for a typical day in February are given in the table in 3-h intervals. The Trombe wall has single glazing with an absorptivity-transmissivity product of \( \kappa = 0.76 \) (that is, 76 percent of the solar energy incident is absorbed by the exposed surface of the Trombe wall), and the average combined heat transfer coefficient for heat loss from the Trombe wall to the ambient is determined to be \( h_{\text{out}} = 3.4 \text{ W/m}^2 \cdot \text{°C} \). The interior of the house is maintained at \( T_{\text{in}} = 20 \text{ °C} \) at all times, and the heat transfer coefficient at the interior surface of the Trombe wall is \( h_{\text{in}} = 9.1 \text{ W/m}^2 \cdot \text{°C} \). Also, the vents on the Trombe wall are kept closed, and thus the only heat transfer between the air in the house and the Trombe wall is through the...
Consider two-dimensional transient heat transfer in an L-shaped solid bar that is initially at a uniform temperature of 140°C and whose cross section is given in the figure. The thermal conductivity and diffusivity of the body are \( k = 15 \text{ W/m} \cdot \text{°C} \) and \( \alpha = 3.2 \times 10^{-6} \text{ m}^2/\text{s} \), respectively, and heat is generated in it uniformly at a rate of \( q = 8 \times 10^5 \text{ W/m}^3 \). All four sides of the bar are subjected to convection to the ambient air at \( T_a = 30°C \) with a heat transfer coefficient of \( h = 45 \text{ W/m}^2 \cdot \text{°C} \). Using the explicit finite difference method with a mesh size of \( \Delta x = \Delta y = 10 \text{ cm} \), determine the centerline temperature of the bar (a) after 10 min and (b) after steady conditions are established.

### TABLE P5–86

<table>
<thead>
<tr>
<th>Time of Day</th>
<th>Ambient Temperature, °C</th>
<th>Solar Insolation, W/m²</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 AM–10 AM</td>
<td>0</td>
<td>375</td>
</tr>
<tr>
<td>10 AM–1 PM</td>
<td>4</td>
<td>750</td>
</tr>
<tr>
<td>1 PM–4 PM</td>
<td>6</td>
<td>580</td>
</tr>
<tr>
<td>4 PM–7 PM</td>
<td>1</td>
<td>95</td>
</tr>
<tr>
<td>7 PM–10 PM</td>
<td>−2</td>
<td>0</td>
</tr>
<tr>
<td>10 PM–1 AM</td>
<td>−3</td>
<td>0</td>
</tr>
<tr>
<td>1 AM–4 AM</td>
<td>−4</td>
<td>0</td>
</tr>
<tr>
<td>4 AM–7 AM</td>
<td>4</td>
<td>0</td>
</tr>
</tbody>
</table>

Interior surface of the wall. Assuming the temperature of the Trombe wall to vary linearly between 20°C at the interior surface and 0°C at the exterior surface at 7 AM and using the explicit finite difference method with a uniform nodal spacing of \( \Delta x = 5 \text{ cm} \), determine the temperature distribution along the thickness of the Trombe wall after 6, 12, 18, 24, 30, 36, 42, and 48 hours and plot the results. Also, determine the net amount of heat transferred to the house from the Trombe wall during the first day if the wall is 2.8 m high and 7 m long.

Consider a house whose windows are made of 0.375-in.-thick glass \( (k = 0.48 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F} \) and \( \alpha = 4.2 \times 10^{-6} \text{ ft}^2/\text{s} \)). Initially, the entire house, including the walls and the windows, is at the outdoor temperature of \( T_o = 35°F \). It is observed that the windows are fogged because the indoor temperature is below the dew-point temperature of 54°F. Now the heater is turned on and the air temperature in the house is raised to \( T_i = 72°F \) at a rate of 2°F rise per minute. The heat transfer coefficients at the inner and outer surfaces of the wall can be taken to be \( h_i = 1.2 \) and \( h_o = 2.6 \text{ Btu/h} \cdot \text{ft}^2 \cdot \text{°F} \), respectively, and the outdoor temperature can be assumed to remain constant. Using the explicit finite difference method with a mesh size of \( \Delta x = 0.125 \text{ in.} \), determine how long it will take for fog to appear on the interior surface of the windows.
for the fog on the windows to clear up (i.e., for the inner surface temperature of the window glass to reach 54°F).

5–91 A common annoyance in cars in winter months is the formation of fog on the glass surfaces that blocks the view. A practical way of solving this problem is to blow hot air or to attach electric resistance heaters to the inner surfaces. Consider the rear window of a car that consists of a 0.4-cm-thick glass \((k = 0.84 \text{ W/m} \cdot \text{°C} \text{ and } \alpha = 0.39 \times 10^{-6} \text{ m}^2/\text{s})\). Strip heater wires of negligible thickness are attached to the inner surface of the glass, 4 cm apart. Each wire generates heat at a rate of 10 W/m length. Initially the entire car, including its windows, is at the outdoor temperature of \(T_o = -3\text{°C}\). The heat transfer coefficients at the inner and outer surfaces of the glass can be taken to be \(h_i = 6\) and \(h_o = 20 \text{ W/m}^2 \cdot \text{°C} \), respectively. Using the explicit finite difference method with a mesh size of \(\Delta x = 0.2 \text{ cm}\) along the thickness and \(\Delta y = 1 \text{ cm}\) in the direction normal to the heater wires, determine the temperature distribution throughout the glass 15 min after the strip heaters are turned on. Also, determine the temperature distribution when steady conditions are reached.

5–92 Repeat Problem 5–91 using the implicit method with a time step of 1 min.

5–93 The roof of a house consists of a 15-cm-thick concrete slab \((k = 1.4 \text{ W/m} \cdot \text{°C} \text{ and } \alpha = 0.69 \times 10^{-6} \text{ m}^2/\text{s})\) that is 20 m wide and 20 m long. One evening at 6 PM, the slab is observed to be at a uniform temperature of 18°C. The average ambient air and the night sky temperatures for the entire night are predicted to be 6°C and 260 K, respectively. The convection heat transfer coefficients at the inner and outer surfaces of the roof can be taken to be \(h_i = 5\) and \(h_o = 12 \text{ W/m}^2 \cdot \text{°C} \), respectively. The house and the interior surfaces of the walls and the floor are maintained at a constant temperature of 20°C during the night, and the emissivity of both surfaces of the concrete roof is 0.9. Considering both radiation and convection heat transfers and using the explicit finite difference method with a time step of \(\Delta t = 5\) min and a mesh size of \(\Delta x = 3\) cm, determine the temperatures of the inner and outer surfaces of the roof at 6 AM. Also, determine the average rate of heat transfer through the roof during that night.

5–94 Consider a refrigerator whose outer dimensions are 1.80 m \(\times\) 0.8 m \(\times\) 0.7 m. The walls of the refrigerator are constructed of 3-cm-thick urethane insulation \((k = 0.026 \text{ W/m} \cdot \text{°C} \text{ and } \alpha = 0.36 \times 10^{-6} \text{ m}^2/\text{s})\) sandwiched between two layers of sheet metal with negligible thickness. The refrigerated space is maintained at 3°C and the average heat transfer coefficients at the inner and outer surfaces of the wall are 6 W/m\(^2\) \cdot °C and 9 W/m\(^2\) \cdot °C, respectively. Heat transfer through the bottom surface of the refrigerator is negligible. The kitchen temperature remains constant at about 25°C. Initially, the refrigerator contains 15 kg of food items at an average specific heat of 3.6 kJ/kg \cdot °C. Now a malfunction occurs and the refrigerator stops running for 6 h as a result. Assuming the
temperature of the contents of the refrigerator, including the air inside, rises uniformly during this period, predict the temperature inside the refrigerator after 6 h when the repairman arrives. Use the explicit finite difference method with a time step of $\Delta t = 1$ min and a mesh size of $\Delta x = 1$ cm and disregard corner effects (i.e., assume one-dimensional heat transfer in the walls).

5–95 Reconsider Problem 5–94. Using EES (or other) software, plot the temperature inside the refrigerator as a function of heating time as time varies from 1 h to 10 h, and discuss the results.

Special Topic: Controlling the Numerical Error

5–96C Why do the results obtained using a numerical method differ from the exact results obtained analytically? What are the causes of this difference?

5–97C What is the cause of the discretization error? How does the global discretization error differ from the local discretization error?

5–98C Can the global (accumulated) discretization error be less than the local error during a step? Explain.

5–99C How is the finite difference formulation for the first derivative related to the Taylor series expansion of the solution function?

5–100C Explain why the local discretization error of the finite difference method is proportional to the square of the step size. Also explain why the global discretization error is proportional to the step size itself.

5–101C What causes the round-off error? What kind of calculations are most susceptible to round-off error?

5–102C What happens to the discretization and the round-off errors as the step size is decreased?

5–103C Suggest some practical ways of reducing the round-off error.

5–104C What is a practical way of checking if the round-off error has been significant in calculations?

5–105C What is a practical way of checking if the discretization error has been significant in calculations?

**Review Problems**

5–106 Starting with an energy balance on the volume element, obtain the steady three-dimensional finite difference equation for a general interior node in rectangular coordinates for $T(x, y, z)$ for the case of constant thermal conductivity and uniform heat generation.

5–107 Starting with an energy balance on the volume element, obtain the three-dimensional transient explicit finite difference equation for a general interior node in rectangular coordinates for $T(x, y, z, t)$ for the case of constant thermal conductivity and no heat generation.

5–108 Consider steady one-dimensional heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, 2, and 3 with a uniform nodal spacing of $\Delta x$. The temperature at the left boundary (node 0) is specified. Using the energy balance approach, obtain the finite difference formulation of boundary node 3 at the right boundary for the case of combined convection and radiation with an emissivity of $\varepsilon$, convection coefficient of $h$, ambient temperature of $T_a$, and surrounding temperature of $T_{sur}$. Also, obtain the finite difference formulation for the rate of heat transfer at the left boundary.

5–109 Consider one-dimensional transient heat conduction in a plane wall with variable heat generation and constant thermal conductivity. The nodal network of the medium consists of nodes 0, 1, and 2 with a uniform nodal spacing of $\Delta x$. Using the energy balance approach, obtain the explicit finite difference formulation of this problem for the case of specified heat flux $q_0$ and convection at the left boundary (node 0) with a convection coefficient of $h$ and ambient temperature of $T_a$, and radiation at the right boundary (node 2) with an emissivity of $\varepsilon$ and surrounding temperature of $T_{sur}$.

5–110 Repeat Problem 5–109 for the case of implicit formulation.

5–111 Consider steady one-dimensional heat conduction in a pin fin of constant diameter $D$ with constant thermal conductivity. The fin is losing heat by convection with the ambient air...
5–112 Starting with an energy balance on the volume element, obtain the two-dimensional transient explicit finite difference equation for a general interior node in rectangular coordinates for \( T(x, y, t) \) for the case of constant thermal conductivity and uniform heat generation.

5–113 Starting with an energy balance on a disk volume element, derive the one-dimensional transient implicit finite difference equation for a general interior node for \( T(z, t) \) in a cylinder whose side surface is subjected to convection with a convection coefficient of \( h \) and an ambient temperature of \( T_a \) for the case of constant thermal conductivity with uniform heat generation.

5–114E The roof of a house consists of a 5-in.-thick concrete slab \( (k = 0.81 \text{ Btu/h} \cdot \text{ft} \cdot {^\circ}\text{F} \) and \( \alpha = 7.4 \times 10^{-6} \text{ ft}^2/\text{s} \) that is 45 ft wide and 55 ft long. One evening at 6 PM, the slab is observed to be at a uniform temperature of 70 \( ^\circ\text{F} \). The ambient temperature is predicted to be at about 50 \( ^\circ\text{F} \) from 6 PM to 10 PM, 42 \( ^\circ\text{F} \) from 10 PM to 2 AM, and 38 \( ^\circ\text{F} \) from 2 AM to 6 AM, while the night sky temperature is expected to be about 445 \( ^\circ\text{R} \) for the entire night. The convection heat transfer coefficients at the inner and outer surfaces of the roof can be taken to be \( h_i = 0.9 \) and \( h_o = 2.1 \) \( \text{Btu/h} \cdot \text{ft}^2 \cdot ^\circ\text{F} \), respectively. The house and the interior surfaces of the walls and the floor are maintained at a constant temperature of 70 \( ^\circ\text{F} \) during the night, and the emissivity of both surfaces of the concrete roof is 0.9. Considering both radiation and convection heat transfers and using the explicit finite difference method with a mesh size of \( \Delta x = 1 \text{ in.} \) and a time step of \( \Delta t = 5 \text{ min} \), determine the temperatures of the inner and outer surfaces of the roof at 6 AM. Also, determine the average rate of heat transfer through the roof during that night.

5–115 Solar radiation incident on a large body of clean water \( (k = 0.61 \text{ W/m} \cdot ^\circ\text{C} \) and \( \alpha = 0.15 \times 10^{-6} \text{ m}^2/\text{s} \) such as a lake, a river, or a pond is mostly absorbed by water, and the amount of absorption varies with depth. For solar radiation incident at a 45\(^\circ\) angle on a 1-m-deep large pond whose bottom surface is black (zero reflectivity), for example, 2.8 percent of the solar energy is reflected back to the atmosphere, 37.9 percent is absorbed by the bottom surface, and the remaining 59.3 percent is absorbed by the water body. If the pond is considered to be four layers of equal thickness (0.25 m in this case), it can be shown that 47.3 percent of the incident solar energy is absorbed by the top layer, 6.1 percent by the upper mid layer, 3.6 percent by the lower mid layer, and 2.4 percent by the bottom layer [for more information see Çengel and Özişik, *Solar Energy*, 33, no. 6 (1984), pp. 581–591]. The radiation absorbed by the water can be treated conveniently as heat generation in the heat transfer analysis of the pond.

Consider a large 1-m-deep pond that is initially at a uniform temperature of 15\(^\circ\text{C} \) throughout. Solar energy is incident on the pond surface at 45\(^\circ\) at an average rate of 500 W/m\(^2\) for a period of 4 h. Assuming no convection currents in the water and using the explicit finite difference method with a mesh size of \( \Delta x = 0.25 \text{ m} \) and a time step of \( \Delta t = 15 \text{ min} \), determine the temperature distribution in the pond under the most favorable conditions (i.e., no heat losses from the top or bottom surfaces of the pond). The solar energy absorbed by the bottom surface of the pond can be treated as a heat flux to the water at that surface in this case.
5–116 Reconsider Problem 5–115. The absorption of solar radiation in that case can be expressed more accurately as a fourth-degree polynomial as

\[ g(x) = q_x(0.859 - 3.415x + 6.704x^2 - 6.339x^3 + 2.278x^4), \text{ W/m}^2 \]

where \( q_x \) is the solar flux incident on the surface of the pond in W/m² and \( x \) is the distance from the free surface of the pond in m. Solve Problem 5–115 using this relation for the absorption of solar radiation.

5–117 A hot surface at 120°C is to be cooled by attaching 8 cm long, 0.8 cm in diameter aluminum pin fins \( (k = 237 \text{ W/m} \cdot \text{°C}, \alpha = 97.1 \times 10^{-6} \text{ m}^2/\text{s}) \) to it with a center-to-center distance of 1.6 cm. The temperature of the surrounding medium is 15°C, and the heat transfer coefficient on the surfaces is 35 W/m²·°C. Initially, the fins are at a uniform temperature of 30°C, and at time \( t = 0 \), the temperature of the hot surface is raised to 120°C. Assuming one-dimensional heat conduction along the fin and taking the nodal spacing to be \( \Delta x = 2 \text{ cm} \) and a time step to be \( \Delta t = 0.5 \text{ s} \), determine the nodal temperatures after 5 min by using the explicit finite difference method. Also, determine how long it will take for steady conditions to be reached.

5–118E Consider a large plane wall of thickness \( L = 0.3 \text{ ft} \) and thermal conductivity \( k = 1.2 \text{ Btu/h} \cdot \text{ ft} \cdot \text{°F} \) in space. The wall is covered with a material having an emissivity of \( \varepsilon = 0.80 \) and a solar absorptivity of \( \alpha_s = 0.45 \). The inner surface of the wall is maintained at 520 R at all times, while the outer surface is exposed to solar radiation that is incident at a rate of \( q_x = 300 \text{ Btu/h} \cdot \text{ ft}^2 \). The outer surface is also losing heat by radiation to deep space at 0 R. Using a uniform nodal spacing of \( \Delta x = 0.1 \text{ ft} \), (a) obtain the finite difference formulation for steady one-dimensional heat conduction and (b) determine the nodal temperatures by solving those equations.

Answers: (b) 522 R, 525 R, 527 R

5–119 Frozen food items can be defrosted by simply leaving them on the counter, but it takes too long. The process can be speeded up considerably for flat items such as steaks by placing them on a large piece of highly conducting metal, called the defrosting plate, which serves as a fin. The increased surface area enhances heat transfer and thus reduces the defrosting time.

Consider two 1.5-cm-thick frozen steaks at −18°C that resemble a 15-cm-diameter circular object when placed next to each other. The steaks are now placed on a 1-cm-thick black-anodized circular aluminum defrosting plate \( (k = 237 \text{ W/m} \cdot \text{°C}, \alpha = 97.1 \times 10^{-6} \text{ m}^2/\text{s}, \varepsilon = 0.90) \) whose outer diameter is 30 cm. The properties of the frozen steaks are \( \rho = 970 \text{ kg/m}^3, C_p = 1.55 \text{ kJ/kg} \cdot \text{°C}, k = 1.40 \text{ W/m} \cdot \text{°C}, \alpha = 0.93 \times 10^{-6} \text{ m}^2/\text{s}, \varepsilon = 0.95, \) and the heat of fusion is \( h_f = 187 \text{ kJ/kg} \). The steaks can be considered to be defrosted when their average temperature is 0°C and all of the ice in the steaks is melted. Initially, the defrosting plate is at the room temperature of 20°C, and the wooden countertop it is placed on can be treated as insulation. Also, the surrounding surfaces can be taken to be at the same temperature as the ambient air, and the convection heat transfer coefficient for all exposed surfaces can be taken to be 12 W/m²·°C. Heat transfer from the lateral surfaces of the steaks and the defrosting plate can be neglected. Assuming one-dimensional heat conduction in both the steaks and the defrosting plate and using the explicit finite difference method, determine how long it will take to defrost the steaks. Use four nodes with a nodal spacing of \( \Delta x = 0.5 \text{ cm} \) for the steaks, and three nodes with a nodal spacing of \( \Delta x = 3.75 \text{ cm} \) for the exposed portion of the defrosting plate. Also, use a time step of \( \Delta t = 5 \text{ s} \). Hint: First, determine the total amount of heat transfer needed to defrost the steaks, and then determine how long it will take to transfer that much heat.
5–120  Repeat Problem 5–119 for a copper defrosting plate using a time step of \( \Delta t = 3 \) s.

Design and Essay Problems

5–121  Write a two-page essay on the finite element method, and explain why it is used in most commercial engineering software packages. Also explain how it compares to the finite difference method.

5–122  Numerous professional software packages are available in the market for performing heat transfer analysis, and they are widely advertised in professional magazines such as the *Mechanical Engineering* magazine published by the American Society of Mechanical Engineers (ASME). Your company decides to purchase such a software package and asks you to prepare a report on the available packages, their costs, capabilities, ease of use, and compatibility with the available hardware, and other software as well as the reputation of the software company, their history, financial health, customer support, training, and future prospects, among other things. After a preliminary investigation, select the top three packages and prepare a full report on them.

5–123  Design a defrosting plate to speed up defrosting of flat food items such as frozen steaks and packaged vegetables and evaluate its performance using the finite difference method (see Prob. 5–119). Compare your design to the defrosting plates currently available on the market. The plate must perform well, and it must be suitable for purchase and use as a household utensil, durable, easy to clean, easy to manufacture, and affordable. The frozen food is expected to be at an initial temperature of \(-18\)°C at the beginning of the thawing process and 0°C at the end with all the ice melted. Specify the material, shape, size, and thickness of the proposed plate. Justify your recommendations by calculations. Take the ambient and surrounding surface temperatures to be 20°C and the convection heat transfer coefficient to be 15 W/m²·°C in your analysis. For a typical case, determine the defrosting time with and without the plate.

5–124  Design a fire-resistant safety box whose outer dimensions are 0.5 m \times 0.5 m \times 0.5 m that will protect its combustible contents from fire which may last up to 2 h. Assume the box will be exposed to an environment at an average temperature of 700°C with a combined heat transfer coefficient of 70 W/m²·°C and the temperature inside the box must be below 150°C at the end of 2 h. The cavity of the box must be as large as possible while meeting the design constraints, and the insulation material selected must withstand the high temperatures to which it will be exposed. Cost, durability, and strength are also important considerations in the selection of insulation materials.
So far, we have considered conduction, which is the mechanism of heat transfer through a solid or a quiescent fluid. We now consider convection, which is the mechanism of heat transfer through a fluid in the presence of bulk fluid motion.

Convection is classified as natural (or free) and forced convection, depending on how the fluid motion is initiated. In forced convection, the fluid is forced to flow over a surface or in a pipe by external means such as a pump or a fan. In natural convection, any fluid motion is caused by natural means such as the buoyancy effect, which manifests itself as the rise of warmer fluid and the fall of the cooler fluid. Convection is also classified as external and internal, depending on whether the fluid is forced to flow over a surface or in a channel.

We start this chapter with a general physical description of the convection mechanism. We then discuss the velocity and thermal boundary layers, and laminar and turbulent flows. We continue with the discussion of the dimensionless Reynolds, Prandtl, and Nusselt numbers, and their physical significance. Next we derive the convection equations of on the basis of mass, momentum, and energy conservation, and obtain solutions for flow over a flat plate. We then nondimensionalize the convection equations, and obtain functional forms of friction and convection coefficients. Finally, we present analogies between momentum and heat transfer.
We mentioned earlier that there are three basic mechanisms of heat transfer: conduction, convection, and radiation. Conduction and convection are similar in that both mechanisms require the presence of a material medium. But they are different in that convection requires the presence of fluid motion.

Heat transfer through a solid is always by conduction, since the molecules of a solid remain at relatively fixed positions. Heat transfer through a liquid or gas, however, can be by conduction or convection, depending on the presence of any bulk fluid motion. Heat transfer through a fluid is by convection in the presence of bulk fluid motion and by conduction in the absence of it. Therefore, conduction in a fluid can be viewed as the limiting case of convection, corresponding to the case of quiescent fluid (Fig. 6–1).

Convection heat transfer is complicated by the fact that it involves fluid motion as well as heat conduction. The fluid motion enhances heat transfer, since it brings hotter and cooler chunks of fluid into contact, initiating higher rates of conduction at a greater number of sites in a fluid. Therefore, the rate of heat transfer through a fluid is much higher by convection than it is by conduction. In fact, the higher the fluid velocity, the higher the rate of heat transfer.

To clarify this point further, consider steady heat transfer through a fluid contained between two parallel plates maintained at different temperatures, as shown in Figure 6–2. The temperatures of the fluid and the plate will be the same at the points of contact because of the continuity of temperature. Assuming no fluid motion, the energy of the hotter fluid molecules near the hot plate will be transferred to the adjacent cooler fluid molecules. This energy will then be transferred to the next layer of the cooler fluid molecules. This energy will then be transferred to the next layer of the cooler fluid, and so on, until it is finally transferred to the other plate. This is what happens during conduction through a fluid. Now let us use a syringe to draw some fluid near the hot plate and inject it near the cold plate repeatedly. You can imagine that this will speed up the heat transfer process considerably, since some energy is carried to the other side as a result of fluid motion.

Consider the cooling of a hot iron block with a fan blowing air over its top surface, as shown in Figure 6–3. We know that heat will be transferred from the hot block to the surrounding cooler air, and the block will eventually cool. We also know that the block will cool faster if the fan is switched to a higher speed. Replacing air by water will enhance the convection heat transfer even more.

Experience shows that convection heat transfer strongly depends on the fluid properties dynamic viscosity $\mu$, thermal conductivity $k$, density $\rho$, and specific heat $C_p$, as well as the fluid velocity $V$. It also depends on the geometry and the roughness of the solid surface, in addition to the type of fluid flow (such as being streamlined or turbulent). Thus, we expect the convection heat transfer relations to be rather complex because of the dependence of convection on so many variables. This is not surprising, since convection is the most complex mechanism of heat transfer.

Despite the complexity of convection, the rate of convection heat transfer is observed to be proportional to the temperature difference and is conveniently expressed by Newton’s law of cooling as
$$q_{\text{conv}} = h(T_i - T_s) \quad (\text{W/m}^2) \quad (6-1)$$

or

$$Q_{\text{conv}} = hA_s(T_i - T_s) \quad (\text{W}) \quad (6-2)$$

where

- $h$ = convection heat transfer coefficient, W/m$^2$·°C
- $A_s$ = heat transfer surface area, m$^2$
- $T_i$ = temperature of the surface, °C
- $T_s$ = temperature of the fluid sufficiently far from the surface, °C

Judging from its units, the **convection heat transfer coefficient** $h$ can be defined as the rate of heat transfer between a solid surface and a fluid per unit surface area per unit temperature difference.

You should not be deceived by the simple appearance of this relation, because the convection heat transfer coefficient $h$ depends on the several of the mentioned variables, and thus is difficult to determine.

When a fluid is forced to flow over a solid surface that is nonporous (i.e., impermeable to the fluid), it is observed that the fluid in motion comes to a complete stop at the surface and assumes a zero velocity relative to the surface. That is, the fluid layer in direct contact with a solid surface "sticks" to the surface and there is no slip. In fluid flow, this phenomenon is known as the **no-slip condition**, and it is due to the viscosity of the fluid (Fig. 6–4).

The no-slip condition is responsible for the development of the velocity profile for flow. Because of the friction between the fluid layers, the layer that sticks to the wall slows the adjacent fluid layer, which slows the next layer, and so on. A consequence of the no-slip condition is that all velocity profiles must have zero values at the points of contact between a fluid and a solid. The only exception to the no-slip condition occurs in extremely rarified gases.

A similar phenomenon occurs for the temperature. When two bodies at different temperatures are brought into contact, heat transfer occurs until both bodies assume the same temperature at the point of contact. Therefore, a fluid and a solid surface will have the same temperature at the point of contact. This is known as the **no-temperature-jump condition**.

An implication of the no-slip and the no-temperature jump conditions is that heat transfer from the solid surface to the fluid layer adjacent to the surface is by pure conduction, since the fluid layer is motionless, and can be expressed as

$$q_{\text{conv}} = q_{\text{cond}} = -k_{\text{fluid}} \frac{\partial T}{\partial y}\bigg|_{y=0} \quad (\text{W/m}^2) \quad (6-3)$$

where $T$ represents the temperature distribution in the fluid and $(\partial T/\partial y)_{y=0}$ is the temperature gradient at the surface. This heat is then convected away from the surface as a result of fluid motion. Note that convection heat transfer from a solid surface to a fluid is merely the conduction heat transfer from the solid surface to the fluid layer adjacent to the surface. Therefore, we can equate Eqs. 6-1 and 6-3 for the heat flux to obtain
for the determination of the convection heat transfer coefficient when the temperature distribution within the fluid is known.

The convection heat transfer coefficient, in general, varies along the flow (or \( x \)-) direction. The average or mean convection heat transfer coefficient for a surface in such cases is determined by properly averaging the local convection heat transfer coefficients over the entire surface.

**Nusselt Number**

In convection studies, it is common practice to nondimensionalize the governing equations and combine the variables, which group together into dimensionless numbers in order to reduce the number of total variables. It is also common practice to nondimensionalize the heat transfer coefficient \( h \) with the Nusselt number, defined as

\[
Nu = \frac{hL_c}{k}
\]  

(6-5)

where \( k \) is the thermal conductivity of the fluid and \( L_c \) is the characteristic length. The Nusselt number is named after Wilhelm Nusselt, who made significant contributions to convective heat transfer in the first half of the twentieth century, and it is viewed as the dimensionless convection heat transfer coefficient.

To understand the physical significance of the Nusselt number, consider a fluid layer of thickness \( L \) and temperature difference \( \Delta T = T_2 - T_1 \), as shown in Fig. 6–5. Heat transfer through the fluid layer will be by convection when the fluid involves some motion and by conduction when the fluid layer is motionless. Heat flux (the rate of heat transfer per unit time per unit surface area) in either case will be

\[
q_{\text{conv}} = h\Delta T
\]  

(6-6)

and

\[
q_{\text{cond}} = k \frac{\Delta T}{L}
\]  

(6-7)

Taking their ratio gives

\[
\frac{q_{\text{conv}}}{q_{\text{cond}}} = \frac{h\Delta T}{k \Delta T/L} = \frac{hL}{k} = Nu
\]  

(6-8)

which is the Nusselt number. Therefore, the Nusselt number represents the enhancement of heat transfer through a fluid layer as a result of convection relative to conduction across the same fluid layer. The larger the Nusselt number, the more effective the convection. A Nusselt number of \( Nu = 1 \) for a fluid layer represents heat transfer across the layer by pure conduction.

We use forced convection in daily life more often than you might think (Fig. 6–6). We resort to forced convection whenever we want to increase the rate of heat transfer.
rate of heat transfer from a hot object. For example, we turn on the fan on hot summer days to help our body cool more effectively. The higher the fan speed, the better we feel. We stir our soup and blow on a hot slice of pizza to make them cool faster. The air on windy winter days feels much colder than it actually is. The simplest solution to heating problems in electronics packaging is to use a large enough fan.

6–2 - CLASSIFICATION OF FLUID FLOWS

Convection heat transfer is closely tied with fluid mechanics, which is the science that deals with the behavior of fluids at rest or in motion, and the interaction of fluids with solids or other fluids at the boundaries. There are a wide variety of fluid flow problems encountered in practice, and it is usually convenient to classify them on the basis of some common characteristics to make it feasible to study them in groups. There are many ways to classify the fluid flow problems, and below we present some general categories.

Viscous versus Inviscid Flow

When two fluid layers move relative to each other, a friction force develops between them and the slower layer tries to slow down the faster layer. This internal resistance to flow is called the viscosity, which is a measure of internal stickiness of the fluid. Viscosity is caused by cohesive forces between the molecules in liquids, and by the molecular collisions in gases. There is no fluid with zero viscosity, and thus all fluid flows involve viscous effects to some degree. Flows in which the effects of viscosity are significant are called viscous flows. The effects of viscosity are very small in some flows, and neglecting those effects greatly simplifies the analysis without much loss in accuracy. Such idealized flows of zero-viscosity fluids are called frictionless or inviscid flows.

Internal versus External Flow

A fluid flow is classified as being internal and external, depending on whether the fluid is forced to flow in a confined channel or over a surface. The flow of an unbounded fluid over a surface such as a plate, a wire, or a pipe is external flow. The flow in a pipe or duct is internal flow if the fluid is completely bounded by solid surfaces. Water flow in a pipe, for example, is internal flow, and air flow over an exposed pipe during a windy day is external flow (Fig. 6–7). The flow of liquids in a pipe is called open-channel flow if the pipe is partially filled with the liquid and there is a free surface. The flow of water in rivers and irrigation ditches are examples of such flows.

Compressible versus Incompressible Flow

A fluid flow is classified as being compressible or incompressible, depending on the density variation of the fluid during flow. The densities of liquids are essentially constant, and thus the flow of liquids is typically incompressible. Therefore, liquids are usually classified as incompressible substances. A pressure of 210 atm, for example, will cause the density of liquid water at 1 atm to change by just 1 percent. Gases, on the other hand, are highly compressible. A
pressure change of just 0.01 atm, for example, will cause a change of 1 percent in the density of atmospheric air. However, gas flows can be treated as incompressible if the density changes are under about 5 percent, which is usually the case when the flow velocity is less than 30 percent of the velocity of sound in that gas (i.e., the Mach number of flow is less than 0.3). The velocity of sound in air at room temperature is 346 m/s. Therefore, the compressibility effects of air can be neglected at speeds under 100 m/s. Note that the flow of a gas is not necessarily a compressible flow.

Laminar versus Turbulent Flow
Some flows are smooth and orderly while others are rather chaotic. The highly ordered fluid motion characterized by smooth streamlines is called laminar. The flow of high-viscosity fluids such as oils at low velocities is typically laminar. The highly disordered fluid motion that typically occurs at high velocities characterized by velocity fluctuations is called turbulent. The flow of low-viscosity fluids such as air at high velocities is typically turbulent. The flow regime greatly influences the heat transfer rates and the required power for pumping.

Natural (or Unforced) versus Forced Flow
A fluid flow is said to be natural or forced, depending on how the fluid motion is initiated. In forced flow, a fluid is forced to flow over a surface or in a pipe by external means such as a pump or a fan. In natural flows, any fluid motion is due to a natural means such as the buoyancy effect, which manifests itself as the rise of the warmer (and thus lighter) fluid and the fall of cooler (and thus denser) fluid. This thermosiphoning effect is commonly used to replace pumps in solar water heating systems by placing the water tank sufficiently above the solar collectors (Fig. 6–8).

Steady versus Unsteady (Transient) Flow
The terms steady and uniform are used frequently in engineering, and thus it is important to have a clear understanding of their meanings. The term steady implies no change with time. The opposite of steady is unsteady, or transient. The term uniform, however, implies no change with location over a specified region.

Many devices such as turbines, compressors, boilers, condensers, and heat exchangers operate for long periods of time under the same conditions, and they are classified as steady-flow devices. During steady flow, the fluid properties can change from point to point within a device, but at any fixed point they remain constant.

One-, Two-, and Three-Dimensional Flows
A flow field is best characterized by the velocity distribution, and thus a flow is said to be one-, two-, or three-dimensional if the flow velocity \( \vec{V} \) varies in one, two, or three primary dimensions, respectively. A typical fluid flow involves a three-dimensional geometry and the velocity may vary in all three dimensions rendering the flow three-dimensional \( \vec{V}(x, y, z) \) in rectangular or \( \vec{V}(r, \theta, z) \) in cylindrical coordinates. However, the variation of velocity in
certain direction can be small relative to the variation in other directions, and
can be ignored with negligible error. In such cases, the flow can be modeled
conveniently as being one- or two-dimensional, which is easier to analyze.

When the entrance effects are disregarded, fluid flow in a circular pipe is
one-dimensional since the velocity varies in the radial \( r \) direction but not in
the angular \( \theta \)- or axial \( z \)-directions (Fig. 6–9). That is, the velocity profile is
the same at any axial \( z \)-location, and it is symmetric about the axis of the pipe.
Note that even in this simplest flow, the velocity cannot be uniform across the
cross section of the pipe because of the no-slip condition. However, for con-
venience in calculations, the velocity can be assumed to be constant and thus
uniform at a cross section. Fluid flow in a pipe usually approximated as one-
dimensional uniform flow.

6–3 VELOCITY BOUNDARY LAYER

Consider the parallel flow of a fluid over a flat plate, as shown in Fig. 6–10.
Surfaces that are slightly contoured such as turbine blades can also be ap-
proximated as flat plates with reasonable accuracy. The \( x \)-coordinate is mea-
sured along the plate surface from the leading edge of the plate in the
direction of the flow, and \( y \) is measured from the surface in the normal direc-
tion. The fluid approaches the plate in the \( x \)-direction with a uniform upstream
velocity of \( \dot{V} \), which is practically identical to the free-stream velocity \( u_\infty \) over
the plate away from the surface (this would not be the case for cross flow over
blunt bodies such as a cylinder).

For the sake of discussion, we can consider the fluid to consist of adjacent
layers piled on top of each other. The velocity of the particles in the first fluid
layer adjacent to the plate becomes zero because of the no-slip condition. This
motionless layer slows down the particles of the neighboring fluid layer as a
result of friction between the particles of these two adjoining fluid layers at
different velocities. This fluid layer then slows down the molecules of the next
layer, and so on. Thus, the presence of the plate is felt up to some normal dis-
tance \( \delta \) from the plate beyond which the free-stream velocity \( u_\infty \) remains es-
tially unchanged. As a result, the \( x \)-component of the fluid velocity, \( u \), will
vary from 0 at \( y = 0 \) to nearly \( u_\infty \) at \( y = \delta \) (Fig. 6–11).

![FIGURE 6–9](image)

One-dimensional flow in a circular pipe.

![FIGURE 6–10](image)

The development of the boundary layer for flow over a flat plate, and the different flow regimes.
The region of the flow above the plate bounded by $\delta$ in which the effects of the viscous shearing forces caused by fluid viscosity are felt is called the velocity boundary layer. The boundary layer thickness, $\delta$, is typically defined as the distance $y$ from the surface at which $u = 0.99u_c$.

The hypothetical line of $u = 0.99u_c$ divides the flow over a plate into two regions: the boundary layer region, in which the viscous effects and the velocity changes are significant, and the inviscid flow region, in which the frictional effects are negligible and the velocity remains essentially constant.

**Surface Shear Stress**

Consider the flow of a fluid over the surface of a plate. The fluid layer in contact with the surface will try to drag the plate along via friction, exerting a friction force on it. Likewise, a faster fluid layer will try to drag the adjacent slower layer and exert a friction force because of the friction between the two layers. Friction force per unit area is called shear stress, and is denoted by $\tau_s$.

Experimental studies indicate that the shear stress for most fluids is proportional to the velocity gradient, and the shear stress at the wall surface is as $\tau_s = \mu \frac{\partial u}{\partial y} |_{y=0}$ (N/m²) (6-9)

where the constant of proportionality $\mu$ is called the dynamic viscosity of the fluid, whose unit is kg/m ⋅ s (or equivalently, N ⋅ s/m², or Pa ⋅ s, or poise = 0.1 Pa ⋅ s).

The fluids that obey the linear relationship above are called Newtonian fluids, after Sir Isaac Newton who expressed it first in 1687. Most common fluids such as water, air, gasoline, and oils are Newtonian fluids. Blood and liquid plastics are examples of non-Newtonian fluids. In this text we will consider Newtonian fluids only.

In fluid flow and heat transfer studies, the ratio of dynamic viscosity to density appears frequently. For convenience, this ratio is given the name kinematic viscosity $\nu$ and is expressed as $\nu = \mu/\rho$. Two common units of kinematic viscosity are m²/s and stoke (1 stoke = 1 cm²/s = 0.001 m²/s).

The viscosity of a fluid is a measure of its resistance to flow, and it is a strong function of temperature. The viscosities of liquids decrease with temperature, whereas the viscosities of gases increase with temperature (Fig. 6–12). The viscosities of some fluids at 20°C are listed in Table 6–1. Note that the viscosities of different fluids differ by several orders of magnitude.

The determination of the surface shear stress $\tau_s$ from Eq. 6-9 is not practical since it requires a knowledge of the flow velocity profile. A more practical approach in external flow is to relate $\tau_s$ to the upstream velocity $\mathbf{V}$ as

$$\tau_s = C_f \frac{\rho \mathbf{V}^2}{2} \text{ (N/m}^2\text{)} \quad (6-10)$$

where $C_f$ is the dimensionless friction coefficient, whose value in most cases is determined experimentally, and $\rho$ is the density of the fluid. Note that the friction coefficient, in general, will vary with location along the surface. Once the average friction coefficient over a given surface is available, the friction force over the entire surface is determined from
where $A_s$ is the surface area.

The friction coefficient is an important parameter in heat transfer studies since it is directly related to the heat transfer coefficient and the power requirements of the pump or fan.

**6–4 THERMAL BOUNDARY LAYER**

We have seen that a velocity boundary layer develops when a fluid flows over a surface as a result of the fluid layer adjacent to the surface assuming the surface velocity (i.e., zero velocity relative to the surface). Also, we defined the velocity boundary layer as the region in which the fluid velocity varies from zero to $0.99\frac{u}{H}$.

Likewise, a thermal boundary layer develops when a fluid at a specified temperature flows over a surface that is at a different temperature, as shown in Fig. 6–13.

Consider the flow of a fluid at a uniform temperature of $T_\infty$ over an isothermal flat plate at temperature $T_s$. The fluid particles in the layer adjacent to the surface will reach thermal equilibrium with the plate and assume the surface temperature $T_s$. These fluid particles will then exchange energy with the particles in the adjoining-fluid layer, and so on. As a result, a temperature profile will develop in the flow field that ranges from $T_s$ at the surface to $T_\infty$ sufficiently far from the surface. The flow region over the surface in which the temperature variation in the direction normal to the surface is significant is the thermal boundary layer.

The thickness of the thermal boundary layer increases in the flow direction, since the effects of heat transfer are felt at greater distances from the surface further downstream.

The convection heat transfer rate anywhere along the surface is directly related to the temperature gradient at that location. Therefore, the shape of the temperature profile in the thermal boundary layer dictates the convection heat transfer between a solid surface and the fluid flowing over it. In flow over a heated (or cooled) surface, both velocity and thermal boundary layers will develop simultaneously. Noting that the fluid velocity will have a strong influence on the temperature profile, the development of the velocity boundary layer relative to the thermal boundary layer will have a strong effect on the convection heat transfer.

### Prandtl Number

The relative thickness of the velocity and the thermal boundary layers is best described by the **dimensionless parameter Prandtl number**, defined as

$$ Pr = \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}} = \frac{\nu}{\alpha} = \frac{\mu C_p}{k} \quad (6-12) $$

**TABLE 6–1**

<table>
<thead>
<tr>
<th>Fluid</th>
<th>Dynamic viscosity $\mu$, kg/m · s</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glycerin:</td>
<td></td>
</tr>
<tr>
<td>$-20^\circ C$</td>
<td>134.0</td>
</tr>
<tr>
<td>$0^\circ C$</td>
<td>12.1</td>
</tr>
<tr>
<td>$20^\circ C$</td>
<td>1.49</td>
</tr>
<tr>
<td>$40^\circ C$</td>
<td>0.27</td>
</tr>
<tr>
<td>Engine oil:</td>
<td></td>
</tr>
<tr>
<td>SAE 10W</td>
<td>0.10</td>
</tr>
<tr>
<td>SAE 10W30</td>
<td>0.17</td>
</tr>
<tr>
<td>SAE 30</td>
<td>0.29</td>
</tr>
<tr>
<td>SAE 50</td>
<td>0.86</td>
</tr>
<tr>
<td>Mercury</td>
<td>0.0015</td>
</tr>
<tr>
<td>Ethyl alcohol</td>
<td>0.0012</td>
</tr>
<tr>
<td>Water:</td>
<td></td>
</tr>
<tr>
<td>$0^\circ C$</td>
<td>0.0018</td>
</tr>
<tr>
<td>$20^\circ C$</td>
<td>0.0010</td>
</tr>
<tr>
<td>$100^\circ C$ (liquid)</td>
<td>0.0003</td>
</tr>
<tr>
<td>$100^\circ C$ (vapor)</td>
<td>0.000013</td>
</tr>
<tr>
<td>Blood, $37^\circ C$</td>
<td>0.0004</td>
</tr>
<tr>
<td>Gasoline</td>
<td>0.00029</td>
</tr>
<tr>
<td>Ammonia</td>
<td>0.00022</td>
</tr>
<tr>
<td>Air</td>
<td>0.000018</td>
</tr>
<tr>
<td>Hydrogen, $0^\circ C$</td>
<td>0.000009</td>
</tr>
</tbody>
</table>

**FIGURE 6–13**

Thermal boundary layer on a flat plate (the fluid is hotter than the plate surface).
It is named after Ludwig Prandtl, who introduced the concept of boundary layer in 1904 and made significant contributions to boundary layer theory. The Prandtl numbers of fluids range from less than 0.01 for liquid metals to more than 100,000 for heavy oils (Table 6–2). Note that the Prandtl number is in the order of 10 for water.

The Prandtl numbers of gases are about 1, which indicates that both momentum and heat dissipate through the fluid at about the same rate. Heat diffuses very quickly in liquid metals (Pr \( \approx 11270 \)) and very slowly in oils (Pr \( \approx 11271 \)) relative to momentum. Consequently the thermal boundary layer is much thicker for liquid metals and much thinner for oils relative to the velocity boundary layer.

6–5 LAMINAR AND TURBULENT FLOWS

If you have been around smokers, you probably noticed that the cigarette smoke rises in a smooth plume for the first few centimeters and then starts fluctuating randomly in all directions as it continues its journey toward the lungs of others (Fig. 6–14). Likewise, a careful inspection of flow in a pipe reveals that the fluid flow is streamlined at low velocities but turns chaotic as the velocity is increased above a critical value, as shown in Figure 6–15. The flow regime in the first case is said to be laminar, characterized by smooth streamlines and highly-ordered motion, and turbulent in the second case, where it is characterized by velocity fluctuations and highly-disordered motion. The transition from laminar to turbulent flow does not occur suddenly; rather, it occurs over some region in which the flow fluctuates between laminar and turbulent flows before it becomes fully turbulent.

We can verify the existence of these laminar, transition, and turbulent flow regimes by injecting some dye streak into the flow in a glass tube, as the British scientist Osborn Reynolds (1842–1912) did over a century ago. We will observe that the dye streak will form a straight and smooth line at low velocities when the flow is laminar (we may see some blurring because of molecular diffusion), will have bursts of fluctuations in the transition regime, and will zigzag rapidly and randomly when the flow becomes fully turbulent. These zigzags and the dispersion of the dye are indicative of the fluctuations in the main flow and the rapid mixing of fluid particles from adjacent layers.

Typical velocity profiles in laminar and turbulent flow are also given in Figure 6–10. Note that the velocity profile is approximately parabolic in laminar flow and becomes flatter in turbulent flow, with a sharp drop near the surface. The turbulent boundary layer can be considered to consist of three layers. The very thin layer next to the wall where the viscous effects are dominant is the laminar sublayer. The velocity profile in this layer is nearly linear, and the flow is streamlined. Next to the laminar sublayer is the buffer layer, in which the turbulent effects are significant but not dominant of the diffusion effects, and next to it is the turbulent layer, in which the turbulent effects dominate.

The intense mixing of the fluid in turbulent flow as a result of rapid fluctuations enhances heat and momentum transfer between fluid particles, which increases the friction force on the surface and the convection heat transfer rate. It also causes the boundary layer to enlarge. Both the friction and heat transfer coefficients reach maximum values when the flow becomes fully turbulent. So it will come as no surprise that a special effort is made in the design

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### Table 6–2

<table>
<thead>
<tr>
<th>Fluid</th>
<th>Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquid metals</td>
<td>0.004–0.030</td>
</tr>
<tr>
<td>Gases</td>
<td>0.7–1.0</td>
</tr>
<tr>
<td>Water</td>
<td>1.7–13.7</td>
</tr>
<tr>
<td>Light organic fluids</td>
<td>5–50</td>
</tr>
<tr>
<td>Oils</td>
<td>50–100,000</td>
</tr>
<tr>
<td>Glycerin</td>
<td>2000–100,000</td>
</tr>
</tbody>
</table>

---

**FIGURE 6–14**
Laminar and turbulent flow regimes of cigarette smoke.

**FIGURE 6–15**
The behavior of colored fluid injected into the flow in laminar and turbulent flows in a tube.
of heat transfer coefficients associated with turbulent flow. The enhancement in heat transfer in turbulent flow does not come for free, however. It may be necessary to use a larger pump to overcome the larger friction forces accompanying the higher heat transfer rate.

**Reynolds Number**

The transition from laminar to turbulent flow depends on the surface geometry, surface roughness, free-stream velocity, surface temperature, and type of fluid, among other things. After exhaustive experiments in the 1880s, Osborn Reynolds discovered that the flow regime depends mainly on the ratio of the *inertia forces* to *viscous forces* in the fluid. This ratio is called the Reynolds number, which is a *dimensionless* quantity, and is expressed for external flow as (Fig. 6–16)

$$Re = \frac{\text{Inertia forces}}{\text{Viscous forces}} = \frac{\mathcal{V}L}{\nu} = \frac{\rho \mathcal{V}^2 L}{\mu}$$

(6-13)

where $\mathcal{V}$ is the upstream velocity (equivalent to the free-stream velocity $u_\infty$ for a flat plate), $L$ is the characteristic length of the geometry, and $\nu = \mu/\rho$ is the kinematic viscosity of the fluid. For a flat plate, the characteristic length is the distance $x$ from the leading edge. Note that kinematic viscosity has the unit m$^2$/s, which is identical to the unit of thermal diffusivity, and can be viewed as *viscous diffusivity* or *diffusivity for momentum*.

At large Reynolds numbers, the inertia forces, which are proportional to the density and the velocity of the fluid, are large relative to the viscous forces, and thus the viscous forces cannot prevent the random and rapid fluctuations of the fluid. At small Reynolds numbers, however, the viscous forces are large enough to overcome the inertia forces and to keep the fluid “in line.” Thus the flow is turbulent in the first case and laminar in the second.

The Reynolds number at which the flow becomes turbulent is called the **critical Reynolds number**. The value of the critical Reynolds number is different for different geometries. For flow over a flat plate, the generally accepted value of the critical Reynolds number is $Re_{cr} = \mathcal{V}x_{cr}/\nu = u_\infty x_{cr}/\nu = 5 \times 10^5$, where $x_{cr}$ is the distance from the leading edge of the plate at which transition from laminar to turbulent flow occurs. The value of $Re_{cr}$ may change substantially, however, depending on the level of turbulence in the free stream.

**6–6 • HEAT AND MOMENTUM TRANSFER IN TURBULENT FLOW**

Most flows encountered in engineering practice are turbulent, and thus it is important to understand how turbulence affects wall shear stress and heat transfer. Turbulent flow is characterized by random and rapid fluctuations of groups of fluid particles, called *eddies*, throughout the boundary layer. These fluctuations provide an additional mechanism for momentum and heat transfer. In laminar flow, fluid particles flow in an orderly manner along streamlines, and both momentum and heat are transferred across streamlines by molecular diffusion. In turbulent flow, the transverse motion of eddies transport momentum and heat to other regions of flow before they mix with the rest of the fluid and lose their identity, greatly enhancing momentum and heat
transfer. As a result, turbulent flow is associated with much higher values of friction and heat transfer coefficients (Fig. 6–17).

Even when the mean flow is steady, the eddying motion in turbulent flow causes significant fluctuations in the values of velocity, temperature, pressure, and even density (in compressible flow). Figure 6–18 shows the variation of the instantaneous velocity component \( u \) with time at a specified location, as can be measured with a hot-wire anemometer probe or other sensitive device. We observe that the instantaneous values of the velocity fluctuate about a mean value \( \bar{u} \) and a fluctuating component \( u' \),

\[
\begin{align*}
u = \bar{u} + u' 
\end{align*}
\] (6-14)

This is also the case for other properties such as the velocity component \( v \) in the \( y \) direction, and thus \( v = \bar{v} + v' \), \( P = P + P' \), and \( T = \bar{T} + T' \). The mean value of a property at some location is determined by averaging it over a time interval that is sufficiently large so that the net effect of fluctuations is zero. Therefore, the time average of fluctuating components is zero, e.g., \( \bar{u}' = 0 \). The magnitude of \( u' \) is usually just a few percent of \( \bar{u} \), but the high frequencies of eddies (in the order of a thousand per second) makes them very effective for the transport of momentum and thermal energy. In steady turbulent flow, the mean values of properties (indicated by an overbar) are independent of time.

Consider the upward eddy motion of a fluid during flow over a surface. The mass flow rate of fluid per unit area normal to flow is \( \rho u' \). Noting that \( h = C_p T \) represents the energy of the fluid and \( T' \) is the eddy temperature relative to the mean value, the rate of thermal energy transport by turbulent eddies is \( \dot{q}_t = \rho C_p u' v' \). By a similar argument on momentum transfer, the turbulent shear stress can be shown to be \( \tau_t = -\rho u' v' \). Note that \( u' v' \not= 0 \) even though \( u' = 0 \) and \( v' = 0 \), and experimental results show that \( u' v' \) is a negative quantity. Terms such as \(-\rho u' v'\) are called Reynolds stresses.

The random eddy motion of groups of particles resembles the random motion of molecules in a gas—colliding with each other after traveling a certain distance and exchanging momentum and heat in the process. Therefore, momentum and heat transport by eddies in turbulent boundary layers is analogous to the molecular momentum and heat diffusion. Then turbulent wall shear stress and turbulent heat transfer can be expressed in an analogous manner as

\[
\begin{align*}
\tau_t &= -\rho u' v' = \mu_t \frac{\partial \bar{u}}{\partial y} \quad \text{and} \quad \dot{q}_t = \rho C_p \bar{v} T' = -k_t \frac{\partial T}{\partial y} \, (6-15)
\end{align*}
\]

where \( \mu_t \) is called the turbulent viscosity, which accounts for momentum transport by turbulent eddies, and \( k_t \) is called the turbulent thermal conductivity, which accounts for thermal energy transport by turbulent eddies. Then the total shear stress and total heat flux can be expressed conveniently as

\[
\begin{align*}
\tau_{\text{total}} &= (\mu + \mu_t) \frac{\partial \bar{u}}{\partial y} = \rho (v + \epsilon_d) \frac{\partial \bar{u}}{\partial y} \, (6-16)
\end{align*}
\]

and

\[
\begin{align*}
\dot{q}_{\text{total}} &= -(k + k_t) \frac{\partial T}{\partial y} = -\rho C_p (\alpha + \epsilon_d) \frac{\partial T}{\partial y} \, (6-17)
\end{align*}
\]
where \( \varepsilon_M = \mu_r / \rho \) is the **eddy diffusivity of momentum** and \( \varepsilon_H = k / \rho C_p \) is the **eddy diffusivity of heat**.

Eddy motion and thus eddy diffusivities are much larger than their molecular counterparts in the core region of a turbulent boundary layer. The eddy motion loses its intensity close to the wall, and diminishes at the wall because of the no-slip condition. Therefore, the velocity and temperature profiles are nearly uniform in the core region of a turbulent boundary layer, but very steep in the thin layer adjacent to the wall. So it is no surprise that the wall shear stress and wall heat flux are much larger in turbulent flow than they are in laminar flow (Fig. 6–19).

Note that molecular diffusivities \( \nu \) and \( \alpha \) (as well as \( \mu \) and \( k \)) are fluid properties, and their values can be found listed in fluid handbooks. Eddy diffusivities \( \varepsilon_M \) and \( \varepsilon_H \) (as well as \( \mu_r \) and \( k_r \)), however are not fluid properties and their values depend on flow conditions. Eddy diffusivities \( \varepsilon_M \) and \( \varepsilon_H \) decrease towards the wall, becoming zero at the wall.

### 6–7 DERIVATION OF DIFFERENTIAL CONVECTION EQUATIONS*

In this section we derive the governing equations of fluid flow in the boundary layers. To keep the analysis at a manageable level, we assume the flow to be steady and two-dimensional, and the fluid to be Newtonian with constant properties (density, viscosity, thermal conductivity, etc.).

Consider the parallel flow of a fluid over a surface. We take the flow direction along the surface to be \( x \) and the direction normal to the surface to be \( y \), and we choose a differential volume element of length \( dx \), height \( dy \), and unit depth in the \( z \)-direction (normal to the paper) for analysis (Fig. 6–20). The fluid flows over the surface with a uniform free-stream velocity \( u_{\infty} \), but the velocity within boundary layer is two-dimensional: the \( x \)-component of the velocity is \( u \), and the \( y \)-component is \( v \). Note that \( u = u(x, y) \) and \( v = v(x, y) \) in steady two-dimensional flow.

Next we apply three fundamental laws to this fluid element: Conservation of mass, conservation of momentum, and conservation of energy to obtain the continuity, momentum, and energy equations for laminar flow in boundary layers.

#### Conservation of Mass Equation

The conservation of mass principle is simply a statement that mass cannot be created or destroyed, and all the mass must be accounted for during an analysis. In steady flow, the amount of mass within the control volume remains constant, and thus the conservation of mass can be expressed as

\[
\left( \text{Rate of mass flow into the control volume} \right) = \left( \text{Rate of mass flow out of the control volume} \right)
\]

**Figure 6–19**

The velocity and temperature gradients at the wall, and thus the wall shear stress and heat transfer rate, are much larger for turbulent flow than they are for laminar flow (\( T \) is shown relative to \( T_w \)).

**Figure 6–20**

Differential control volume used in the derivation of mass balance in velocity boundary layer in two-dimensional flow over a surface.

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*This and the upcoming sections of this chapter deal with theoretical aspects of convection, and can be skipped and be used as a reference if desired without a loss in continuity.*
Noting that mass flow rate is equal to the product of density, mean velocity, and cross-sectional area normal to flow, the rate at which fluid enters the control volume from the left surface is \( \rho u (dy \cdot 1) \). The rate at which the fluid leaves the control volume from the right surface can be expressed as

\[
\rho \left( u + \frac{\partial u}{\partial x} dx \right) (dy \cdot 1) \tag{6-19}
\]

Repeating this for the \( y \) direction and substituting the results into Eq. 6-18, we obtain

\[
\rho u (dy \cdot 1) + \rho v (dx \cdot 1) = \rho \left( u + \frac{\partial u}{\partial x} dx \right)(dy \cdot 1) + \rho \left( v + \frac{\partial v}{\partial y} dy \right)(dx \cdot 1) \tag{6-20}
\]

Simplifying and dividing by \( dx \cdot dy \cdot 1 \) gives

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{6-21}
\]

This is the *conservation of mass* relation, also known as the *continuity equation*, or *mass balance* for steady two-dimensional flow of a fluid with constant density.

**Conservation of Momentum Equations**

The differential forms of the equations of motion in the velocity boundary layer are obtained by applying Newton’s second law of motion to a differential control volume element in the boundary layer. Newton’s second law is an expression for the conservation of momentum, and can be stated as *the net force acting on the control volume is equal to the mass times the acceleration of the fluid element within the control volume, which is also equal to the net rate of momentum outflow from the control volume*.

The forces acting on the control volume consist of *body forces* that act throughout the entire body of the control volume (such as gravity, electric, and magnetic forces) and are proportional to the volume of the body, and *surface forces* that act on the control surface (such as the pressure forces due to hydrostatic pressure and shear stresses due to viscous effects) and are proportional to the surface area. The surface forces appear as the control volume is isolated from its surroundings for analysis, and the effect of the detached body is replaced by a force at that location. Note that pressure represents the compressive force applied on the fluid element by the surrounding fluid, and is always directed to the surface.

We express Newton’s second law of motion for the control volume as

\[
\begin{bmatrix}
\text{(Mass)}
\end{bmatrix}
\begin{bmatrix}
\text{Acceleration}
\end{bmatrix}
\text{in a specified direction} = \begin{bmatrix}
\text{(Net force (body and surface))}
\end{bmatrix}
\text{acting in that direction} \tag{6-22}
\]

or

\[
\delta m \cdot a_x = F_{\text{surface},x} + F_{\text{body},x} \tag{6-23}
\]

where the mass of the fluid element within the control volume is

\[
\delta m = \rho (dx \cdot dy \cdot 1) \tag{6-24}
\]
Noting that flow is steady and two-dimensional and thus \( u = u(x, y) \), the total differential of \( u \) is

\[
du = \frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy \tag{6-25}
\]

Then the acceleration of the fluid element in the \( x \) direction becomes

\[
a_x = \frac{du}{dt} = \frac{\partial u}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial y} \frac{dy}{dt} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \tag{6-26}
\]

You may be tempted to think that acceleration is zero in steady flow since acceleration is the rate of change of velocity with time, and in steady flow there is no change with time. Well, a garden hose nozzle will tell us that this understanding is not correct. Even in steady flow and thus constant mass flow rate, water will accelerate through the nozzle (Fig. 6–21). Steady simply means no change with time at a specified location (and thus \( \partial u/\partial t = 0 \)), but the value of a quantity may change from one location to another (and thus \( \partial u/\partial x \) and \( \partial u/\partial y \) may be different from zero). In the case of a nozzle, the velocity of water remains constant at a specified point, but it changes from inlet to the exit (water accelerates along the nozzle, which is the reason for attaching a nozzle to the garden hose in the first place).

The forces acting on a surface are due to pressure and viscous effects. In two-dimensional flow, the viscous stress at any point on an imaginary surface within the fluid can be resolved into two perpendicular components: one normal to the surface called normal stress (which should not be confused with pressure) and another along the surface called shear stress. The normal stress is related to the velocity gradients \( \partial u/\partial x \) and \( \partial v/\partial y \), that are much smaller than \( \partial u/\partial y \), to which shear stress is related. Neglecting the normal stresses for simplicity, the surface forces acting on the control volume in the \( x \)-direction will be as shown in Fig. 6–22. Then the net surface force acting in the \( x \)-direction becomes

\[
F_{\text{surface, } x} = \left( \frac{\partial \tau}{\partial y} \right)(dx \cdot 1) - \left( \frac{\partial P}{\partial x} \right)(dy \cdot 1) = \left( \frac{\partial \tau}{\partial y} - \frac{\partial P}{\partial x} \right)(dx \cdot dy \cdot 1)
\]

\[
= \left( \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} \right)(dx \cdot dy \cdot 1) \tag{6-27}
\]

since \( \tau = \mu(\partial u/\partial y) \). Substituting Eqs. 6-21, 6-23, and 6-24 into Eq. 6-20 and dividing by \( dx \cdot dy \cdot 1 \) gives

\[
p \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} \tag{6-28}
\]

This is the relation for the conservation of momentum in the \( x \)-direction, and is known as the \textit{x-momentum equation}. Note that we would obtain the same result if we used momentum flow rates for the left-hand side of this equation instead of mass times acceleration. If there is a body force acting in the \( x \)-direction, it can be added to the right side of the equation provided that it is expressed per unit volume of the fluid.

In a boundary layer, the velocity component in the flow direction is much larger than that in the normal direction, and thus \( u \gg v \), and \( \partial v/\partial x \) and \( \partial v/\partial y \) are
Boundary layer approximations.

**FIGURE 6–23**
Boundary layer approximations.

1) Velocity components:
\[ u \ll u_\infty \]

2) Velocity gradients:
\[ \frac{\partial u}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 0 \]

3) Temperature gradients:
\[ \frac{\partial T}{\partial x} \ll \frac{\partial T}{\partial y} \]

These approximations greatly simplify the analysis usually with little loss in accuracy, and make it possible to obtain analytical solutions for certain types of flow problems (Fig. 6–23).

When gravity effects and other body forces are negligible and the boundary layer approximations are valid, applying Newton’s second law of motion on the volume element in the y-direction gives the y-momentum equation to be

\[ \frac{\partial P}{\partial y} = 0 \quad (6-29) \]

That is, the variation of pressure in the direction normal to the surface is negligible, and thus \( P = P(x) \) and \( \partial P/\partial x = dP/\partial x \). Then it follows that for a given \( x \), the pressure in the boundary layer is equal to the pressure in the free stream, and the pressure determined by a separate analysis of fluid flow in the free stream (which is typically easier because of the absence of viscous effects) can readily be used in the boundary layer analysis.

The velocity components in the free stream region of a flat plate are \( u = u_\infty = \) constant and \( v = 0 \). Substituting these into the x-momentum equations (Eq. 6-28) gives \( \partial P/\partial x = 0 \). Therefore, for flow over a flat plate, the pressure remains constant over the entire plate (both inside and outside the boundary layer).

**Conservation of Energy Equation**

The energy balance for any system undergoing any process is expressed as \( E_{\text{in}} - E_{\text{out}} = \Delta E_{\text{system}} \), which states that the change in the energy content of a system during a process is equal to the difference between the energy input and the energy output. During a steady-flow process, the total energy content of a control volume remains constant (and thus \( \Delta E_{\text{system}} = 0 \)), and the amount of energy entering a control volume in all forms must be equal to the amount of energy leaving it. Then the rate form of the general energy equation reduces for a steady-flow process to \( E_{\text{in}} - E_{\text{out}} = 0 \).

Noting that energy can be transferred by heat, work, and mass only, the energy balance for a steady-flow control volume can be written explicitly as

\[ (E_{\text{in}} - E_{\text{out}})_{\text{by heat}} + (E_{\text{in}} - E_{\text{out}})_{\text{by work}} + (E_{\text{in}} - E_{\text{out}})_{\text{by mass}} = 0 \quad (6-30) \]

The total energy of a flowing fluid stream per unit mass is \( e_{\text{stream}} = h + ke + pe \) where \( h \) is the enthalpy (which is the sum of internal energy and flow energy), \( pe = gz \) is the potential energy, and \( ke = \gamma^2/2 = (u^2 + v^2)/2 \) is the kinetic energy of the fluid per unit mass. The kinetic and potential energies are usually very small relative to enthalpy, and therefore it is common practice to neglect them (besides, it can be shown that if kinetic energy is included in the analysis below, all the terms due to this inclusion cancel each other).
assume the density \( \rho \), specific heat \( C_p \), viscosity \( \mu \), and the thermal conductivity \( k \) of the fluid to be constant. Then the energy of the fluid per unit mass can be expressed as \( e_{\text{stream}} = h = C_p T \).

Energy is a scalar quantity, and thus energy interactions in all directions can be combined in one equation. Noting that mass flow rate of the fluid entering the control volume from the left is \( \rho u(dy \cdot 1) \), the rate of energy transfer to the control volume by mass is determined to be

\[
\dot{E}_{\text{in}} - \dot{E}_{\text{out}} \text{by mass,} \, x = (\dot{m} e_{\text{stream}})_x - \left( (\dot{m} e_{\text{stream}})_x + \frac{\partial (\dot{m} e_{\text{stream}})}{\partial x} dx \right)
\]

\[
= -\frac{\partial [\rho u(dy \cdot 1) C_p T]}{\partial x} dx = -\rho C_p \left( \frac{\partial T}{\partial x} + T \frac{\partial u}{\partial x} \right) dx dy \quad (6-31)
\]

Repeating this for the \( y \)-direction and adding the results, the net rate of energy transfer to the control volume by mass is determined to be

\[
(\dot{E}_{\text{in}} - \dot{E}_{\text{out}}) \text{by mass} = -\rho C_p \left( \frac{\partial T}{\partial x} + u \frac{\partial T}{\partial y} \right) dx dy
\]

\[
= -\rho C_p \left( \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) dx dy \quad (6-32)
\]

since \( \partial u/\partial x + \partial v/\partial y = 0 \) from the continuity equation.

The net rate of heat conduction to the volume element in the \( x \)-direction is

\[
(\dot{E}_{\text{in}} - \dot{E}_{\text{out}}) \text{by heat,} \, x = \dot{Q}_x - \left( \dot{Q}_x + \frac{\partial \dot{Q}_x}{\partial x} dx \right)
\]

\[
= -\frac{\partial}{\partial x} \left( -k(dy \cdot 1) \frac{\partial T}{\partial x} \right) dx = k \frac{\partial^2 T}{\partial x^2} dx dy \quad (6-33)
\]

Repeating this for the \( y \)-direction and adding the results, the net rate of energy transfer to the control volume by heat conduction becomes

\[
(\dot{E}_{\text{in}} - \dot{E}_{\text{out}}) \text{by heat} = k \frac{\partial^2 T}{\partial x^2} dx dy + k \frac{\partial^2 T}{\partial y^2} dx dy = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) dx dy \quad (6-34)
\]

Another mechanism of energy transfer to and from the fluid in the control volume is the work done by the body and surface forces. The work done by a body force is determined by multiplying this force by the velocity in the direction of the force and the volume of the fluid element, and this work needs to be considered only in the presence of significant gravitational, electric, or magnetic effects. The surface forces consist of the forces due to fluid pressure and the viscous shear stresses. The work done by pressure (the flow work) is already accounted for in the analysis above by using enthalpy for the microscopic energy of the fluid instead of internal energy. The shear stresses that result from viscous effects are usually very small, and can be neglected in many cases. This is especially the case for applications that involve low or moderate velocities.

Then the energy equation for the steady two-dimensional flow of a fluid with constant properties and negligible shear stresses is obtained by substituting Eqs. 6-32 and 6-34 into 6-30 to be

\[
\rho C_p \left( \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \quad (6-35)
\]
which states that the net energy convected by the fluid out of the control volume is equal to the net energy transferred into the control volume by heat conduction.

When the viscous shear stresses are not negligible, their effect is accounted for by expressing the energy equation as

\[ \rho C_p \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \mu \Phi \]  

(6-36)

where the viscous dissipation function \( \Phi \) is obtained after a lengthy analysis (see an advanced book such as the one by Schlichting (Ref. 9) for details) to be

\[ \Phi = 2 \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right) + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \]  

(6-37)

Viscous dissipation may play a dominant role in high-speed flows, especially when the viscosity of the fluid is high (like the flow of oil in journal bearings). This manifests itself as a significant rise in fluid temperature due to the conversion of the kinetic energy of the fluid to thermal energy. Viscous dissipation is also significant for high-speed flights of aircraft.

For the special case of a stationary fluid, \( u = v = 0 \) and the energy equation reduces, as expected, to the steady two-dimensional heat conduction equation,

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \]  

(6-38)

---

**EXAMPLE 6–1 Temperature Rise of Oil in a Journal Bearing**

The flow of oil in a journal bearing can be approximated as parallel flow between two large plates with one plate moving and the other stationary. Such flows are known as Couette flow.

Consider two large isothermal plates separated by 2-mm-thick oil film. The upper plate moves at a constant velocity of 12 m/s, while the lower plate is stationary. Both plates are maintained at 20˚C. (a) Obtain relations for the velocity and temperature distributions in the oil. (b) Determine the maximum temperature in the oil and the heat flux from the oil to each plate (Fig. 6–25).

**SOLUTION** Parallel flow of oil between two plates is considered. The velocity and temperature distributions, the maximum temperature, and the total heat transfer rate are to be determined.

**Assumptions** 1 Steady operating conditions exist. 2 Oil is an incompressible substance with constant properties. 3 Body forces such as gravity are negligible. 4 The plates are large so that there is no variation in the z direction.

**Properties** The properties of oil at 20˚C are (Table A-10):

\[ k = 0.145 \text{ W/m} \cdot \text{K} \quad \text{and} \quad \mu = 0.800 \text{ kg/m} \cdot \text{s} = 0.800 \text{ N} \cdot \text{s/m}^2 \]

**Analysis** (a) We take the x-axis to be the flow direction, and y to be the normal direction. This is parallel flow between two plates, and thus \( v = 0 \). Then the continuity equation (Eq. 6-21) reduces to

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \rightarrow \quad \frac{\partial u}{\partial x} = 0 \quad \rightarrow \quad u = u(y) \]
Therefore, the $x$-component of velocity does not change in the flow direction (i.e., the velocity profile remains unchanged). Noting that $u = u(y)$, $v = 0$, and $\partial P/\partial x = 0$ (flow is maintained by the motion of the upper plate rather than the pressure gradient), the $x$-momentum equation (Eq. 6-28) reduces to

$$x\text{-momentum: } \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} \rightarrow \frac{\partial^2 u}{\partial y^2} = 0 \tag{6-29}$$

This is a second-order ordinary differential equation, and integrating it twice gives

$$u(y) = C_1 y + C_2 \tag{6-30}$$

The fluid velocities at the plate surfaces must be equal to the velocities of the plates because of the no-slip condition. Therefore, the boundary conditions are $u(0) = 0$ and $u(L) = \mathcal{V}$, and applying them gives the velocity distribution to be

$$u(y) = \frac{\mathcal{V}}{L} y \tag{6-31}$$

Frictional heating due to viscous dissipation in this case is significant because of the high viscosity of oil and the large plate velocity. The plates are isothermal and there is no change in the flow direction, and thus the temperature depends on $y$ only, $T = T(y)$. Also, $u = u(y)$ and $v = 0$. Then the energy equation with dissipation (Eqs. 6-36 and 6-37) reduce to

$$\text{Energy: } 0 = k \frac{\partial^2 T}{\partial y^2} + \mu \left( \frac{\partial u}{\partial y} \right)^2 \rightarrow k \frac{\partial^2 T}{\partial y^2} = -\mu \left( \frac{\mathcal{V}}{L} \right)^2 \tag{6-32}$$

since $\partial u/\partial y = \mathcal{V}/L$. Dividing both sides by $k$ and integrating twice give

$$T(y) = -\frac{\mu}{2k} \left( \frac{\mathcal{V}}{L} y \right)^2 + C_3 y + C_4 \tag{6-33}$$

Applying the boundary conditions $T(0) = T_0$ and $T(L) = T_0$ gives the temperature distribution to be

$$T(y) = T_0 + \frac{\mu \mathcal{V}^2}{2k} \left( \frac{y}{L} - \frac{y^2}{L^2} \right) \tag{6-34}$$

(b) The temperature gradient is determined by differentiating $T(y)$ with respect to $y$,

$$\frac{dT}{dy} = \frac{\mu \mathcal{V}^2}{2k} \left( 1 - \frac{2 y}{L} \right) \tag{6-35}$$

The location of maximum temperature is determined by setting $dT/dy = 0$ and solving for $y$,

$$\frac{dT}{dy} = \frac{\mu \mathcal{V}^2}{2kL} \left( 1 - \frac{2 y}{L} \right) = 0 \quad \rightarrow \quad y = \frac{L}{2} \tag{6-36}$$

Therefore, maximum temperature will occur at mid plane, which is not surprising since both plates are maintained at the same temperature. The maximum temperature is the value of temperature at $y = L/2$,

$$T_{\text{max}} = T \left( \frac{L}{2} \right) = T_0 + \frac{\mu \mathcal{V}^2}{2k} \left( \frac{L/2}{L} - \frac{(L/2)^2}{L^2} \right) = T_0 + \frac{\mu \mathcal{V}^2}{8k} \tag{6-37}$$

$$= 20 + \frac{(0.8 \text{ N} \cdot \text{m/s}^2)(12 \text{m/s})^2}{8(0.145 \text{ W/m} \cdot \text{C})} \left( \frac{1 \text{ W}}{1 \text{ N} \cdot \text{m/s}} \right) = 119^\circ \text{C} \tag{6-38}$$
Heat flux at the plates is determined from the definition of heat flux,
\[
\dot{q}_0 = -k \frac{dT}{dy} \bigg|_{y=0} = -k \frac{\mu \gamma^2}{2kL} (1 - 0) = \frac{\mu \gamma^2}{2L} \\
= \frac{(0.8 \text{ N} \cdot \text{s/m}^2)(12 \text{ m/s})^2}{2(0.002 \text{ m})} \left( \frac{1 \text{ W}}{1 \text{ N} \cdot \text{m/s}} \right) = -28,800 \text{ W/m}^2
\]
\[
\dot{q}_L = -k \frac{dT}{dy} \bigg|_{y=L} = -k \frac{\mu \gamma^2}{2kL} (1 - 2) = \frac{\mu \gamma^2}{2L} = -\dot{q}_0 = 28,800 \text{ W/m}^2
\]
Therefore, heat fluxes at the two plates are equal in magnitude but opposite in sign.

**Discussion** A temperature rise of 99°C confirms our suspicion that viscous dissipation is very significant. Also, the heat flux is equivalent to the rate of mechanical energy dissipation. Therefore, mechanical energy is being converted to thermal energy at a rate of 57.2 kW/m² of plate area to overcome friction in the oil. Finally, calculations are done using oil properties at 20°C, but the oil temperature turned out to be much higher. Therefore, knowing the strong dependence of viscosity on temperature, calculations should be repeated using properties at the average temperature of 70°C to improve accuracy.

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**6–8 SOLUTIONS OF CONVECTION EQUATIONS FOR A FLAT PLATE**

Consider laminar flow of a fluid over a flat plate, as shown in Fig. 6–19. Surfaces that are slightly contoured such as turbine blades can also be approximated as flat plates with reasonable accuracy. The x-coordinate is measured along the plate surface from the leading edge of the plate in the direction of the flow, and y is measured from the surface in the normal direction. The fluid approaches the plate in the x-direction with a uniform upstream velocity, which is equivalent to the free stream velocity \(u_\infty\).

When viscous dissipation is negligible, the continuity, momentum, and energy equations (Eqs. 6-21, 6-28, and 6-35) reduce for steady, incompressible, laminar flow of a fluid with constant properties over a flat plate to

**Continuity:**
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6-39)
\]

**Momentum:**
\[
u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = v \frac{\partial^2 u}{\partial y^2} \quad (6-40)
\]

**Energy:**
\[
u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (6-41)
\]

with the boundary conditions (Fig. 6–26)

At \(x = 0\):
\[
u(0, y) = u_\infty, \quad T(0, y) = T_\infty
\]

At \(y = 0\):
\[
u(x, 0) = 0, \quad T(x, 0) = T_\infty
\]

As \(y \to \infty\):
\[
u(x, \infty) = u_\infty, \quad T(x, \infty) = T_\infty
\]

When fluid properties are assumed to be constant and thus independent of temperature, the first two equations can be solved separately for the velocity components \(u\) and \(v\). Once the velocity distribution is available, we can determine
the friction coefficient and the boundary layer thickness using their definitions. Also, knowing \( u \) and \( v \), the temperature becomes the only unknown in the last equation, and it can be solved for temperature distribution.

The continuity and momentum equations were first solved in 1908 by the German engineer H. Blasius, a student of L. Prandtl. This was done by transforming the two partial differential equations into a single ordinary differential equation by introducing a new independent variable, called the similarity variable. The finding of such a variable, assuming it exists, is more of an art than science, and it requires to have a good insight of the problem.

Noticing that the general shape of the velocity profile remains the same along the plate, Blasius reasoned that the nondimensional velocity profile \( u/ \sqrt{u u_c} \) should remain unchanged when plotted against the nondimensional distance \( y/\delta \), where \( \delta \) is the thickness of the local velocity boundary layer at a given \( x \). That is, although both \( \delta \) and \( u \) at a given \( y \) vary with \( x \), the velocity \( u \) at a fixed \( y/\delta \) remains constant. Blasius was also aware from the work of Stokes that \( \delta \) is proportional to \( \sqrt{u x/ u_c} \), and thus he defined a dimensionless similarity variable as

\[
\eta = y/ \sqrt{u x/ u_c} \quad (6-43)
\]

and thus \( u/ \sqrt{u u_c} = \text{function}(\eta) \). He then introduced a stream function \( \psi(x, y) \) as

\[
\begin{align*}
    u &= \frac{\partial \psi}{\partial y} \quad \text{and} \quad v &= -\frac{\partial \psi}{\partial x}
\end{align*}
\]

so that the continuity equation (Eq. 6-39) is automatically satisfied and thus eliminated (this can be verified easily by direct substitution). He then defined a function \( f(\eta) \) as the dependent variable as

\[
    f(\eta) = \frac{\psi}{u_c \sqrt{u x/ u_c}} \quad (6-45)
\]

Then the velocity components become

\[
\begin{align*}
    u &= \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta} \frac{\partial \eta}{\partial y} = u_c \sqrt{\frac{v x}{u_c x}} \frac{df}{d\eta} = u_c \frac{df}{d\eta} \\
    v &= \frac{\partial \psi}{\partial x} = -u_c \sqrt{\frac{v x}{u_c x}} \frac{df}{d\eta} = -u_c \frac{df}{d\eta} \left( \frac{1}{2} \frac{u_c v}{x} \left( \eta \frac{df}{d\eta} - f \right) \right)
\end{align*}
\]

By differentiating these \( u \) and \( v \) relations, the derivatives of the velocity components can be shown to be

\[
\begin{align*}
    \frac{\partial u}{\partial x} &= \frac{u_c x}{2 x} \frac{d^2 f}{d\eta^2}, \quad \frac{\partial u}{\partial y} = \frac{u_c x}{v x} \frac{d^2 f}{d\eta^2}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{u_c x}{v x} \frac{d^3 f}{d\eta^3}
\end{align*}
\]

Substituting these relations into the momentum equation and simplifying, we obtain

\[
2 \frac{d f}{d\eta} + f \frac{d^2 f}{d\eta^2} = 0 \quad (6-49)
\]

which is a third-order nonlinear differential equation. Therefore, the system of two partial differential equations is transformed into a single ordinary
differential equation by the use of a similarity variable. Using the definitions of \( f \) and \( \eta \), the boundary conditions in terms of the similarity variables can be expressed as

\[
f(0) = 0, \quad \frac{df}{d\eta} \bigg|_{\eta=0} = 0, \quad \text{and} \quad \frac{d^2f}{d\eta^2} \bigg|_{\eta=\infty} = 1 \quad (6-50)
\]

The transformed equation with its associated boundary conditions cannot be solved analytically, and thus an alternative solution method is necessary. The problem was first solved by Blasius in 1908 using a power series expansion approach, and this original solution is known as the Blasius solution. The problem is later solved more accurately using different numerical approaches, and results from such a solution are given in Table 6–3. The nondimensional velocity profile can be obtained by plotting \( u/u_w \) against \( \eta \). The results obtained by this simplified analysis are in excellent agreement with experimental results.

Recall that we defined the boundary layer thickness as the distance from the surface for which \( u/u_w = 0.99 \). We observe from Table 6–3 that the value of \( \eta \) corresponding to \( u/u_w = 0.992 \) is \( \eta = 5.0 \). Substituting \( \eta = 5.0 \) and \( \eta = \delta \) into the definition of the similarity variable (Eq. 6-43) gives \( 5.0 = \delta \sqrt{u_w/\nu} \). Then the velocity boundary layer thickness becomes

\[
\delta = \frac{5.0}{\sqrt{u_w/\nu}} = \frac{5.0x}{\sqrt{Re_x}} \quad (6-51)
\]

since \( Re_x = u_w x/\nu \), where \( x \) is the distance from the leading edge of the plate. Note that the boundary layer thickness increases with increasing kinematic viscosity \( \nu \) and with increasing distance from the leading edge \( x \), but it decreases with increasing free-stream velocity \( u_w \). Therefore, a large free-stream velocity will suppress the boundary layer and cause it to be thinner.

The shear stress on the wall can be determined from its definition and the \( \partial u/\partial y \) relation in Eq. 6-48:

\[
\tau_w = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} = \mu u_w \sqrt{\frac{u_w}{\nu}} \frac{1}{\sqrt{\eta^2}} \bigg|_{\eta=0} \quad (6-52)
\]

Substituting the value of the second derivative of \( f \) at \( \eta = 0 \) from Table 6–3 gives

\[
\tau_w = 0.332 u_w \sqrt{\frac{\rho u_w^2}{x}} = \frac{0.332 \rho u_w^2}{\sqrt{Re_x}} \quad (6-53)
\]

Then the local skin friction coefficient becomes

\[
C_{f,l} = \frac{\tau_w}{\rho V^2/2} = \frac{\tau_w}{\rho u_w^2/2} = 0.664 Re^{-1/2} \quad (6-54)
\]

Note that unlike the boundary layer thickness, wall shear stress and the skin friction coefficient decrease along the plate as \( x^{-1/2} \).

### The Energy Equation

Knowing the velocity profile, we are now ready to solve the energy equation for temperature distribution for the case of constant wall temperature \( T_w \). First we introduce the dimensionless temperature \( \theta \) as

\[
\theta = \frac{T - T_w}{T_{out} - T_w}
\]
\[
\theta(x, y) = \frac{T(x, y) - T_s}{T_x - T_s} \quad (6-55)
\]

Noting that both \( T_s \) and \( T_x \) are constant, substitution into the energy equation gives

\[
u \frac{\partial \theta}{\partial x} + v \frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2} \quad (6-56)
\]

Temperature profiles for flow over an isothermal flat plate are similar, just like the velocity profiles, and thus we expect a similarity solution for temperature to exist. Further, the thickness of the thermal boundary layer is proportional to \( \sqrt{\nu x/u_x} \), just like the thickness of the velocity boundary layer, and thus the similarity variable is also \( \eta \), and \( \theta = \theta(\eta) \). Using the chain rule and substituting the \( u \) and \( v \) expressions into the energy equation gives

\[
\frac{df}{d\eta} \frac{d\theta}{\partial \eta} + \frac{1}{2} \frac{u_x v}{x} \left( \eta \frac{df}{d\eta} - f \right) \frac{d\theta}{d\eta} \frac{\partial \eta}{\partial y} = \alpha \frac{d^2 \theta}{d\eta^2} \left( \frac{\partial \eta}{\partial y} \right)^2 \quad (6-57)
\]

Simplifying and noting that \( Pr = v/\alpha \) gives

\[
2 \frac{d^2 \theta}{d\eta^2} + Pr \frac{d\theta}{d\eta} = 0 \quad (6-58)
\]

with the boundary conditions \( \theta(0) = 0 \) and \( \theta(\infty) = 1 \). Obtaining an equation for \( \theta \) as a function of \( \eta \) alone confirms that the temperature profiles are similar, and thus a similarity solution exists. Again a closed-form solution cannot be obtained for this boundary value problem, and it must be solved numerically.

It is interesting to note that for \( Pr = 1 \), this equation reduces to Eq. 6-49 when \( \theta \) is replaced by \( df/d\eta \), which is equivalent to \( u/u_x \) (see Eq. 6-46). The boundary conditions for \( \theta \) and \( df/d\eta \) are also identical. Thus we conclude that the velocity and thermal boundary layers coincide, and the nondimensional velocity and temperature profiles \( (u/u_x \) and \( \theta \) \) are identical for steady, incompressible, laminar flow of a fluid with constant properties and \( Pr = 1 \) over an isothermal flat plate (Fig. 6–27). The value of the temperature gradient at the surface \( (y = 0 \) or \( \eta = 0 \) \) in this case is, from Table 6–3, \( d\theta/d\eta = d^2 f/d\eta^2 = 0.332 \).

Equation 6-58 is solved for numerous values of Prandtl numbers. For \( Pr > 0.6 \), the nondimensional temperature gradient at the surface is found to be proportional to \( Pr^{1/3} \), and is expressed as

\[
\frac{d\theta}{d\eta} \bigg|_{\eta=0} = 0.332 Pr^{1/3} \quad (6-59)
\]

The temperature gradient at the surface is

\[
\left. \frac{\partial T}{\partial y} \right|_{y=0} = \left. (T_x - T_s) \frac{\partial \theta}{\partial y} \right|_{y=0} = \left. (T_x - T_s) \frac{d\theta}{d\eta} \right|_{\eta=0} \frac{\partial \eta}{\partial y} \bigg|_{y=0} = 0.332 Pr^{1/3} (T_x - T_s) \sqrt{\frac{u_x}{v x}} \quad (6-60)
\]

Then the local convection coefficient and Nusselt number become

\[
h_s = \frac{q_s}{T_s - T_x} = \frac{-k(\partial T/\partial y)_{y=0}}{T_s - T_x} = 0.332 Pr^{1/3} k \sqrt{\frac{u_x}{v x}} \quad (6-61)
\]
and

\[ \text{Nu}_x = \frac{h_x}{k} = 0.332 \text{Pr}^{1/3} \text{Re}^{1/2} \quad \text{Pr} > 0.6 \quad (6-62) \]

The Nu_x values obtained from this relation agree well with measured values. Solving Eq. 6-58 numerically for the temperature profile for different Prandtl numbers, and using the definition of the thermal boundary layer, it is determined that \( \delta/\delta_t \approx \text{Pr}^{1/3} \). Then the thermal boundary layer thickness becomes

\[ \delta_t = \frac{\delta}{\text{Pr}^{1/3}} = \frac{5.0x}{\text{Pr}^{1/3} \sqrt{\text{Re}_x}} \quad (6-63) \]

Note that these relations are valid only for laminar flow over an isothermal flat plate. Also, the effect of variable properties can be accounted for by evaluating all such properties at the film temperature defined as \( T_f = (T_r + T_s)/2 \).

The Blasius solution gives important insights, but its value is largely historical because of the limitations it involves. Nowadays both laminar and turbulent flows over surfaces are routinely analyzed using numerical methods.

**6–9 NONDIMENSIONALIZED CONVECTION EQUATIONS AND SIMILARITY**

When viscous dissipation is negligible, the continuity, momentum, and energy equations for steady, incompressible, laminar flow of a fluid with constant properties are given by Eqs. 6-21, 6-28, and 6-35.

These equations and the boundary conditions can be nondimensionalized by dividing all dependent and independent variables by relevant and meaningful constant quantities: all lengths by a characteristic length \( L \) (which is the length for a plate), all velocities by a reference velocity \( \bar{v} \) (which is the free stream velocity for a plate), pressure by \( \rho \bar{v}^2 \) (which is twice the free stream dynamic pressure for a plate), and temperature by a suitable temperature difference (which is \( T_\infty - T_s \) for a plate). We get

\[ x^* = \frac{x}{L}, \quad y^* = \frac{y}{L}, \quad u^* = \frac{u}{\bar{v}}, \quad v^* = \frac{v}{\bar{v}}, \quad P^* = \frac{P}{\rho \bar{v}^2}, \quad \text{and} \quad T^* = \frac{T - T_s}{T_\infty - T_s} \]

where the asterisks are used to denote nondimensional variables. Introducing these variables into Eqs. 6-21, 6-28, and 6-35 and simplifying give

**Continuity:**

\[ \frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0 \quad (6-64) \]

**Momentum:**

\[ u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = \frac{1}{\text{Re}_L} \frac{\partial^2 u^*}{\partial y^*} - \frac{dP^*}{dx^*} \quad (6-65) \]

**Energy:**

\[ u^* \frac{\partial T^*}{\partial x^*} + v^* \frac{\partial T^*}{\partial y^*} = \frac{1}{\text{Re}_L \text{Pr}} \frac{\partial^2 T^*}{\partial y^*} \quad (6-66) \]

with the boundary conditions

\[ u^*(0, y^*) = 1, \quad u^*(x^*, 0) = 0, \quad u^*(x^*, \infty) = 1, \quad v^*(x^*, 0) = 0, \quad (6-67) \]

\[ T^*(0, y^*) = 1, \quad T^*(x^*, 0) = 0, \quad T^*(x^*, \infty) = 1 \]
where Re = \( \frac{V L}{\nu} \) is the dimensionless Reynolds number and Pr = \( \frac{\nu}{\alpha} \) is the Prandtl number. For a given type of geometry, the solutions of problems with the same Re and Nu numbers are similar, and thus Re and Nu numbers serve as similarity parameters. Two physical phenomena are similar if they have the same dimensionless forms of governing differential equations and boundary conditions (Fig. 6–28).

A major advantage of nondimensionalizing is the significant reduction in the number of parameters. The original problem involves 6 parameters \( (L, V, T_w, T_v, \nu, \alpha) \), but the nondimensionalized problem involves just 2 parameters (Re and Pr). For a given geometry, problems that have the same values for the similarity parameters have identical solutions. For example, determining the convection heat transfer coefficient for flow over a given surface will require numerical solutions or experimental investigations for several fluids, with several sets of velocities, surface lengths, wall temperatures, and free stream temperatures. The same information can be obtained with far fewer investigations by grouping data into the dimensionless Re and Pr numbers. Another advantage of similarity parameters is that they enable us to group the results of a large number of experiments and to report them conveniently in terms of such parameters (Fig. 6–29).

### 6–10 FUNCTIONAL FORMS OF FRICTION AND CONVECTION COEFFICIENTS

The three nondimensionalized boundary layer equations (Eqs. 6-64, 6-65, and 6-66) involve three unknown functions \( u^*, v^*, \) and \( T^* \), two independent variables \( x^* \) and \( y^* \), and two parameters ReL and Pr. The pressure \( P^*(x^*) \) depends on the geometry involved (it is constant for a flat plate), and it has the same value inside and outside the boundary layer at a specified \( x^* \). Therefore, it can be determined separately from the free stream conditions, and \( \frac{dP^*}{dx^*} \) in Eq. 6-65 can be treated as a known function of \( x^* \). Note that the boundary conditions do not introduce any new parameters.

For a given geometry, the solution for \( u^* \) can be expressed as
\[
 u^* = f_1(x^*, y^*, \text{Re}_L) \tag{6-68}
\]

Then the shear stress at the surface becomes
\[
 \tau_s = \mu \frac{\partial u^*}{\partial y^*} \bigg|_{y^*=0} = \frac{\mu V}{L} \frac{\partial u^*}{\partial y^*} \bigg|_{y^*=0} = \frac{\mu V}{L} f_2{x^*, \text{Re}_L} \tag{6-69}
\]

Substituting into its definition gives the local friction coefficient,
\[
 C_{f.s} = \frac{\tau_s}{\rho V^2/2} = \frac{\mu V/L}{\rho V^2/2} f_2(x^*, \text{Re}_L) = \frac{2}{\text{Re}_L} f_2(x^*, \text{Re}_L) = f_3(x^*, \text{Re}_L) \tag{6-70}
\]

Thus we conclude that the friction coefficient for a given geometry can be expressed in terms of the Reynolds number Re and the dimensionless space variable \( x^* \) alone (instead of being expressed in terms of \( x, L, V, \rho, \) and \( \mu \)). This is a very significant finding, and shows the value of nondimensionalized equations.

Similarly, the solution of Eq. 6-66 for the dimensionless temperature \( T^* \) for a given geometry can be expressed as

---

**FIGURE 6–28**

Two geometrically similar bodies have the same value of friction coefficient at the same Reynolds number.

**FIGURE 6–29**

The number of parameters is reduced greatly by nondimensionalizing the convection equations.
Using the definition of \( T^* \), the convection heat transfer coefficient becomes

\[
\frac{hL}{k} = \frac{k \partial T^*}{\partial y^*} \bigg|_{y^*=0} = \frac{k}{L} \frac{\partial T^*}{\partial y^*} \bigg|_{y^*=0}
\]  

(6-72)

Substituting this into the Nusselt number relation gives \([\text{or alternately, we can rearrange the relation above in dimensionless form as} hL/k = (\partial T^*/\partial y^*)]_{y^*=0}\) and define the dimensionless group \( hL/k \) as the Nusselt number

\[
\text{Nu} = \frac{hL}{k} = \frac{\partial T^*}{\partial y^*} \bigg|_{y^*=0} = g_{3}(x^*, \text{Re}_L, \text{Pr})
\]  

(6-73)

Note that the Nusselt number is equivalent to the dimensionless temperature gradient at the surface, and thus it is properly referred to as the dimensionless heat transfer coefficient (Fig. 6–30). Also, the Nusselt number for a given geometry can be expressed in terms of the Reynolds number \( \text{Re}_L \), the Prandtl number \( \text{Pr} \), and the space variable \( x^* \), and such a relation can be used for different fluids flowing at different velocities over similar geometries of different lengths.

The average friction and heat transfer coefficients are determined by integrating \( C_f(x) \) and \( \text{Nu}(x) \) over the surface of the given body with respect to \( x^* \) from 0 to 1. Integration will remove the dependence on \( x^* \), and the average friction coefficient and Nusselt number can be expressed as

\[
C_f = f_{3}(\text{Re}_L) \quad \text{and} \quad \text{Nu} = g_{3}(\text{Re}_L, \text{Pr})
\]  

(6-74)

These relations are extremely valuable as they state that for a given geometry, the friction coefficient can be expressed as a function of Reynolds number alone, and the Nusselt number as a function of Reynolds and Prandtl numbers alone (Fig. 6–31). Therefore, experimentalists can study a problem with a minimum number of experiments, and report their friction and heat transfer coefficient measurements conveniently in terms of Reynolds and Prandtl numbers. For example, a friction coefficient relation obtained with air for a given surface can also be used for water at the same Reynolds number. But it should be kept in mind that the validity of these relations is limited by the limitations on the boundary layer equations used in the analysis.

The experimental data for heat transfer is often represented with reasonable accuracy by a simple power-law relation of the form

\[
\text{Nu} = C \text{Re}_L^m \text{Pr}^n
\]  

(6-75)

where \( m \) and \( n \) are constant exponents (usually between 0 and 1), and the value of the constant \( C \) depends on geometry. Sometimes more complex relations are used for better accuracy.

### 6–11 ANALOGIES BETWEEN MOMENTUM AND HEAT TRANSFER

In forced convection analysis, we are primarily interested in the determination of the quantities \( C_f \) (to calculate shear stress at the wall) and \( \text{Nu} \) (to calculate heat transfer rates). Therefore, it is very desirable to have a relation between
Consider the nondimensionalized momentum and energy equations for steady, incompressible, laminar flow of a fluid with constant properties and negligible viscous dissipation (Eqs. 6-65 and 6-66). When \( Pr = 1 \) (which is approximately the case for gases) and \( \partial P^*/\partial x^* = 0 \) (which is the case when, \( u = u_\infty = \gamma = \text{constant} \) in the free stream, as in flow over a flat plate), these equations simplify to

\[
\text{Momentum:} \quad u \frac{\partial u^*}{\partial x^*} + v \frac{\partial u^*}{\partial y^*} = \frac{1}{\text{Re}_L} \frac{\partial^2 u^*}{\partial y^*} \tag{6-76}
\]

\[
\text{Energy:} \quad u \frac{\partial T^*}{\partial x^*} + v \frac{\partial T^*}{\partial y^*} = \frac{1}{\text{Re}_L} \frac{\partial^2 T^*}{\partial y^*} \tag{6-77}
\]

which are exactly of the same form for the dimensionless velocity \( u^* \) and temperature \( T^* \). The boundary conditions for \( u^* \) and \( T^* \) are also identical. Therefore, the functions \( u^* \) and \( T^* \) must be identical, and thus the first derivatives of \( u^* \) and \( T^* \) at the surface must be equal to each other,

\[
\left. \frac{\partial u^*}{\partial y^*} \right|_{y^*=0} = \left. \frac{\partial T^*}{\partial y^*} \right|_{y^*=0} \tag{6-78}
\]

Then from Eqs. 6-69, 6-70, and 6-73 we have

\[
\frac{C_f}{\text{Re}_L} \frac{\text{Re}_L}{2} = \text{Nu}_L \quad (\text{Pr} = 1) \tag{6-79}
\]

which is known as the Reynolds analogy (Fig. 6–32). This is an important analogy since it allows us to determine the heat transfer coefficient for fluids with \( Pr = 1 \) from a knowledge of friction coefficient which is easier to measure. Reynolds analogy is also expressed alternately as

\[
\frac{C_f}{\text{Re}_L} \frac{\text{Re}_L}{2} = \text{St}_L \quad (\text{Pr} = 1) \tag{6-80}
\]

where

\[
\text{St} = \frac{h}{\rho C_p \gamma} = \frac{\text{Nu}_L}{\text{Re}_L \text{Pr}} \tag{6-81}
\]

is the Stanton number, which is also a dimensionless heat transfer coefficient.

Reynolds analogy is of limited use because of the restrictions \( Pr = 1 \) and \( \partial P^*/\partial x^* = 0 \) on it, and it is desirable to have an analogy that is applicable over a wide range of \( Pr \). This is done by adding a Prandtl number correction. The friction coefficient and Nusselt number for a flat plate are determined in Section 6-8 to be

\[
C_f = 0.664 \text{Re}_L^{-1/2} \quad \text{and} \quad \text{Nu}_L = 0.332 \text{Pr}^{1/3} \text{Re}_L^{1/2} \tag{6-82}
\]

Taking their ratio and rearranging give the desired relation, known as the modified Reynolds analogy or Chilton–Colburn analogy,
for $0.6 < \text{Pr} < 60$. Here $j_H$ is called the \textit{Colburn j-factor}. Although this relation is developed using relations for laminar flow over a flat plate (for which $\partial P^*/\partial x^* = 0$), experimental studies show that it is also applicable approximately for turbulent flow over a surface, even in the presence of pressure gradients. For laminar flow, however, the analogy is not applicable unless $\partial P^*/\partial x^* = 0$. Therefore, it does not apply to laminar flow in a pipe. Analogies between $C_f$ and $Nu$ that are more accurate are also developed, but they are more complex and beyond the scope of this book. The analogies given above can be used for both local and average quantities.

\begin{equation}
\frac{C_f}{2} \frac{Re_L}{\text{Nu}} = \frac{h}{\rho C_p V^2} \text{Pr}^{2/3} = j_H \tag{6-83}
\end{equation}

\[ \text{EXAMPLE 6–2} \quad \text{Finding Convection Coefficient from Drag Measurement} \]

A 2-m $\times$ 3-m flat plate is suspended in a room, and is subjected to air flow parallel to its surfaces along its 3-m-long side. The free stream temperature and velocity of air are 20°C and 7 m/s. The total drag force acting on the plate is measured to be 0.86 N. Determine the average convection heat transfer coefficient for the plate (Fig. 6–33).

**SOLUTION**

A flat plate is subjected to air flow, and the drag force acting on it is measured. The average convection coefficient is to be determined.

**Assumptions**

1. Steady operating conditions exist.
2. The edge effects are negligible.
3. The local atmospheric pressure is 1 atm.

**Properties**

The properties of air at 20°C and 1 atm are (Table A-15):

- $\rho = 1.204 \text{ kg/m}^3$,
- $C_p = 1.007 \text{ kJ/kg} \cdot \text{K}$,
- $\text{Pr} = 0.7309$

**Analysis**

The flow is along the 3-m side of the plate, and thus the characteristic length is $L = 3$ m. Both sides of the plate are exposed to air flow, and thus the total surface area is

$$A_s = 2WL = 2(2 \text{ m})(3 \text{ m}) = 12 \text{ m}^2$$

For flat plates, the drag force is equivalent to friction force. The average friction coefficient $C_f$ can be determined from Eq. 6-11,

$$F_f = C_f A_s \rho V^2$$

Solving for $C_f$ and substituting,

$$C_f = \frac{F_f}{\rho A_s V^2} = \frac{0.86 \text{ N}}{(1.204 \text{ kg/m}^3)(12 \text{ m}^2)(7 \text{ m/s})^2/2} = 0.00243$$

Then the average heat transfer coefficient can be determined from the modified Reynolds analogy (Eq. 6-83) to be

$$h = \frac{C_f \rho Y C_p}{2 \text{ Pr}^{2/3}} = \frac{0.00243 \times (1.204 \text{ kg/m}^3)(7 \text{ m/s})(1007 \text{ J/kg} \cdot \text{K})}{2 \times 0.7309^{2/3}} = 12.7 \text{ W/m}^2 \cdot \text{°C}$$

**Discussion**

This example shows the great utility of momentum-heat transfer analogies in that the convection heat transfer coefficient can be determined from a knowledge of friction coefficient, which is easier to determine.
Convection heat transfer is expressed by Newton’s law of cooling as

\[ \dot{Q}_{\text{conv}} = hA_s(T_s - T_x) \]

where \( h \) is the convection heat transfer coefficient, \( T_s \) is the surface temperature, and \( T_x \) is the free-stream temperature. The convection coefficient is also expressed as

\[ h = \frac{-k_{\text{fluid}}(\partial T/\partial y)_{y=0}}{T_s - T_x} \]

The Nusselt number, which is the dimensionless heat transfer coefficient, is defined as

\[ \text{Nu} = \frac{hL_s}{k} \]

where \( k \) is the thermal conductivity of the fluid and \( L_s \) is the characteristic length.

The highly ordered fluid motion characterized by smooth streamlines is called laminar. The highly disordered fluid motion is called turbulent. The random and rapid fluctuations of groups of fluid particles, called eddies, provide an additional mechanism for momentum and heat transfer.

The region of the flow above the plate bounded by \( \delta \) in which the effects of the viscous shearing forces caused by fluid viscosity are felt is called the velocity boundary layer. The boundary layer thickness, \( \delta \), is defined as the distance from the surface at which \( u = 0.99u_{\text{in}} \). The hypothetical line of \( u = 0.99u_{\text{in}} \) divides the flow over a plate into the boundary layer region in which the viscous effects and the velocity changes are significant, and the inviscid flow region, in which the frictional effects are negligible.

The friction force per unit area is called the shear stress, and the shear stress at the wall surface is expressed as

\[ \tau_s = \mu \left. \frac{\partial u}{\partial y} \right|_{y=0} \quad \text{or} \quad \tau_s = C_f \frac{\rho V^2}{2} \]

where \( \mu \) is the dynamic viscosity, \( V \) is the upstream velocity, and \( C_f \) is the dimensionless friction coefficient. The property \( \nu = \mu/\rho \) is the kinematic viscosity. The friction force over the entire surface is determined from

\[ F_f = C_f A_s \frac{\rho V^2}{2} \]

The flow region over the surface in which the temperature variation in the direction normal to the surface is significant is the thermal boundary layer. The thickness of the thermal boundary layer \( \delta_t \), at any location along the surface is the distance from the surface at which the temperature difference \( T - T_s \) equals 0.99(\( T_x - T_s \)). The relative thickness of the velocity and the thermal boundary layers is best described by the dimensionless Prandtl number, defined as

\[ \text{Pr} = \frac{\text{Molecular diffusivity of momentum}}{\text{Molecular diffusivity of heat}} = \frac{\nu}{\alpha} = \frac{\mu C_p}{k} \]

For external flow, the dimensionless Reynolds number is expressed as

\[ \text{Re} = \frac{\text{Inertia forces}}{\text{Viscous forces}} = \frac{\nu L_s}{\mu} \]

For a flat plate, the characteristic length is the distance \( x \) from the leading edge. The Reynolds number at which the flow becomes turbulent is called the critical Reynolds number. For flow over a flat plate, its value is taken to be \( \text{Re}_{\text{cr}} = \frac{y_{\text{in}}}{v} = 5 \times 10^5 \).

The continuity, momentum, and energy equations for steady two-dimensional incompressible flow with constant properties are determined from mass, momentum, and energy balances to be

**Continuity:**

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]

**x-momentum:**

\[ \rho \left( \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \mu \frac{\partial^2 u}{\partial y^2} - \frac{\partial P}{\partial x} \]

**Energy:**

\[ \rho C_p \left( \frac{\partial u}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \mu \Phi \]

where the viscous dissipation function \( \Phi \) is

\[ \Phi = 2 \left( \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 \right) + \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \]

Using the boundary layer approximations and a similarity variable, these equations can be solved for parallel steady incompressible flow over a flat plate, with the following results:

**Velocity boundary layer thickness:**

\[ \delta = \frac{5.0}{\sqrt{Pr^{1/3}}} \]

**Local friction coefficient:**

\[ C_{f,x} = \frac{\tau_w}{\rho V^2/2} = 0.664 \text{Re}_{x}^{-1/2} \]

**Local Nusselt number:**

\[ \text{Nu}_x = \frac{h_s x}{k} = 0.332 \text{Pr}^{1/3} \text{Re}_{x}^{1/2} \]

**Thermal boundary layer thickness:**

\[ \delta_t = \frac{\delta}{\text{Pr}^{1/3}} = \frac{5.0}{\text{Pr}^{1/3} \sqrt{\text{Re}_{x}}} \]

The average friction coefficient and Nusselt number are expressed in functional form as
\[ C_f = f_d(Re_d) \quad \text{and} \quad Nu = g_d(Re_d, Pr) \]

The Nusselt number can be expressed by a simple power-law relation of the form

\[ Nu = C Re_d^m Pr^n \]

where \( m \) and \( n \) are constant exponents, and the value of the constant \( C \) depends on geometry. The Reynolds analogy relates the convection coefficient to the friction coefficient for fluids with \( Pr \approx 1 \), and is expressed as

\[ \frac{C_f}{2} = \frac{Nu}{Re} \quad \text{or} \quad \frac{C_f}{2} = St \]

where

\[ St = \frac{h}{\rho C_v V} = \frac{Nu}{Re D Pr} \]

**REFERENCES AND SUGGESTED READING**


**PROBLEMS**

**Mechanism and Types of Convection**

6–1C What is forced convection? How does it differ from natural convection? Is convection caused by winds forced or natural convection?

6–2C What is external forced convection? How does it differ from internal forced convection? Can a heat transfer system involve both internal and external convection at the same time? Give an example.

*Problems designated by a “C” are concept questions, and students are encouraged to answer them all. Problems designated by an “E” are in English units, and the SI users can ignore them. Problems with an EES-CD icon are solved using EES, and complete solutions together with parametric studies are included on the enclosed CD. Problems with a computer-EES icon are comprehensive in nature, and are intended to be solved with a computer, preferably using the EES software that accompanies this text.*
6–3C In which mode of heat transfer is the convection heat transfer coefficient usually higher, natural convection or forced convection? Why?

6–4C Consider a hot baked potato. Will the potato cool faster or slower when we blow the warm air coming from our lungs on it instead of letting it cool naturally in the cooler air in the room? Explain.

6–5C What is the physical significance of the Nusselt number? How is it defined?

6–6C When is heat transfer through a fluid conduction and when is it convection? For what case is the rate of heat transfer higher? How does the convection heat transfer coefficient differ from the thermal conductivity of a fluid?

6–7C Define incompressible flow and incompressible fluid. Must the flow of a compressible fluid necessarily be treated as compressible?

6–8 During air cooling of potatoes, the heat transfer coefficient for combined convection, radiation, and evaporation is determined experimentally to be as shown:

<table>
<thead>
<tr>
<th>Air Velocity, m/s</th>
<th>Heat Transfer Coefficient, W/m² · °C</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.66</td>
<td>14.0</td>
</tr>
<tr>
<td>1.00</td>
<td>19.1</td>
</tr>
<tr>
<td>1.36</td>
<td>20.2</td>
</tr>
<tr>
<td>1.73</td>
<td>24.4</td>
</tr>
</tbody>
</table>

Consider a 10-cm-diameter potato initially at 20°C with a thermal conductivity of 0.49 W/m · °C. Potatoes are cooled by refrigerated air at 5°C at a velocity of 1 m/s. Determine the initial rate of heat transfer from a potato, and the initial value of the temperature gradient in the potato at the surface.

Answers: 9.0 W/m², –585°C/m

6–9 An average man has a body surface area of 1.8 m² and a skin temperature of 33°C. The convection heat transfer coefficient for a clothed person walking in still air is expressed as

\[ h = 8.6 \sqrt{v} \]  

for 0.5 < \( v \) < 2 m/s, where \( v \) is the walking velocity in m/s. Assuming the average surface temperature of the clothed person to be 30°C, determine the rate of heat loss from an average man walking in still air at 10°C by convection at a walking velocity of (a) 0.5 m/s, (b) 1.0 m/s, (c) 1.5 m/s, and (d) 2.0 m/s.

6–10 The convection heat transfer coefficient for a clothed person standing in moving air is expressed as

\[ h = 14.8 \sqrt{v} \]  

for 0.15 < \( v \) < 1.5 m/s, where \( v \) is the air velocity. For a person with a body surface area of 1.7 m² and an average surface temperature of 29°C, determine the rate of heat loss from the person in windy air at 10°C by convection for air velocities of (a) 0.5 m/s, (b) 1.0 m/s, and (c) 1.5 m/s.

6–11 During air cooling of oranges, grapefruit, and tangelos, the heat transfer coefficient for combined convection, radiation, and evaporation for air velocities of 0.11 < \( v \) < 0.33 m/s is determined experimentally and is expressed as

\[ h = 5.05 \sqrt{v} \cdot \frac{k_w \cdot Re^{1/3}}{D} \]  

where the diameter \( D \) is the characteristic length. Oranges are cooled by refrigerated air at 5°C and 1 atm at a velocity of 0.5 m/s. Determine (a) the initial rate of heat transfer from a 7-cm-diameter orange initially at 15°C with a thermal conductivity of 0.50 W/m · °C, (b) the value of the initial temperature gradient inside the orange at the surface, and (c) the value of the Nusselt number.

**Velocity and Thermal Boundary Layers**

6–12C What is viscosity? What causes viscosity in liquids and in gases? Is dynamic viscosity typically higher for a liquid or for a gas?

6–13C What is Newtonian fluid? Is water a Newtonian fluid?

6–14C What is the no-slip condition? What causes it?

6–15C Consider two identical small glass balls dropped into two identical containers, one filled with water and the other with oil. Which ball will reach the bottom of the container first? Why?

6–16C How does the dynamic viscosity of (a) liquids and (b) gases vary with temperature?

6–17C What fluid property is responsible for the development of the velocity boundary layer? For what kind of fluids will there be no velocity boundary layer on a flat plate?

6–18C What is the physical significance of the Prandtl number? Does the value of the Prandtl number depend on the type of flow or the flow geometry? Does the Prandtl number of air change with pressure? Does it change with temperature?

6–19C Will a thermal boundary layer develop in flow over a surface even if both the fluid and the surface are at the same temperature?

**Laminar and Turbulent Flows**

6–20C How does turbulent flow differ from laminar flow? For which flow is the heat transfer coefficient higher?

6–21C What is the physical significance of the Reynolds number? How is it defined for external flow over a plate of length \( L \)?

6–22C What does the friction coefficient represent in flow over a flat plate? How is it related to the drag force acting on the plate?
Convection Equations and Similarity Solutions

6–26C Under what conditions can a curved surface be treated as a flat plate in fluid flow and convection analysis?

6–27C Express continuity equation for steady two-dimensional flow with constant properties, and explain what each term represents.

6–28C Is the acceleration of a fluid particle necessarily zero in steady flow? Explain.

6–29C For steady two-dimensional flow, what are the boundary layer approximations?

6–30C For what types of fluids and flows is the viscous dissipation term in the energy equation likely to be significant?

6–31C For steady two-dimensional flow over an isothermal flat plate in the x-direction, express the boundary conditions for the velocity components $u$ and $v$, and the temperature $T$ at the plate surface and at the edge of the boundary layer.

6–32C What is a similarity variable, and what is it used for? For what kinds of functions can we expect a similarity solution for a set of partial differential equations to exist?

6–33C Consider steady, laminar, two-dimensional flow over an isothermal plate. Does the thickness of the velocity boundary layer increase or decrease with (a) distance from the leading edge, (b) free-stream velocity, and (c) kinematic viscosity?

6–34C Consider steady, laminar, two-dimensional flow over an isothermal plate. Does the wall shear stress increase, decrease, or remain constant with distance from the leading edge?

6–35C What are the advantages of nondimensionalizing the convection equations?

6–36C Consider steady, laminar, two-dimensional, incompressible flow with constant properties and a Prandtl number of unity. For a given geometry, is it correct to say that both the average friction and heat transfer coefficients depend on the Reynolds number only?

6–37 Oil flow in a journal bearing can be treated as parallel flow between two large isothermal plates with one plate moving at a constant velocity of 12 m/s and the other stationary. Consider such a flow with a uniform spacing of 0.7 mm between the plates. The temperatures of the upper and lower plates are 40°C and 15°C, respectively. By simplifying and solving the continuity, momentum, and energy equations, determine (a) the velocity and temperature distributions in the oil, (b) the maximum temperature and where it occurs, and (c) the heat flux from the oil to each plate.

6–38 Repeat Problem 6–37 for a spacing of 0.4 mm.

6–39 A 6-cm-diameter shaft rotates at 3000 rpm in a 20-cm-long bearing with a uniform clearance of 0.2 mm. At steady operating conditions, both the bearing and the shaft in the vicinity of the oil gap are at 50°C, and the viscosity and thermal conductivity of lubricating oil are 0.05 N · s/m² and 0.17 W/m · K. By simplifying and solving the continuity, momentum, and energy equations, determine (a) the maximum temperature of the oil, (b) the rates of heat transfer to the bearing and the shaft, and (c) the mechanical power wasted by the viscous dissipation in the oil. Answers: (a) 53.3°C, (b) 419 W, (c) 838 W

6–40 Repeat Problem 6–39 by assuming the shaft to have reached peak temperature and thus heat transfer to the shaft to be negligible, and the bearing surface still to be maintained at 50°C.

6–41 Reconsider Problem 6–39. Using EES (or other) software, investigate the effect of shaft velocity on the mechanical power wasted by viscous dissipation. Let the shaft rotation vary from 0 rpm to 5000 rpm. Plot the power wasted versus the shaft rpm, and discuss the results.

6–42 Consider a 5-cm-diameter shaft rotating at 2500 rpm in a 10-cm-long bearing with a clearance of 0.5 mm. Determine the power required to rotate the shaft if the fluid in the gap is (a) air, (b) water, and (c) oil at 40°C and 1 atm.

6–43 Consider the flow of fluid between two large parallel isothermal plates separated by a distance $L$. The upper plate is moving at a constant velocity of $V$ and maintained at temperature $T_u$ while the lower plate is stationary and insulated. By simplifying and solving the continuity, momentum, and energy equations, obtain relations for the maximum temperature of fluid, the location where it occurs, and heat flux at the upper plate.

6–44 Reconsider Problem 6–43. Using the results of this problem, obtain a relation for the volumetric heat generation rate $g$, in W/m³. Then express the convection problem as an equivalent
conduction problem in the oil layer. Verify your model by solving the conduction problem and obtaining a relation for the maximum temperature, which should be identical to the one obtained in the convection analysis.

6–45 A 5-cm-diameter shaft rotates at 4500 rpm in a 15-cm-long, 8-cm-outer-diameter cast iron bearing \( (k = 70 \text{ W/m} \cdot \text{K}) \) with a uniform clearance of 0.6 mm filled with lubricating oil \( (\mu = 0.03 \text{ N} \cdot \text{s/m}^2 \text{ and } k = 0.14 \text{ W/m} \cdot \text{K}) \). The bearing is cooled externally by a liquid, and its outer surface is maintained at 40°C. Disregarding heat conduction through the shaft and assuming one-dimensional heat transfer, determine (a) the rate of heat transfer to the coolant, (b) the surface temperature of the shaft, and (c) the mechanical power wasted by the viscous dissipation in oil.

![Diagram of a shaft and bearing](image.png)

**FIGURE P6–45**

6–46 Repeat Problem 6–45 for a clearance of 1 mm.

### Momentum and Heat Transfer Analogies

6–47C How is Reynolds analogy expressed? What is the value of it? What are its limitations?

6–48C How is the modified Reynolds analogy expressed? What is the value of it? What are its limitations?

6–49 A 4-m × 4-m flat plate maintained at a constant temperature of 80°C is subjected to parallel flow of air at 1 atm, 20°C, and 10 m/s. The total drag force acting on the upper surface of the plate is measured to be 2.4 N. Using momentum-heat transfer analogy, determine the average convection heat transfer coefficient, and the rate of heat transfer between the upper surface of the plate and the air.

6–50 A metallic airfoil of elliptical cross section has a mass of 50 kg, surface area of 12 m², and a specific heat of 0.50 kJ/kg · °C. The airfoil is subjected to air flow at 1 atm, 25°C, and 8 m/s along its 3-m-long side. The average temperature of the airfoil is observed to drop from 160°C to 150°C within 2 min of cooling. Assuming the surface temperature of the airfoil to be equal to its average temperature and using momentum-heat transfer analogy, determine the average friction coefficient of the airfoil surface. **Answer: 0.000227**

6–51 Repeat Problem 6–50 for an air-flow velocity of 12 m/s.

6–52 The electrically heated 0.6-m-high and 1.8-m-long windshield of a car is subjected to parallel winds at 1 atm, 0°C, and 80 km/h. The electric power consumption is observed to be 50 W when the exposed surface temperature of the windshield is 4°C. Disregarding radiation and heat transfer from the inner surface and using the momentum-heat transfer analogy, determine drag force the wind exerts on the windshield.

6–53 Consider an airplane cruising at an altitude of 10 km where standard atmospheric conditions are −50°C and 26.5 kPa at a speed of 800 km/h. Each wing of the airplane can be modeled as a 25-m × 3-m flat plate, and the friction coefficient of the wings is 0.0016. Using the momentum-heat transfer analogy, determine the heat transfer coefficient for the wings at cruising conditions. **Answer: 89.6 W/m² · °C**

### Design and Essay Problems

6–54 Design an experiment to measure the viscosity of liquids using a vertical funnel with a cylindrical reservoir of height \( h \) and a narrow flow section of diameter \( D \) and length \( L \). Making appropriate assumptions, obtain a relation for viscosity in terms of easily measurable quantities such as density and volume flow rate.

6–55 A facility is equipped with a wind tunnel, and can measure the friction coefficient for flat surfaces and airfoils. Design an experiment to determine the mean heat transfer coefficient for a surface using friction coefficient data.
In Chapter 6 we considered the general and theoretical aspects of forced convection, with emphasis on differential formulation and analytical solutions. In this chapter we consider the practical aspects of forced convection to or from flat or curved surfaces subjected to *external flow*, characterized by the freely growing boundary layers surrounded by a free flow region that involves no velocity and temperature gradients.

We start this chapter with an overview of external flow, with emphasis on friction and pressure drag, flow separation, and the evaluation of average drag and convection coefficients. We continue with *parallel flow over flat plates*. In Chapter 6, we solved the boundary layer equations for steady, laminar, parallel flow over a flat plate, and obtained relations for the local friction coefficient and the Nusselt number. Using these relations as the starting point, we determine the average friction coefficient and Nusselt number. We then extend the analysis to turbulent flow over flat plates with and without an unheated starting length.

Next we consider *cross flow over cylinders and spheres*, and present graphs and empirical correlations for the drag coefficients and the Nusselt numbers, and discuss their significance. Finally, we consider *cross flow over tube banks* in aligned and staggered configurations, and present correlations for the pressure drop and the average Nusselt number for both configurations.
Fluid flow over solid bodies frequently occurs in practice, and it is responsible for numerous physical phenomena such as the drag force acting on the automobiles, power lines, trees, and underwater pipelines; the lift developed by airplane wings; upward draft of rain, snow, hail, and dust particles in high winds; and the cooling of metal or plastic sheets, steam and hot water pipes, and extruded wires. Therefore, developing a good understanding of external flow and external forced convection is important in the mechanical and thermal design of many engineering systems such as aircraft, automobiles, buildings, electronic components, and turbine blades.

The flow fields and geometries for most external flow problems are too complicated to be solved analytically, and thus we have to rely on correlations based on experimental data. The availability of high-speed computers has made it possible to conduct series of “numerical experimentations” quickly by solving the governing equations numerically, and to resort to the expensive and time-consuming testing and experimentation only in the final stages of design. In this chapter we will mostly rely on relations developed experimentally.

The velocity of the fluid relative to an immersed solid body sufficiently far from the body (outside the boundary layer) is called the free-stream velocity, and is denoted by \( u_* \). It is usually taken to be equal to the upstream velocity \( V \), also called the approach velocity, which is the velocity of the approaching fluid far ahead of the body. This idealization is nearly exact for very thin bodies, such as a flat plate parallel to flow, but approximate for blunt bodies such as a large cylinder. The fluid velocity ranges from zero at the surface (no-slip condition) to the free-stream value away from the surface, and the subscript “infinity” serves as a reminder that this is the value at a distance where the presence of the body is not felt. The upstream velocity, in general, may vary with location and time (e.g., the wind blowing past a building). But in the design and analysis, the upstream velocity is usually assumed to be uniform and steady for convenience, and this is what we will do in this chapter.

**Friction and Pressure Drag**

You may have seen high winds knocking down trees, power lines, and even trailers, and have felt the strong “push” the wind exerts on your body. You experience the same feeling when you extend your arm out of the window of a moving car. The force a flowing fluid exerts on a body in the flow direction is called drag (Fig. 7–1).

A stationary fluid exerts only normal pressure forces on the surface of a body immersed in it. A moving fluid, however, also exerts tangential shear forces on the surface because of the no-slip condition caused by viscous effects. Both of these forces, in general, have components in the direction of flow, and thus the drag force is due to the combined effects of pressure and wall shear forces in the flow direction. The components of the pressure and wall shear forces in the normal direction to flow tend to move the body in that direction, and their sum is called lift.

In general, both the skin friction (wall shear) and pressure contribute to the drag and the lift. In the special case of a thin flat plate aligned parallel to the flow direction, the drag force depends on the wall shear only and is
independent of pressure. When the flat plate is placed normal to the flow direction, however, the drag force depends on the pressure only and is independent of the wall shear since the shear stress in this case acts in the direction normal to flow (Fig. 7–2). For slender bodies such as wings, the shear force acts nearly parallel to the flow direction. The drag force for such slender bodies is mostly due to shear forces (the skin friction).

The drag force depends on the density \( \rho \) of the fluid, the upstream velocity \( V \), and the size, shape, and orientation of the body, among other things. The drag characteristics of a body is represented by the dimensionless drag coefficient \( C_D \) defined as

\[
C_D = \frac{F_D}{\frac{1}{2} \rho V^2 A}
\]

(7-1)

where \( A \) is the frontal area (the area projected on a plane normal to the direction of flow) for blunt bodies—bodies that tends to block the flow. The frontal area of a cylinder of diameter \( D \) and length \( L \), for example, is \( A = LD \). For parallel flow over flat plates or thin airfoils, \( A \) is the surface area. The drag coefficient is primarily a function of the shape of the body, but it may also depend on the Reynolds number and the surface roughness.

The drag force is the net force exerted by a fluid on a body in the direction of flow due to the combined effects of wall shear and pressure forces. The part of drag that is due directly to wall shear stress \( \tau_w \) is called the skin friction drag (or just friction drag) since it is caused by frictional effects, and the part that is due directly to pressure \( P \) is called the pressure drag (also called the form drag because of its strong dependence on the form or shape of the body). When the friction and pressure drag coefficients are available, the total drag coefficient is determined by simply adding them,

\[
C_D = C_{D, \text{friction}} + C_{D, \text{pressure}}
\]

(7-2)

The friction drag is the component of the wall shear force in the direction of flow, and thus it depends on the orientation of the body as well as the magnitude of the wall shear stress \( \tau_w \). The friction drag is zero for a surface normal to flow, and maximum for a surface parallel to flow since the friction drag in this case equals the total shear force on the surface. Therefore, for parallel flow over a flat plate, the drag coefficient is equal to the friction coefficient (Fig. 7–3). That is,

\[
C_D = C_{D, \text{friction}} = C_f
\]

(7-3)

Once the average friction coefficient \( C_f \) is available, the drag (or friction) force over the surface can be determined from Eq. 7-1. In this case \( A \) is the surface area of the plate exposed to fluid flow. When both sides of a thin plate are subjected to flow, \( A \) becomes the total area of the top and bottom surfaces. Note that the friction coefficient, in general, will vary with location along the surface.

Friction drag is a strong function of viscosity, and an “idealized” fluid with zero viscosity would produce zero friction drag since the wall shear stress would be zero (Fig. 7–4). The pressure drag would also be zero in this case during steady flow regardless of the shape of the body since there will be no pressure losses. For flow in the horizontal direction, for example, the pressure along a horizontal line will be constant (just like stationary fluids) since the

\[
F_D = \frac{1}{2} \rho V^2 A C_D
\]
upstream velocity is constant, and thus there will be no net pressure force acting on the body in the horizontal direction. Therefore, the total drag is zero for the case of ideal inviscid fluid flow.

At low Reynolds numbers, most drag is due to friction drag. This is especially the case for highly streamlined bodies such as airfoils. The friction drag is also proportional to the surface area. Therefore, bodies with a larger surface area will experience a larger friction drag. Large commercial airplanes, for example, reduce their total surface area and thus drag by retracting their wing extensions when they reach the cruising altitudes to save fuel. The friction drag coefficient is independent of surface roughness in laminar flow, but is a strong function of surface roughness in turbulent flow due to surface roughness elements protruding further into the highly viscous laminar sublayer.

The pressure drag is proportional to the difference between the pressures acting on the front and back of the immersed body, and the frontal area. Therefore, the pressure drag is usually dominant for blunt bodies, negligible for streamlined bodies such as airfoils, and zero for thin flat plates parallel to the flow.

When a fluid is forced to flow over a curved surface at sufficiently high velocities, it will detach itself from the surface of the body. The low-pressure region behind the body where recirculating and back flows occur is called the separation region. The larger the separation area is, the larger the pressure drag will be. The effects of flow separation are felt far downstream in the form of reduced velocity (relative to the upstream velocity). The region of flow trailing the body where the effect of the body on velocity is felt is called the wake (Fig. 7–5). The separated region comes to an end when the two flow streams reattach, but the wake keeps growing behind the body until the fluid in the wake region regains its velocity. The viscous effects are the most significant in the boundary layer, the separated region, and the wake. The flow outside these regions can be considered to be inviscid.

**Heat Transfer**

The phenomena that affect drag force also affect heat transfer, and this effect appears in the Nusselt number. By nondimensionalizing the boundary layer equations, it was shown in Chapter 6 that the local and average Nusselt numbers have the functional form

\[
\text{Nu}_x = f_1(x^*, \text{Re}_x, \text{Pr}) \quad \text{and} \quad \text{Nu} = f_2(\text{Re}_L, \text{Pr})
\]

(7-4a, b)

The experimental data for heat transfer is often represented conveniently with reasonable accuracy by a simple power-law relation of the form

\[
\text{Nu} = C \text{Re}^m \text{Pr}^n
\]

(7-5)

where \(m\) and \(n\) are constant exponents, and the value of the constant \(C\) depends on geometry and flow.

The fluid temperature in the thermal boundary layer varies from \(T_s\) at the surface to about \(T_w\) at the outer edge of the boundary. The fluid properties also vary with temperature, and thus with position across the boundary layer. In order to account for the variation of the properties with temperature, the fluid properties are usually evaluated at the so-called film temperature, defined as
which is the arithmetic average of the surface and the free-stream temperatures. The fluid properties are then assumed to remain constant at those values during the entire flow. An alternative way of accounting for the variation of properties with temperature is to evaluate all properties at the free stream temperature and to multiply the Nusselt number relation in Eq. 7-5 by \( \frac{Pr_s}{Pr_s^r} \) or \( \frac{\mu_s}{\mu_s^r} \).

The local drag and convection coefficients vary along the surface as a result of the changes in the velocity boundary layers in the flow direction. We are usually interested in the drag force and the heat transfer rate for the entire surface, which can be determined using the average friction and convection coefficient. Therefore, we present correlations for both local (identified with the subscript \( x \)) and average friction and convection coefficients. When relations for local friction and convection coefficients are available, the average friction and convection coefficients for the entire surface can be determined by integration from

\[
C_D = \frac{1}{L} \int_0^L C_{D,x} \, dx \quad (7-7)
\]

and

\[
h = \frac{1}{L} \int_0^L h_x \, dx \quad (7-8)
\]

When the average drag and convection coefficients are available, the drag force can be determined from Eq. 7-1 and the rate of heat transfer to or from an isothermal surface can be determined from

\[
\dot{Q} = hA_s(T_s - T_*) \quad (7-9)
\]

where \( A_s \) is the surface area.

7–2 • PARALLEL FLOW OVER FLAT PLATES

Consider the parallel flow of a fluid over a flat plate of length \( L \) in the flow direction, as shown in Figure 7–6. The \( x \)-coordinate is measured along the plate surface from the leading edge in the direction of the flow. The fluid approaches the plate in the \( x \)-direction with uniform upstream velocity \( V \) and temperature \( T_* \). The flow in the velocity boundary layer starts out as laminar, but if the plate is sufficiently long, the flow will become turbulent at a distance \( x_{cr} \) from the leading edge where the Reynolds number reaches its critical value for transition.

The transition from laminar to turbulent flow depends on the surface geometry, surface roughness, upstream velocity, surface temperature, and the type of fluid, among other things, and is best characterized by the Reynolds number. The Reynolds number at a distance \( x \) from the leading edge of a flat plate is expressed as

\[
T_f = \frac{T_s + T_*}{2}
\]
Note that the value of the Reynolds number varies for a flat plate along the flow, reaching $Re = \frac{\nu L}{v}$ at the end of the plate.

For flow over a flat plate, transition from laminar to turbulent is usually taken to occur at the critical Reynolds number of

$$Re_{cr} = \frac{\nu x_{cr}}{v} = 5 \times 10^5$$

(7-11)

The value of the critical Reynolds number for a flat plate may vary from $10^5$ to $3 \times 10^6$, depending on the surface roughness and the turbulence level of the free stream.

**Friction Coefficient**

Based on analysis, the boundary layer thickness and the local friction coefficient at location $x$ for laminar flow over a flat plate were determined in Chapter 6 to be

**Laminar:**

$$\delta_{y,x} = \frac{5x}{Re^{1/2}} \quad \text{and} \quad C_{f,x} = \frac{0.664}{Re^{1/2}}, \quad Re < 5 \times 10^5 \quad (7-12a, b)$$

The corresponding relations for turbulent flow are

**Turbulent:**

$$\delta_{y,x} = \frac{0.382x}{Re^{1/5}} \quad \text{and} \quad C_{f,x} = \frac{0.0592}{Re^{1/5}}, \quad 5 \times 10^5 \leq Re \leq 10^7 \quad (7-13a, b)$$

where $x$ is the distance from the leading edge of the plate and $Re = \nu x/v$ is the Reynolds number at location $x$. Note that $C_{f,x}$ is proportional to $Re^{-1/2}$ and thus to $x^{-1/2}$ for laminar flow. Therefore, $C_{f,x}$ is supposedly infinite at the leading edge ($x = 0$) and decreases by a factor of $x^{-1/2}$ in the flow direction. The local friction coefficients are higher in turbulent flow than they are in laminar flow because of the intense mixing that occurs in the turbulent boundary layer. Note that $C_{f,x}$ reaches its highest values when the flow becomes fully turbulent, and then decreases by a factor of $x^{-1/5}$ in the flow direction.

The average friction coefficient over the entire plate is determined by substituting the relations above into Eq. 7-7 and performing the integrations (Fig. 7–7). We get

**Laminar:**

$$C_f = \frac{1.328}{Re^{1/2}} \quad Re < 5 \times 10^5 \quad (7-14)$$

**Turbulent:**

$$C_f = \frac{0.074}{Re^{1/5}} \quad 5 \times 10^5 \leq Re \leq 10^7 \quad (7-15)$$

The first relation gives the average friction coefficient for the entire plate when the flow is laminar over the entire plate. The second relation gives the average friction coefficient for the entire plate only when the flow is turbulent over the entire plate, or when the laminar flow region of the plate is too small relative to the turbulent flow region (that is, $x_{cr} \ll L$ where the length of the plate $x_{cr}$ over which the flow is laminar can be determined from $Re_{cr} = 5 \times 10^5 = \nu x_{cr}/v$).
In some cases, a flat plate is sufficiently long for the flow to become turbulent, but not long enough to disregard the laminar flow region. In such cases, the average friction coefficient over the entire plate is determined by performing the integration in Eq. 7-7 over two parts: the laminar region \( 0 \leq x \leq x_{tr} \) and the turbulent region \( x_{tr} < x \leq L \) as

\[
C_f = \frac{1}{L} \left( \int_0^{x_{tr}} C_{f,x,\text{laminar}} \, dx + \int_{x_{tr}}^L C_{f,x,\text{turbulent}} \, dx \right) \quad (7-16)
\]

Note that we included the transition region with the turbulent region. Again taking the critical Reynolds number to be \( \text{Re}_{cr} = 5 \times 10^5 \) and performing the integrations of Eq. 7-16 after substituting the indicated expressions, the average friction coefficient over the entire plate is determined to be

\[
C_f = \frac{0.074}{\text{Re}_L^{1/5}} - \frac{1742}{\text{Re}_L} \quad 5 \times 10^5 \leq \text{Re}_L \leq 10^7 \quad (7-17)
\]

The constants in this relation will be different for different critical Reynolds numbers. Also, the surfaces are assumed to be smooth, and the free stream to be turbulent free. For laminar flow, the friction coefficient depends on only the Reynolds number, and the surface roughness has no effect. For turbulent flow, however, surface roughness causes the friction coefficient to increase severalfold, to the point that in fully turbulent regime the friction coefficient is a function of surface roughness alone, and independent of the Reynolds number (Fig. 7–8).

A curve fit of experimental data for the average friction coefficient in this regime is given by Schlichting as

\[
Rough \text{ surface, turbulent:} \quad C_f = \left( 1.89 - 1.62 \log \frac{\varepsilon}{L} \right)^{-2.5} \quad (7-18)
\]

where \( \varepsilon \) is the surface roughness, and \( L \) is the length of the plate in the flow direction. In the absence of a better relation, the relation above can be used for turbulent flow on rough surfaces for \( \text{Re} > 10^6 \), especially when \( \varepsilon/L > 10^{-4} \).

### Heat Transfer Coefficient

The local Nusselt number at a location \( x \) for laminar flow over a flat plate was determined in Chapter 6 by solving the differential energy equation to be

\[
\text{Laminar:} \quad \text{Nu}_x = \frac{h_x x}{k} = 0.332 \text{Re}_x^{0.5} \text{Pr}^{1/3} \quad \text{Pr} > 0.60 \quad (7-19)
\]

The corresponding relation for turbulent flow is

\[
\text{Turbulent:} \quad \text{Nu}_x = \frac{h_x x}{k} = 0.0296 \text{Re}_x^{0.8} \text{Pr}^{1/3} \quad 0.6 \leq \text{Pr} \leq 60 \quad 5 \times 10^3 \leq \text{Re}_x \leq 10^7 \quad (7-20)
\]

Note that \( h_x \) is proportional to \( \text{Re}_x^{0.5} \) and thus to \( x^{-0.5} \) for laminar flow. Therefore, \( h_x \) is infinite at the leading edge (\( x = 0 \)) and decreases by a factor of \( x^{-0.5} \) in the flow direction. The variation of the boundary layer thickness \( \delta \) and the friction and heat transfer coefficients along an isothermal flat plate are shown in Figure 7–9. The local friction and heat transfer coefficients are higher in
turbulent flow than they are in laminar flow. Also, \( h_x \) reaches its highest values when the flow becomes fully turbulent, and then decreases by a factor of \( x^{-0.2} \) in the flow direction, as shown in the figure.

The average Nusselt number over the entire plate is determined by substituting the relations above into Eq. 7-8 and performing the integrations. We get

\[
\text{Laminar:} \quad \frac{Nu}{k} = 0.664 \frac{Re_{L}^{0.5} Pr^{1/3}}{Re_L < 5 \times 10^5} \tag{7-21}
\]

\[
\text{Turbulent:} \quad \frac{Nu}{k} = 0.037 \frac{Re_{L}^{0.8} Pr^{1/3}}{0.6 \leq Pr \leq 60, \quad 5 \times 10^5 \leq Re_L \leq 10^7} \tag{7-22}
\]

The first relation gives the average heat transfer coefficient for the entire plate when the flow is laminar over the entire plate. The second relation gives the average heat transfer coefficient for the entire plate only when the flow is turbulent over the entire plate, or when the laminar flow region of the plate is too small relative to the turbulent flow region.

In some cases, a flat plate is sufficiently long for the flow to become turbulent, but not long enough to disregard the laminar flow region. In such cases, the average heat transfer coefficient over the entire plate is determined by performing the integration in Eq. 7-8 over two parts as

\[
h = \frac{1}{L} \left( \int_0^{x_{cr}} h_x, \text{laminar} \, dx + \int_{x_{cr}}^L h_x, \text{turbulent} \, dx \right) \tag{7-23}
\]

Again taking the critical Reynolds number to be \( Re_{cr} = 5 \times 10^5 \) and performing the integrations in Eq. 7-23 after substituting the indicated expressions, the average Nusselt number over the entire plate is determined to be (Fig. 7–10)

\[
\frac{Nu}{k} = (0.037 Re_{L}^{0.8} - 871) Pr^{1/3} \quad 0.6 \leq Pr \leq 60, \quad 5 \times 10^5 \leq Re_L \leq 10^7 \tag{7-24}
\]

The constants in this relation will be different for different critical Reynolds numbers.

**Liquid metals** such as mercury have high thermal conductivities, and are commonly used in applications that require high heat transfer rates. However, they have very small Prandtl numbers, and thus the thermal boundary layer develops much faster than the velocity boundary layer. Then we can assume the velocity in the thermal boundary layer to be constant at the free stream value and solve the energy equation. It gives

\[
\frac{Nu}{k} = 0.565 (Re, Pr)^{1/2} \quad Pr < 0.05 \tag{7-25}
\]

It is desirable to have a single correlation that applies to all fluids, including liquid metals. By curve-fitting existing data, Churchill and Ozoe (Ref. 3) proposed the following relation which is applicable for all Prandtl numbers and is claimed to be accurate to \( \pm 1\% \),

\[
\frac{Nu}{k} = \frac{h_{x,cr}}{k} = \frac{0.3387 Pr^{1/3}}{1 + (0.0468/Pr)^{2/3}} Re_{L}^{1/2} \tag{7-26}
\]

These relations have been obtained for the case of isothermal surfaces but could also be used approximately for the case of nonisothermal surfaces by assuming the surface temperature to be constant at some average value.
Also, the surfaces are assumed to be smooth, and the free stream to be turbulent free. The effect of variable properties can be accounted for by evaluating all properties at the film temperature.

**Flat Plate with Unheated Starting Length**

So far we have limited our consideration to situations for which the entire plate is heated from the leading edge. But many practical applications involve surfaces with an unheated starting section of length $\xi$, shown in Figure 7-11, and thus there is no heat transfer for $0 < x < \xi$. In such cases, the velocity boundary layer starts to develop at the leading edge ($x = 0$), but the thermal boundary layer starts to develop where heating starts ($x = \xi$).

Consider a flat plate whose heated section is maintained at a constant temperature ($T = T_s$, constant for $x > \xi$). Using integral solution methods (see Kays and Crawford, 1994), the local Nusselt numbers for both laminar and turbulent flows are determined to be

\[
\text{Laminar: } \quad \text{Nu}_x = \frac{\text{Nu}_{x_{\text{for } \xi = 0}}}{[1 - (\xi/x)^{0.8}]^{1/3}} = \frac{0.332 \text{ Re}_{\xi}^{0.5} \text{ Pr}^{1/3}}{[1 - (\xi/x)^{0.8}]^{1/3}} \tag{7-27}
\]

\[
\text{Turbulent: } \quad \text{Nu}_x = \frac{\text{Nu}_{x_{\text{for } \xi = 0}}}{[1 - (\xi/x)^{0.8}]^{1/3}} = \frac{0.0296 \text{ Re}_{\xi}^{0.8} \text{ Pr}^{1/3}}{[1 - (\xi/x)^{0.8}]^{1/3}} \tag{7-28}
\]

for $x > \xi$. Note that for $\xi = 0$, these $\text{Nu}_x$ relations reduce to $\text{Nu}_{x_{\text{for } \xi = 0}}$, which is the Nusselt number relation for a flat plate without an unheated starting length. Therefore, the terms in brackets in the denominator serve as correction factors for plates with unheated starting lengths.

The determination of the average Nusselt number for the heated section of a plate requires the integration of the local Nusselt number relations above, which cannot be done analytically. Therefore, integrations must be done numerically. The results of numerical integrations have been correlated for the average convection coefficients [Thomas, (1977) Ref. 11] as

\[
\text{Laminar: } \quad h = \frac{2[1 - (\xi/x)^{0.8}]}{1 - \xi/L} h_{x=L} \tag{7-29}
\]

\[
\text{Turbulent: } \quad h = \frac{5[1 - (\xi/x)^{0.8}]}{4(1 - \xi/L)} h_{x=L} \tag{7-30}
\]

The first relation gives the average convection coefficient for the entire heated section of the plate when the flow is laminar over the entire plate. Note that for $\xi = 0$ it reduces to $h_{L} = 2h_{x=L}$, as expected. The second relation gives the average convection coefficient for the case of turbulent flow over the entire plate or when the laminar flow region is small relative to the turbulent region.

**Uniform Heat Flux**

When a flat plate is subjected to uniform heat flux instead of uniform temperature, the local Nusselt number is given by

\[
\text{Laminar: } \quad \text{Nu}_x = 0.453 \text{ Re}_{\xi}^{0.5} \text{ Pr}^{1/3} \tag{7-31}
\]

\[
\text{Turbulent: } \quad \text{Nu}_x = 0.0308 \text{ Re}_{\xi}^{0.8} \text{ Pr}^{1/3} \tag{7-32}
\]
These relations give values that are 36 percent higher for laminar flow and 4 percent higher for turbulent flow relative to the isothermal plate case. When the plate involves an unheated starting length, the relations developed for the uniform surface temperature case can still be used provided that Eqs. 7-31 and 7-32 are used for \( \text{Nu}_{\text{for } \eta = 0} \) in Eqs. 7-27 and 7-28, respectively.

When heat flux \( q_s \) is prescribed, the rate of heat transfer to or from the plate and the surface temperature at a distance \( x \) are determined from

\[
\dot{Q} = \dot{q}_s A
\]  

(7-33)

and

\[
\dot{q}_s = h_s [T_s(x) - T_w] \quad \rightarrow \quad T_s(x) = T_w + \frac{\dot{q}_s}{h_s}
\]  

(7-34)

where \( A_s \) is the heat transfer surface area.

---

**EXAMPLE 7-1 Flow of Hot Oil over a Flat Plate**

Engine oil at 60°C flows over the upper surface of a 5-m-long flat plate whose temperature is 20°C with a velocity of 2 m/s (Fig. 7–12). Determine the total drag force and the rate of heat transfer per unit width of the entire plate.

**SOLUTION** Engine oil flows over a flat plate. The total drag force and the rate of heat transfer per unit width of the plate are to be determined.

**Assumptions** 1 The flow is steady and incompressible. 2 The critical Reynolds number is \( \text{Re}_c = 5 \times 10^5 \).

**Properties** The properties of engine oil at the film temperature of \( T_f = (T_s + T_w)/2 = (20 + 60)/2 = 40°C \) are (Table A–14).

\[
\begin{align*}
\rho &= 876 \text{ kg/m}^3 & \text{Pr} &= 2870 \\
\frac{k}{\nu} &= 0.144 \text{ W/m} \cdot \degree \text{C} & \nu &= 242 \times 10^{-6} \text{ m}^2/\text{s}
\end{align*}
\]

**Analysis** Noting that \( L = 5 \text{ m} \), the Reynolds number at the end of the plate is

\[
\text{Re}_L = \frac{\nu L}{V} = \frac{(2 \text{ m/s})(5 \text{ m})}{0.242 \times 10^{-3} \text{ m}^2/\text{s}} = 4.13 \times 10^4
\]

which is less than the critical Reynolds number. Thus we have **laminar flow** over the entire plate, and the average friction coefficient is

\[
C_f = 1.328 \text{Re}_L^{-0.3} = 1.328 \times (4.13 \times 10^4)^{-0.3} = 0.0207
\]

Noting that the pressure drag is zero and thus \( C_D = C_f \) for a flat plate, the drag force acting on the plate per unit width becomes

\[
F_D = C_f A_s \frac{V^2}{2} = 0.0207 \times (5 \times 1 \text{ m}^2) (876 \text{ kg/m}^3)(2 \text{ m/s})^2 \left( \frac{1 \text{ N}}{1 \text{ kg} \cdot \text{m}^2/\text{s}^2} \right)
\]

\[
= 181 \text{ N}
\]

The total drag force acting on the entire plate can be determined by multiplying the value obtained above by the width of the plate.

This force per unit width corresponds to the weight of a mass of about 18 kg. Therefore, a person who applies an equal and opposite force to the plate to keep
it from moving will feel like he or she is using as much force as is necessary to hold a 18-kg mass from dropping.

Similarly, the Nusselt number is determined using the laminar flow relations for a flat plate,

\[
\frac{Nu}{H_{11005}} = 0.664 \frac{Re_{L}^{0.5} Pr^{1/3}}{H_{11003}} \times (4.13 \times 10^{4})^{0.5} \times 2870^{1/3} = 1918
\]

Then,

\[
\frac{h}{L} = \frac{k}{Nu} = \frac{0.144 \text{ W/m} \cdot \text{°C}}{5 \text{ m}} (1918) = 55.2 \text{ W/m}^{2} \cdot \text{°C}
\]

and

\[
\dot{Q} = hA_s(T_s - T_e) = (55.2 \text{ W/m}^{2} \cdot \text{°C})(5 \times 1 \text{ m}^{2})(60 - 20)\text{°C} = 11,040 \text{ W}
\]

Discussion  Note that heat transfer is always from the higher-temperature medium to the lower-temperature one. In this case, it is from the oil to the plate. The heat transfer rate is per m width of the plate. The heat transfer for the entire plate can be obtained by multiplying the value obtained by the actual width of the plate.

**EXAMPLE 7–2 Cooling of a Hot Block by Forced Air at High Elevation**

The local atmospheric pressure in Denver, Colorado (elevation 1610 m), is 83.4 kPa. Air at this pressure and 20°C flows with a velocity of 8 m/s over a 1.5 m × 6 m flat plate whose temperature is 140°C (Fig. 7-13). Determine the rate of heat transfer from the plate if the air flows parallel to the (a) 6-m-long side and (b) the 1.5-m side.

**SOLUTION**  The top surface of a hot block is to be cooled by forced air. The rate of heat transfer is to be determined for two cases.

**Assumptions**  1 Steady operating conditions exist. 2 The critical Reynolds number is \(Re_c = 5 \times 10^5\). 3 Radiation effects are negligible. 4 Air is an ideal gas.

**Properties**  The properties \(k\), \(\mu\), \(C_p\), and \(Pr\) of ideal gases are independent of pressure, while the properties \(\nu\) and \(\alpha\) are inversely proportional to density and thus pressure. The properties of air at the film temperature of \(T_f = (T_s + T_e)/2 = (140 + 20)/2 = 80°C\) and 1 atm pressure are (Table A–15)

\[
k = 0.02953 \text{ W/m} \cdot \text{°C} \quad \text{Pr} = 0.7154
\]

\[
\nu @ 1 \text{ atm} = 2.097 \times 10^{-5} \text{ m}^{2}/\text{s}
\]

The atmospheric pressure in Denver is \(P = (83.4 \text{ kPa})/(101.325 \text{ kPa/atm}) = 0.823\) atm. Then the kinematic viscosity of air in Denver becomes

\[
\nu = \nu @ 1 \text{ atm}/P = (2.097 \times 10^{-5} \text{ m}^{2}/\text{s})/0.823 = 2.548 \times 10^{-5} \text{ m}^{2}/\text{s}
\]

**Analysis**  (a) When air flow is parallel to the long side, we have \(L = 6\) m, and the Reynolds number at the end of the plate becomes

\[
Re_{L} = \frac{\nu L}{\nu} = \frac{(8 \text{ m/s})(6 \text{ m})}{2.548 \times 10^{-5} \text{ m}^{2}/\text{s}} = 1.884 \times 10^6
\]
which is greater than the critical Reynolds number. Thus, we have combined laminar and turbulent flow, and the average Nusselt number for the entire plate is determined to be

$$\text{Nu} = \frac{hL}{k} = \left[0.037 \times 10^6 \right]^{0.8} \times 871) \text{Pr}^{1/3}$$

$$= \left[0.037(1.884 \times 10^6 \right]^{0.8} \times 871)0.7154^{1/3}$$

$$= 2687$$

Then

$$h = \frac{k}{L} \text{Nu} = \frac{0.02953 \text{ W/m} \cdot \text{°C}}{6 \text{ m}} (2687) = 13.2 \text{ W/m}^2 \cdot \text{°C}$$

$$A_s = wL = (1.5 \text{ m})(6 \text{ m}) = 9 \text{ m}^2$$

and

$$\dot{Q} = hA_s(T_s - T_w) = (13.2 \text{ W/m}^2 \cdot \text{°C})(9 \text{ m}^2)(140 - 20)\text{°C} = 1.43 \times 10^4 \text{ W}$$

Note that if we disregarded the laminar region and assumed turbulent flow over the entire plate, we would get $\text{Nu} = 3466$ from Eq. 7–22, which is 29 percent higher than the value calculated above.

(b) When air flow is along the short side, we have $L = 1.5 \text{ m}$, and the Reynolds number at the end of the plate becomes

$$\text{Re}_L = \frac{wL}{v} = \frac{(8 \text{ m/s})(1.5 \text{ m})}{2.548 \times 10^{-5} \text{ m}^2/\text{s}} = 4.71 \times 10^5$$

which is less than the critical Reynolds number. Thus we have laminar flow over the entire plate, and the average Nusselt number is

$$\text{Nu} = \frac{hL}{k} = 0.664 \text{ Re}^{0.5} \text{ Pr}^{1/3} = 0.664 \times (4.71 \times 10^5)^{0.5} \times 0.7154^{1/3} = 408$$

Then

$$h = \frac{k}{L} \text{Nu} = \frac{0.02953 \text{ W/m} \cdot \text{°C}}{1.5 \text{ m}} (408) = 8.03 \text{ W/m}^2 \cdot \text{°C}$$

and

$$\dot{Q} = hA_s(T_s - T_w) = (8.03 \text{ W/m}^2 \cdot \text{°C})(9 \text{ m}^2)(140 - 20)\text{°C} = 8670 \text{ W}$$

which is considerably less than the heat transfer rate determined in case (a).

**Discussion**  Note that the *direction of fluid flow* can have a significant effect on convection heat transfer to or from a surface (Fig. 7-14). In this case, we can increase the heat transfer rate by 65 percent by simply blowing the air along the long side of the rectangular plate instead of the short side.

### EXAMPLE 7–3 Cooling of Plastic Sheets by Forced Air

The forming section of a plastics plant puts out a continuous sheet of plastic that is 4 ft wide and 0.04 in. thick at a velocity of 30 ft/min. The temperature of the plastic sheet is 200°F when it is exposed to the surrounding air, and a 2-ft-long section of the plastic sheet is subjected to air flow at 80°F at a velocity of 10 ft/s on both sides along its surfaces normal to the direction of motion.
of the sheet, as shown in Figure 7–15. Determine (a) the rate of heat transfer from the plastic sheet to air by forced convection and radiation and (b) the temperature of the plastic sheet at the end of the cooling section. Take the density, specific heat, and emissivity of the plastic sheet to be \( \rho = 75 \text{ lbm/ft}^3 \), \( C_p = 0.4 \text{ Btu/lbm} \cdot \text{°F} \), and \( \varepsilon = 0.9 \).

**SOLUTION** Plastic sheets are cooled as they leave the forming section of a plastics plant. The rate of heat loss from the plastic sheet by convection and radiation and the exit temperature of the plastic sheet are to be determined.

**Assumptions**
1. Steady operating conditions exist.
2. The critical Reynolds number is \( \text{Re}_{cr} = 5 \times 10^5 \).
3. Air is an ideal gas.
4. The local atmospheric pressure is 1 atm. 
5. The surrounding surfaces are at the temperature of the room air.

**Properties** The properties of the plastic sheet are given in the problem statement. The properties of air at the film temperature of \( T_f = (T_s + T_a)/2 = (200 + 80)/2 = 140\, ^\circ F \) and 1 atm pressure are (Table A–15E)

\[
\begin{align*}
    k &= 0.01623 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F} \quad \text{Pr} = 0.7202 \\
    \nu &= 0.7344 \text{ ft}^2/\text{h} = 0.204 \times 10^{-3} \text{ ft}^2/\text{s}
\end{align*}
\]

**Analysis** (a) We expect the temperature of the plastic sheet to drop somewhat as it flows through the 2-ft-long cooling section, but at this point we do not know the magnitude of that drop. Therefore, we assume the plastic sheet to be isothermal at 200°F to get started. We will repeat the calculations if necessary to account for the temperature drop of the plastic sheet.

Noting that \( L = 4 \text{ ft} \), the Reynolds number at the end of the air flow across the plastic sheet is

\[
\text{Re}_L = \frac{\gamma L}{\nu} = \frac{(10 \text{ ft/s})(4 \text{ ft})}{0.204 \times 10^{-3} \text{ ft}^2/\text{s}} = 1.961 \times 10^5
\]

which is less than the critical Reynolds number. Thus, we have *laminar flow* over the entire sheet, and the Nusselt number is determined from the laminar flow relations for a flat plate to be

\[
\text{Nu} = \frac{hL}{k} = 0.664 \text{ Re}^{0.5}_{L} \text{ Pr}^{1/3} = 0.664 \times (1.961 \times 10^5)^{0.5} \times (0.7202)^{1/3} = 263.6
\]

Then,

\[
\begin{align*}
    h &= \frac{kL}{\text{Nu}} = \frac{0.01623 \text{ Btu/h} \cdot \text{ft} \cdot \text{°F}}{4 \text{ ft}} (263.6) = 1.07 \text{ Btu/h} \cdot \text{ft}^2 \cdot \text{°F} \\
    A_j &= (2 \text{ ft})(4 \text{ ft})(2 \text{ sides}) = 16 \text{ ft}^2
\end{align*}
\]

and

\[
\begin{align*}
    \dot{Q}_{\text{conv}} &= hA_j(T_s - T_a) \\
    &= (1.07 \text{ Btu/h} \cdot \text{ft}^2 \cdot \text{°F})(16 \text{ ft}^2)(200 - 80)\text{°F} \\
    &= 2054 \text{ Btu/h}
\end{align*}
\]

\[
\begin{align*}
    \dot{Q}_{\text{rad}} &= \varepsilon\sigma A_j(T_s^4 - T_{\text{sur}}^4) \\
    &= (0.9)(0.1714 \times 10^{-8} \text{ Btu/h} \cdot \text{ft}^2 \cdot \text{R}^4)(16 \text{ ft}^2)[(660 \text{ R})^4 - (540 \text{ R})^4] \\
    &= 2584 \text{ Btu/h}
\end{align*}
\]
7–3 FLOW ACROSS CYLINDERS AND SPHERES

Flow across cylinders and spheres is frequently encountered in practice. For example, the tubes in a shell-and-tube heat exchanger involve both internal flow through the tubes and external flow over the tubes, and both flows must be considered in the analysis of the heat exchanger. Also, many sports such as soccer, tennis, and golf involve flow over spherical balls.

The characteristic length for a circular cylinder or sphere is taken to be the external diameter \( D \).

Thus, the Reynolds number is defined as

\[
\text{Re} = \frac{\rho V D}{\mu} = \frac{V D}{v}
\]

where \( V \) is the uniform velocity of the fluid as it approaches the cylinder or sphere. The critical Reynolds number for flow across a circular cylinder or sphere is about \( \text{Re}_c = 2 \times 10^5 \). That is, the boundary layer remains laminar for about \( \text{Re} \approx 2 \times 10^5 \) and becomes turbulent for \( \text{Re} \gg 2 \times 10^5 \).

Cross flow over a cylinder exhibits complex flow patterns, as shown in Figure 7–16. The fluid approaching the cylinder branches out and encircles the cylinder, forming a boundary layer that wraps around the cylinder. The fluid particles on the midplane strike the cylinder at the stagnation point, bringing the fluid to a complete stop and thus raising the pressure at that point. The pressure decreases in the flow direction while the fluid velocity increases.

At very low upstream velocities (\( \text{Re} \approx 1 \)), the fluid completely wraps around the cylinder and the two arms of the fluid meet on the rear side of the cylinder.
in an orderly manner. Thus, the fluid follows the curvature of the cylinder. At higher velocities, the fluid still hugs the cylinder on the frontal side, but it is too fast to remain attached to the surface as it approaches the top of the cylinder. As a result, the boundary layer detaches from the surface, forming a separation region behind the cylinder. Flow in the wake region is characterized by random vortex formation and pressures much lower than the stagnation point pressure.

The nature of the flow across a cylinder or sphere strongly affects the total drag coefficient $C_D$. Both the friction drag and the pressure drag can be significant. The high pressure in the vicinity of the stagnation point and the low pressure on the opposite side in the wake produce a net force on the body in the direction of flow. The drag force is primarily due to friction drag at low Reynolds numbers ($Re < 10$) and to pressure drag at high Reynolds numbers ($Re > 5000$). Both effects are significant at intermediate Reynolds numbers. The average drag coefficients $C_D$ for cross flow over a smooth single circular cylinder and a sphere are given in Figure 7–17. The curves exhibit different behaviors in different ranges of Reynolds numbers:

- For $Re \approx 1$, we have creeping flow, and the drag coefficient decreases with increasing Reynolds number. For a sphere, it is $C_D = 24/Re$. There is no flow separation in this regime.

- At about $Re = 10$, separation starts occurring on the rear of the body with vortex shedding starting at about $Re = 90$. The region of separation increases with increasing Reynolds number up to about $Re = 10^3$. At this point, the drag is mostly (about 95 percent) due to pressure drag. The drag coefficient continues to decrease with increasing Reynolds number in this range of $10 < Re < 10^3$. (A decrease in the drag coefficient does not necessarily indicate a decrease in drag. The drag force is proportional to the square of the velocity, and the increase in velocity at higher Reynolds numbers usually more than offsets the decrease in the drag coefficient.)