Chapter 2
Macromechanical Analysis of a Lamina

• Stress, Strain
• Stress–strain relationships for different types of materials
• Stress–strain relationships for a unidirectional/bidirectional lamina
1. Stress and Strain:

Applied loads on an infinitesimal surface on y-z plane

\[ \tau_{xy} = \lim_{\Delta A \to 0} \frac{\Delta P_y}{\Delta A} \]

\[ \sigma_x = \lim_{\Delta A \to 0} \frac{\Delta P_x}{\Delta A} \]

\[ \tau_{xz} = \lim_{\Delta A \to 0} \frac{\Delta P_z}{\Delta A} \]

Stress on an arbitrary surface

\[ \sigma_n = \lim_{\Delta A \to 0} \frac{\Delta P_n}{\Delta A} \]

\[ \tau_s = \lim_{\Delta A \to 0} \frac{\Delta P_s}{\Delta A} \]
Macromechanical Analysis of a Lamina

Stress state at any point can be represented by 9 components, writing equations of equilibrium:

\[ \tau_{xy} = \tau_{yx} \]
\[ \tau_{yz} = \tau_{zy} \]
\[ \tau_{zx} = \tau_{xz} \]

It means that just with 6 independent components, one can show stress state at any point:

\[ \mathbf{\sigma} = [\mathbf{\sigma}] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \]
Macromechanical Analysis of a Lamina

Strain:

\[ u = u(x,y,z) = \text{displacement in } x\text{-direction at point } (x,y,z) \]
\[ v = v(x,y,z) = \text{displacement in } y\text{-direction at point } (x,y,z) \]
\[ w = w(x,y,z) = \text{displacement in } z\text{-direction at point } (x,y,z) \]

\[ \varepsilon_x = \lim_{\Delta x \to 0} \frac{A'B' - AB}{AB} \]
\[ = \sqrt{(x'P')^2 + (y'P')^2}, \]

\[ = \sqrt{\Delta x + u(x + \Delta x, y) - u(x, y)}^2 + [v(x + \Delta x, y) - v(x, y)]^2, \]

\[ \varepsilon_x = \lim_{\Delta x \to 0} \left\{ \left[ 1 + \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} \right]^2 + \left[ \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \right]^2 \right\}^{1/2} - 1 \]
Macromechanical Analysis of a Lamina

\[ \varepsilon_x = \left[ \left( 1 + \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 \right]^{1/2} - 1 \]

\[ \varepsilon_x = \frac{\partial u}{\partial x} \]

And, so on:

\[ \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \]

\[ \gamma_{yz} = \frac{\partial w}{\partial z} + \frac{\partial u}{\partial y} \]

\[ \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \]

\[ \varepsilon_z = \frac{\partial w}{\partial z} \cdot \]

\[ \varepsilon_y = \frac{\partial v}{\partial y} \]

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

\[ \left[ \varepsilon \right] = \begin{bmatrix} \varepsilon_x & \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} \\ \frac{1}{2} \gamma_{yx} & \varepsilon_y & \frac{1}{2} \gamma_{yz} \\ \frac{1}{2} \gamma_{zx} & \frac{1}{2} \gamma_{zy} & \varepsilon_z \end{bmatrix} \]
2. Stress & Strain Transformation:

\[ \sigma'_{ij} = a_{iq} \sigma_{qm} a_{jm} \quad \text{or} \quad \sigma' = A \sigma A^T \]
### Macromechanical Analysis of a Lamina

#### 3. Stress-Strain Relations (Constitutive Equations or Hook’s Law):

\[
\begin{align*}
\sigma_i &= C_{ij} \varepsilon_j \quad &i, j = 1, \ldots, 6 \\
\varepsilon_i &= S_{ij} \sigma_j \quad &i, j = 1, \ldots, 6
\end{align*}
\]

- **Stiffness Matrix**
- **Compliance Matrix**

There are 36 constants relating stress to strain components which reduce to 21 constants due to the symmetry of the stiffness matrix \([C]\).
**Macromechanical Analysis of a Lamina**

**Question:** Prove that stiffness and compliance matrix are symmetric?

Stress–strain relationships for different types of materials:

A. Anisotropic Materials  
B. Monoclinic Materials  
C. Orthotropic Materials  
D. Transversely Isotropic Materials  
E. Isotropic Materials
Macromechanical Analysis of a Lamina

A. Hook’s law for anisotropic (triclinic) materials:
21 independent constants

There are not any planes of symmetry for the material properties in this kind of materials.

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\
C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\
C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}
\end{bmatrix}
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\epsilon_3 \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\
S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\
S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\
S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\
S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\
S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix}
\]

It’s clear that subjecting anisotropic materials to uniaxial stress makes the body under both axial and shear strains *(Coupling)*
B. Hook’s law for monoclinic materials: 13 independent constants

If there is one plane of material property symmetry (for example plane of symmetry is z=0 or 1-2 plane). E.g Feldspar

Material symmetry implies that the material and its mirror image about the plane of symmetry are identical
Macromechanical Analysis of a Lamina

- Apply tensile stress in 3-direction:
  Using the Hooke’s law Equation and the compliance matrix for the monoclinic material, one gets:

\[ \varepsilon_1 = S_{13} \sigma_3 \]

\[ \varepsilon_2 = S_{23} \sigma_3 \]

\[ \varepsilon_3 = S_{33} \sigma_3 \]

\[ \gamma_{23} = 0 \]

\[ \gamma_{31} = 0 \]

\[ \gamma_{12} = S_{36} \sigma_3 \]

Deformation of a cubic element made of monoclinic material under uniaxial stress in 3-direction (plane 1-2 is plane of symmetry)
Macromechanical Analysis of a Lamina

C. Hook’s law for orthotropic materials:
9 independent constants

If there are two orthogonal planes of material property symmetry for a material, symmetry will exist relative to a third mutually orthogonal plane.

A unidirectional lamina as a orthotropic material with fibers, arranged in a rectangular array. A wooden bar, and rolled steel are other examples.
Note that, for orthotropic, there is no interaction between normal stresses and shearing strains such as occurs in anisotropic materials (no coupling).

\[ \varepsilon_1 = S_{13}\sigma_3 \]
\[ \varepsilon_2 = S_{23}\sigma_3 \]
\[ \varepsilon_3 = S_{33}\sigma_3 \]
\[ \gamma_{23} = 0 \]
\[ \gamma_{31} = 0 \]
\[ \gamma_{12} = 0. \]

• Thus, the cube will not deform in shape under any normal load applied in the principal directions. This is unlike the monoclinic material, in which two out of the six faces of the cube changed shape.

• A cube made of isotropic material would not change its shape either; however, the normal strains, \(\varepsilon_1\) and \(\varepsilon_2\), will be different in an orthotropic material and identical in an isotropic material.
D. Hook’s law for transversely isotropic materials: 5 independent constants

If at every point of a material there is one plane in which the mechanical properties are equal in all directions (or if one of 3 symmetry plane is isotropic)

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\
C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\
C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & C_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & C_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & (C_{11} - C_{12})/2
\end{bmatrix} \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix} = \begin{bmatrix}
S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
S_{12} & S_{11} & S_{13} & 0 & 0 & 0 \\
S_{13} & S_{13} & S_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & S_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & S_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & 2(S_{11} - S_{12})
\end{bmatrix} \begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix}
\]

A unidirectional lamina as a transversely isotropic material with fibers arranged in a square array (plane 2–3 is the plane of isotropy)
E. Hook’s law for isotropic materials:  
2 independent constants (E, v)  

If there is an infinite number of planes of material property symmetry  

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{12} \\
C_{12} & C_{11} & C_{12} \\
C_{12} & C_{12} & C_{11} \\
0 & 0 & (C_{11} - C_{12})/2 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix}
\]

\[
\begin{bmatrix}
S_{11} & S_{12} & S_{12} \\
S_{12} & S_{11} & S_{12} \\
S_{12} & S_{12} & S_{11} \\
0 & 0 & 2(S_{11} - S_{12}) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix}
\]

**Question:** Why for isotropic material:  

\[
C_{66} = \frac{C_{11} - C_{12}}{2}
\]
Macromechanical Analysis of a Lamina

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{zx} \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
\frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
\frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & 0 & 0 & 0 \\
0 & 0 & 0 & G & 0 & 0 \\
0 & 0 & 0 & 0 & G & 0 \\
0 & 0 & 0 & 0 & 0 & G
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\varepsilon_z \\
\gamma_{yz} \\
\gamma_{zx} \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\
\frac{-\nu}{E} & \frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\
\frac{-\nu}{E} & \frac{-\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{G} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{G} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{G}
\end{bmatrix} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_z \\
\tau_{yz} \\
\tau_{zx} \\
\tau_{xy}
\end{bmatrix}
\]

- Anisotropic: 21
- Monoclinic: 13
- Orthotropic: 9
- Transversely isotropic: 5
- Isotropic: 2
Macromechanical Analysis of a Lamina

**Example:** Show that there are 13 constants for monoclinic materials in stress–strain relations?

**Solution:**

If direction 3 is perpendicular to the plane of symmetry. Now, in the coordinate system 1–2–3, strain-stress relation with $C_{ij} = C_{ji}$ is:

$$
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\sigma_3 \\
\tau_{23} \\
\tau_{31} \\
\tau_{12}
\end{bmatrix}
= 
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\
C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\
C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\varepsilon_3 \\
\gamma_{23} \\
\gamma_{31} \\
\gamma_{12}
\end{bmatrix}
$$
Macromechanical Analysis of a Lamina

Also, in the coordinate system 1′–2′–3′:

\[
\begin{bmatrix}
\sigma_1' \\
\sigma_2' \\
\sigma_3' \\
\tau_{23}' \\
\tau_{31}' \\
\tau_{12}'
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\
C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\
C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1' \\
\varepsilon_2' \\
\varepsilon_3' \\
\gamma_{23}' \\
\gamma_{31}' \\
\gamma_{12}'
\end{bmatrix}
\]

Because there is a plane of symmetry normal to direction 3, the stresses and strains in the 1–2–3 and 1′–2′–3′ coordinate systems are related by:

\[
\sigma_1 = \sigma_1', \quad \sigma_2 = \sigma_2', \quad \sigma_3 = \sigma_3'
\]

\[
\tau_{23} = -\tau_{23}', \quad \tau_{31} = -\tau_{31}', \quad \tau_{12} = \tau_{12}'
\]

\[
\varepsilon_1 = \varepsilon_1', \quad \varepsilon_2 = \varepsilon_2', \quad \varepsilon_3 = \varepsilon_3'
\]

\[
\gamma_{23} = -\gamma_{23}', \quad \gamma_{31} = -\gamma_{31}', \quad \gamma_{12} = \gamma_{12}'
\]
Macromechanical Analysis of a Lamina

1. \( \sigma_1 = C_{11} \varepsilon_1 + C_{12} \varepsilon_2 + C_{13} \varepsilon_3 + C_{14} \gamma_{23} + C_{15} \gamma_{31} + C_{16} \gamma_{12} \)

2. \( \sigma_1' = C_{11} \varepsilon'_1 + C_{12} \varepsilon'_2 + C_{13} \varepsilon'_3 + C_{14} \gamma'_{23} + C_{15} \gamma'_{31} + C_{16} \gamma'_{12} \)

Substituting eq. 3 in eq. 5:
\[ \sigma_1 = C_{11} \varepsilon_1 + C_{12} \varepsilon_2 + C_{13} \varepsilon_3 - C_{14} \gamma_{23} - C_{15} \gamma_{31} + C_{16} \gamma_{12} \]

Subtracting eq. 6 from eq. 4:
\[ 0 = 2C_{14} \gamma_{23} + 2C_{15} \gamma_{31} \]

Because \( \gamma_{23} \) and \( \gamma_{31} \) are arbitrary:
\[ C_{14} = C_{15} = 0 \]

Similarly, one can show that:
\[ C_{24} = C_{25} = 0 \]
\[ C_{34} = C_{35} = 0 \]
\[ C_{46} = C_{56} = 0 \]

Thus, only 13 independent elastic constants are present in a monoclinic material.
Macromechanical Analysis of a Lamina

Example: Show that there are 9 constants for orthotropic materials in stress–strain relations?
Macromechanical Analysis of a Lamina

4. Stiffnesses, compliances, & engineering constants for orthotropic materials:

9 constants in these materials are:
3 elasticity (Young) moduli $E_1$, $E_2$ & $E_3$ in 1, 2 & 3 directions (material axis)
3 shear moduli $G_{12}$, $G_{23}$ & $G_{31}$ in the 1-2, 2-3 & 3-1 planes
3 Poisson’s ratio $v_{12}$, $v_{23}$ & $v_{31}$

\[ \varepsilon_i = S_{ij} \sigma_j \quad i, j = 1, \ldots, 6 \]  

\[ [S_{ij}] = \begin{bmatrix} \frac{1}{E_1} & \frac{v_{12}}{E_2} & \frac{v_{13}}{E_3} & 0 & 0 & 0 \\ \frac{v_{12}}{E_2} & \frac{1}{E_2} & \frac{v_{23}}{E_3} & 0 & 0 & 0 \\ \frac{v_{13}}{E_3} & \frac{v_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \]
Macromechanical Analysis of a Lamina

Prove:

Apply uniaxial tensile stress \( \sigma_1 \neq 0, \sigma_2 = 0, \sigma_3 = 0, \tau_{23} = 0, \tau_{31} = 0, \tau_{12} = 0 \)

So:

\[
\begin{align*}
\varepsilon_1 &= S_{11}\sigma_1 & \gamma_{23} &= 0 \\
\varepsilon_2 &= S_{12}\sigma_1 & \gamma_{31} &= 0 \\
\varepsilon_3 &= S_{13}\sigma_1 & \gamma_{12} &= 0 
\end{align*}
\]

The Young’s modulus in direction 1, \( E_1 \), is defined as:

\[
E_1 \equiv \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}}
\]

The Poisson’s ratio, \( \nu_{12} \) & \( \nu_{13} \) are defined as:

\[
\begin{align*}
\nu_{12} &\equiv -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{12}}{S_{11}} \\
\nu_{13} &\equiv -\frac{\varepsilon_3}{\varepsilon_1} = -\frac{S_{13}}{S_{11}}
\end{align*}
\]

Note: \( \nu_{ij} \) is defined as the ratio of the negative of the normal strain in direction \( j \) to the normal strain in direction \( i \), when the load is applied in the normal direction \( i \).
Macromechanical Analysis of a Lamina

Similarly for uniaxial tensile stresses in 2 & 3 directions...

\[
\begin{align*}
E_2 &= \frac{1}{S_{22}} & \nu_{21} &= -\frac{S_{12}}{S_{22}} & \nu_{23} &= -\frac{S_{23}}{S_{22}} \\
E_3 &= \frac{1}{S_{33}} & \nu_{31} &= -\frac{S_{13}}{S_{33}} & \nu_{32} &= -\frac{S_{23}}{S_{33}}
\end{align*}
\]

Then apply \( \sigma_1 = 0, \sigma_2 = 0, \sigma_3 = 0, \tau_{23} \neq 0, \tau_{31} = 0, \tau_{12} = 0 \).

So:

\[
\begin{align*}
\varepsilon_1 &= 0 & \gamma_{23} &= S_{44} \tau_{23} \\
\varepsilon_2 &= 0 & \gamma_{31} &= 0 \\
\varepsilon_3 &= 0 & \gamma_{12} &= 0
\end{align*}
\]

The shear modulus in plane 2–3 is defined as:

\[
G_{23} \equiv \frac{\tau_{23}}{\gamma_{23}} = \frac{1}{S_{44}}
\]

And, similarly:

\[
G_{12} = \frac{1}{S_{66}} \quad G_{31} = \frac{1}{S_{55}}
\]
Macromechanical Analysis of a Lamina

**Note:** When an orthotropic material is stressed in principal material coordinates (the 1, 2, and 3 coordinates), it does not exhibit either shear-extension or shear-shear coupling.

Due to symmetry of stiffness & compliance matrices \((S_{ij}=S_{ji})\), last 12 obtained constants will be reduced to 9:

\[
\frac{v_{ij}}{E_i} = \frac{v_{ji}}{E_j} \quad i, j = 1, 2, 3 \quad i \neq j \quad 4-3
\]

(Betti’s law)
Obtaining stiffness matrix from compliance matrix for orthotropic materials: Because these two matrices are mutually inverse, using matrix algebra

\[
C_{11} = \frac{S_{22}S_{33} - S_{23}^2}{S} \quad C_{12} = \frac{S_{13}S_{23} - S_{12}S_{33}}{S} \quad C_{13} = \frac{S_{12}S_{23} - S_{13}S_{22}}{S}
\]

\[
C_{22} = \frac{S_{33}S_{11} - S_{13}^2}{S} \quad C_{23} = \frac{S_{12}S_{13} - S_{23}S_{11}}{S} \quad C_{33} = \frac{S_{11}S_{22} - S_{12}^2}{S}
\]

\[
C_{44} = \frac{1}{S_{44}} \quad C_{55} = \frac{1}{S_{55}} \quad C_{66} = \frac{1}{S_{66}}
\]

where,

\[
S = S_{11}S_{22}S_{33} - S_{11}S_{23}^2 - S_{22}S_{13}^2 - S_{33}S_{12}^2 + 2S_{12}S_{23}S_{13}
\]

\[
[C] = \begin{bmatrix}
\frac{1 - \nu_{23} \nu_{32}}{E_2 E_3 \Delta} & \frac{\nu_{21} + \nu_{23} \nu_{31}}{E_2 E_3 \Delta} & \frac{\nu_{31} + \nu_{21} \nu_{32}}{E_2 E_3 \Delta} & 0 & 0 & 0 \\
\frac{\nu_{21} + \nu_{23} \nu_{31}}{E_2 E_3 \Delta} & \frac{1 - \nu_{13} \nu_{32}}{E_1 E_3 \Delta} & \frac{\nu_{32} + \nu_{12} \nu_{31}}{E_1 E_3 \Delta} & 0 & 0 & 0 \\
\frac{\nu_{31} + \nu_{21} \nu_{32}}{E_2 E_3 \Delta} & \frac{\nu_{32} + \nu_{12} \nu_{31}}{E_1 E_3 \Delta} & \frac{1 - \nu_{12} \nu_{21}}{E_1 E_2 \Delta} & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\nu_{23}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{\nu_{31}} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{\nu_{12}}
\end{bmatrix}
\]

\[
\Delta = \left(1 - \nu_{12} \nu_{21} - \nu_{23} \nu_{32} - \nu_{13} \nu_{31} - 2 \nu_{21} \nu_{32} \nu_{13}\right) / \left(E_1 E_2 E_3\right)
\]

Islamic Azad University, Najafabad Branch, A. Atrian
Macromechanical Analysis of a Lamina

5. Restrictions on engineering constants:

A. Isotropic Materials:

\[ G = \frac{E}{2(1 + \nu)} \quad \text{and} \quad \nu > -1 \]

When an isotropic body is subjected to hydrostatic pressure \( (\sigma_x = \sigma_y = \sigma_z = -p) \):

\[ \theta = \varepsilon_x + \varepsilon_y + \varepsilon_z = \frac{p}{E/3(1-2\nu)} = \frac{p}{K} \]

\[ K = \frac{E}{3(1-2\nu)} \]

Bulk modulus \( K \) is positive, if it was negative, a hydrostatic pressure would cause expansion of a cube of isotropic material.
Macromechanical Analysis of a Lamina

B. Orthotropic Materials:
Based on the first law of thermodynamics, the stiffness and the compliance matrix must be positive definite (i.e., have positive principal values):

\[ S_{11}, S_{22}, S_{33}, S_{44}, S_{55}, S_{66} > 0 \quad \rightarrow \quad E_1, E_2, E_3, G_{23}, G_{31}, G_{12} > 0 \]

\[ C_{11}, C_{22}, C_{33}, C_{44}, C_{55}, C_{66} > 0 \quad \rightarrow \quad 1 - v_{23}v_{32} > 0, \quad 1 - v_{31}v_{13} > 0, \quad 1 - v_{12}v_{21} > 0, \quad \Delta = 1 - v_{12}v_{21} - v_{23}v_{32} - v_{31}v_{13} - 2v_{13}v_{21}v_{32} > 0 \]

The positive nature of the stiffnesses leads to:

\[ |S_{23}| < \sqrt{S_{22}S_{33}} \quad |S_{13}| < \sqrt{S_{11}S_{33}} \quad |S_{12}| < \sqrt{S_{11}S_{22}} \]

\[ \frac{v_{ij}}{E_i} = \frac{v_{ji}}{E_j} \quad \text{for} \ i \neq j \ \text{and} \ i,j = 1,2,3 \]

\[ |v_{21}| < \sqrt{\frac{E_2}{E_1}} \quad |v_{32}| < \sqrt{\frac{E_3}{E_2}} \quad |v_{13}| < \sqrt{\frac{E_1}{E_3}} \]

\[ |v_{12}| < \sqrt{\frac{E_1}{E_2}} \quad |v_{23}| < \sqrt{\frac{E_2}{E_3}} \quad |v_{31}| < \sqrt{\frac{E_3}{E_1}} \]
In order to obtain a constraint on one Poisson's ratio, $v_{21}$, in terms of two others, $v_{32}$ and $v_{13}$, from Eq.(5-12):

$$v_{21}v_{32}v_{13} < \frac{1 - v_{21}^2}{2} \frac{E_1}{E_2} - \frac{v_{32}^2}{2} \frac{E_2}{E_3} - \frac{v_{13}^2}{2} \frac{E_3}{E_1} < \frac{1}{2}$$

$$\left[ 1 - \frac{v_{32}^2}{E_3} \right] \left[ 1 - \frac{v_{13}^2}{E_1} \right] - \left( v_{21} \sqrt{\frac{E_1}{E_2}} + v_{32}v_{13} \sqrt{\frac{E_2}{E_1}} \right)^2 > 0$$
Macromechanical Analysis of a Lamina

Applications & importance of these restrictions on engineering constants for orthotropic materials

1. To optimize properties of a composite because they show that the nine independent properties cannot be varied without influencing the limits of the others.

2. To examine experimental data to see if they are physically consistent within the framework of the mathematical elasticity model.
Macromechanical Analysis of a Lamina

Example 5-1: Dickerson and DiMartino test: They measured Poisson's ratios for boron-epoxy composite materials as high as 1.97 for the negative of the strain in the 2-direction over the strain in the 1-direction due to loading in the 1-direction ($\nu_{12}$). The reported values of the Young's moduli for the two directions are $E_1=11.86 \times 10^6$ psi (81.77 GPa) and $E_2=1.33 \times 10^6$ psi (9.17 GPa).

$$\sqrt{\frac{E_1}{E_2}} = 2.99$$

(For isotropic materials this ratio is nearly 1)

$$|\nu_{12}| < \sqrt{\frac{E_1}{E_2}}$$

1.97 < 2.99: $\nu_{12}$ is a reasonable number even though our intuition based on isotropic materials ($\nu < 1/2$) rejects such a large number.

Also, the 'converse' (or minor) Poisson's ratio, $\nu_{21}'$ was reported as 0.22. This value satisfies the reciprocal relations in eq. (5-9).
Macromechanical Analysis of a Lamina

**Example 5-2:** Find the compliance and stiffness matrix for a graphite/epoxy lamina. The material properties are given as

\[ E_1 = 181 \text{GPa} \quad E_2 = 10.3 \text{GPa} \quad E_3 = 10.3 \text{GPa} \quad \nu_{12} = 0.28 \quad \nu_{23} = 0.60 \quad \nu_{13} = 0.27 \]

\[ G_{12} = 7.17 \text{GPa} \quad G_{23} = 3.0 \text{GPa} \quad G_{31} = 7.00 \text{GPa} \]

**Solution:**

\[ S_{11} = \frac{1}{E_1} = \frac{1}{181 \times 10^9} = 5.525 \times 10^{-12} \text{Pa}^{-1} \]

\[ S_{13} = -\frac{\nu_{13}}{E_1} = -\frac{0.27}{181 \times 10^9} = -1.492 \times 10^{-12} \text{Pa}^{-1} \]

\[ S_{22} = \frac{1}{E_2} = \frac{1}{10.3 \times 10^9} = 9.709 \times 10^{-11} \text{Pa}^{-1} \]

\[ S_{23} = -\frac{\nu_{23}}{E_2} = -\frac{0.6}{10.3 \times 10^9} = -5.825 \times 10^{-11} \text{Pa}^{-1} \]

\[ S_{33} = \frac{1}{E_3} = \frac{1}{10.3 \times 10^9} = 9.709 \times 10^{-11} \text{Pa}^{-1} \]

\[ S_{44} = \frac{1}{G_{23}} = \frac{1}{3 \times 10^8} = 3.333 \times 10^{-10} \text{Pa}^{-1} \]

\[ S_{12} = -\frac{\nu_{12}}{E_1} = -\frac{0.28}{181 \times 10^9} = -1.547 \times 10^{-12} \text{Pa}^{-1} \]

\[ S_{55} = \frac{1}{G_{31}} = \frac{1}{7 \times 10^9} = 1.429 \times 10^{-10} \text{Pa}^{-1} \]

\[ S_{66} = \frac{1}{G_{12}} = \frac{1}{7.17 \times 10^9} = 1.395 \times 10^{-10} \text{Pa}^{-1} \]
Macromechanical Analysis of a Lamina

Solution (continue):

\[
\begin{bmatrix}
S
\end{bmatrix} =
\begin{bmatrix}
5.525 \times 10^{-12} & -1.547 \times 10^{-12} & -1.492 \times 10^{-12} & 0 & 0 & 0 \\
-1.547 \times 10^{-12} & 9.709 \times 10^{-11} & -5.825 \times 10^{-11} & 0 & 0 & 0 \\
-1.492 \times 10^{-12} & -5.825 \times 10^{-11} & 9.709 \times 10^{-11} & 0 & 0 & 0 \\
0 & 0 & 0 & 3.333 \times 10^{-10} & 0 & 0 \\
0 & 0 & 0 & 0 & 1.429 \times 10^{-10} & 0 \\
0 & 0 & 0 & 0 & 0 & 1.395 \times 10^{-10}
\end{bmatrix} \text{ Pa}^{-1}
\]

\[
\begin{bmatrix}
C
\end{bmatrix} = \begin{bmatrix}
S
\end{bmatrix}^{-1} \quad \text{(or by using eq. 4-6)}
\]

\[
\begin{bmatrix}
C
\end{bmatrix} =
\begin{bmatrix}
0.1850 \times 10^{12} & 0.7269 \times 10^{10} & 0.7204 \times 10^{10} & 0 & 0 & 0 \\
0.7269 \times 10^{10} & 0.1638 \times 10^{11} & 0.9938 \times 10^{10} & 0 & 0 & 0 \\
0.7204 \times 10^{10} & 0.9938 \times 10^{10} & 0.1638 \times 10^{11} & 0 & 0 & 0 \\
0 & 0 & 0 & 0.3000 \times 10^{10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0.6998 \times 10^{10} & 0 \\
0 & 0 & 0 & 0 & 0 & 0.7168 \times 10^{10}
\end{bmatrix} \text{ Pa}
\]
Macromechanical Analysis of a Lamina

6. STRESS-STRAIN RELATIONS FOR PLANE STRESS IN AN ORTHOTROPIC MATERIAL

Plane stress conditions for a thin plate

Unidirectionally Reinforced Lamina

\[ \sigma_3 = 0 \quad \tau_{23} = 0 \quad \tau_{31} = 0 \]
\[ \sigma_1 \neq 0 \quad \sigma_2 \neq 0 \quad \tau_{12} \neq 0 \]

Note: A plane stress state on a lamina is not merely an idealization of reality, but instead is a practical and achievable objective of how we must use a lamina with fibers in its plane.
Macromechanical Analysis of a Lamina

A unidirectionally reinforced lamina mainly can withstand in plane loads especially in its fibers direction. So, it would need 'help' carrying in-plane stress perpendicular to its fibers, but that help can be provided by other (parallel) layers that have their fibers in the direction of the stress using laminate.

For orthotropic materials, imposing a state of plane stress results in implied out-of-plane strains of:

\[
\begin{align*}
\epsilon_3 &= S_{13}\epsilon_1 + S_{23}\epsilon_2 \quad \gamma_{23} = 0 \quad \gamma_{31} = 0
\end{align*}
\]

The normal strain, \(\epsilon_3\), is not an independent strain because it is a function of the other two normal strains, \(\epsilon_1\) and \(\epsilon_2\). Therefore, the normal strain, \(\epsilon_3\), can be omitted from the stress–strain relationship. Also, the shearing strains, \(\gamma_{23}\) and \(\gamma_{31}\), can be omitted because they are zero:

\[
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & 0 \\
S_{12} & S_{22} & 0 \\
0 & 0 & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]

\[
\begin{bmatrix}
S_{11} = \frac{1}{E_1} & S_{12} = \frac{v_{12}}{E_1} = -\frac{v_{21}}{E_2} & S_{22} = \frac{1}{E_2} & S_{66} = \frac{1}{G_{12}}
\end{bmatrix}
\]
Reduced stiffnesses for a plane stress state in the 1-2 plane which are determined by:
1. as the components of the inverted compliance matrix in eq. (6.3) or
2. from the $C_{ij}$ directly by applying the condition $\sigma_3 = 0$ to the strain-stress relations to get an expression for \( \varepsilon_3 \) and simplifying the results to get:

\[
Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}} \quad i,j = 1, 2, 6
\]

6-5

\[
\begin{align*}
Q_{11} &= \frac{S_{22}}{S_{11}S_{22} - S_{12}^2} \\
Q_{22} &= \frac{S_{11}}{S_{11}S_{22} - S_{12}^2} \\
Q_{12} &= \frac{S_{12}}{S_{11}S_{22} - S_{12}^2} \\
Q_{66} &= \frac{1}{S_{66}}
\end{align*}
\]

6-6

\[
\begin{align*}
Q_{11} &= \frac{E_1}{1 - \nu_{12}\nu_{21}} \\
Q_{12} &= \frac{\nu_{12}E_2}{1 - \nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1 - \nu_{12}\nu_{21}} \\
Q_{22} &= \frac{E_2}{1 - \nu_{12}\nu_{21}} \\
Q_{66} &= G_{12}
\end{align*}
\]

6-7
Macromechanical Analysis of a Lamina

Relationship of Compliance and Stiffness Matrix to Engineering Elastic Constants of a Lamina:

For a unidirectional lamina, these engineering elastic constants are:
- $E_1 =$ longitudinal Young’s modulus (in direction 1)
- $E_2 =$ transverse Young’s modulus (in direction 2)
- $V_{12} =$ major Poisson’s ratio
- $G_{12} =$ in-plane shear modulus (in plane 1–2)
The unidirectional or bidirectional lamina is a *specially orthotropic lamina* because normal stresses applied in the 1–2 direction do not result in any shearing strains in the 1–2 plane because $Q_{16} = Q_{26} = 0 = S_{16} = S_{26}$. Also, the shearing stresses applied in the 1–2 plane do not result in any normal strains in the 1 and 2 directions because $Q_{16} = Q_{26} = 0 = S_{16} = S_{26}$. 

---

**Macromechanical Analysis of a Lamina**

\[ \sigma_1 = 0, \; \sigma_2 \neq 0, \; \tau_{12} = 0 \quad \Rightarrow \quad \varepsilon_1 = S_{12} \sigma_2 \quad \varepsilon_2 = S_{22} \sigma_2 \quad \gamma_{12} = 0. \]

\[ \varepsilon_1 = 0, \quad \varepsilon_2 = 0, \quad \gamma_{12} = S_{66} \tau_{12} \]

\[ E_2 \equiv \frac{\sigma_2}{\varepsilon_2} = \frac{1}{S_{22}}, \quad \nu_{21} \equiv -\frac{\varepsilon_1}{\varepsilon_2} = -\frac{S_{12}}{S_{22}} \]

(Minor Poisson’s Ratio)
Macromechanical Analysis of a Lamina

For plane stress on isotropic materials:

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & 0 \\
S_{12} & S_{11} & 0 \\
0 & 0 & 2(S_{11} - S_{12})
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & 0 \\
Q_{12} & Q_{11} & 0 \\
0 & 0 & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
\]

\[
S_{11} = \frac{1}{E} \quad S_{12} = -\frac{\nu}{E}
\]

\[
Q_{11} = \frac{E}{1 - \nu^2} \quad Q_{12} = -\frac{\nu E}{1 - \nu^2} \quad Q_{66} = \frac{E}{2(1 + \nu)} = G
\]

**Question:** Obtain stiffness & compliance matrices for orthotropic & isotropic materials under plane stress condition?

**Question:** Show that for an orthotropic material \(Q_{11} \neq C_{11}\). Explain why. Also, show \(Q_{66} = C_{66}\). Explain why
**Macromechanical Analysis of a Lamina**

**Question:** For a graphite/epoxy unidirectional lamina, find the following
1. Compliance matrix
2. Minor Poisson’s ratio
3. Reduced stiffness matrix
4. Strains in the $1–2$ coordinate system

Use the properties of unidirectional graphite/epoxy lamina from next table.
Macromechanical Analysis of a Lamina

Typical Mechanical Properties of a Unidirectional Lamina (SI System of Units)

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Units</th>
<th>Glass/epoxy</th>
<th>Boron/epoxy</th>
<th>Graphite/epoxy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fiber volume fraction</td>
<td>$V_f$</td>
<td></td>
<td>0.45</td>
<td>0.50</td>
<td>0.70</td>
</tr>
<tr>
<td>Longitudinal elastic modulus</td>
<td>$E_1$</td>
<td>GPa</td>
<td>38.6</td>
<td>204</td>
<td>181</td>
</tr>
<tr>
<td>Transverse elastic modulus</td>
<td>$E_2$</td>
<td>GPa</td>
<td>8.27</td>
<td>18.50</td>
<td>10.30</td>
</tr>
<tr>
<td>Major Poisson’s ratio</td>
<td>$v_{12}$</td>
<td></td>
<td>0.26</td>
<td>0.23</td>
<td>0.28</td>
</tr>
<tr>
<td>Shear modulus</td>
<td>$C_{12}$</td>
<td>GPa</td>
<td>4.14</td>
<td>5.59</td>
<td>7.17</td>
</tr>
<tr>
<td>Ultimate longitudinal tensile strength</td>
<td>$(\sigma^T_{1})_{ult}$</td>
<td>MPa</td>
<td>1062</td>
<td>1260</td>
<td>1500</td>
</tr>
<tr>
<td>Ultimate longitudinal compressive strength</td>
<td>$(\sigma^C_1)_{ult}$</td>
<td>MPa</td>
<td>610</td>
<td>2500</td>
<td>1500</td>
</tr>
<tr>
<td>Ultimate transverse tensile strength</td>
<td>$(\sigma^T_2)_{ult}$</td>
<td>MPa</td>
<td>31</td>
<td>61</td>
<td>40</td>
</tr>
<tr>
<td>Ultimate transverse compressive strength</td>
<td>$(\sigma^C_2)_{ult}$</td>
<td>MPa</td>
<td>118</td>
<td>202</td>
<td>246</td>
</tr>
<tr>
<td>Ultimate in-plane shear strength</td>
<td>$(\tau_{12})_{ult}$</td>
<td>MPa</td>
<td>72</td>
<td>67</td>
<td>68</td>
</tr>
<tr>
<td>Longitudinal coefficient of thermal expansion</td>
<td>$\alpha_1$</td>
<td>$\mu$m/m/°C</td>
<td>8.6</td>
<td>6.1</td>
<td>0.02</td>
</tr>
<tr>
<td>Transverse coefficient of thermal expansion</td>
<td>$\alpha_2$</td>
<td>$\mu$m/m/°C</td>
<td>22.1</td>
<td>30.3</td>
<td>22.5</td>
</tr>
<tr>
<td>Longitudinal coefficient of moisture expansion</td>
<td>$\beta_1$</td>
<td>m/m/kg/kg</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Transverse coefficient of moisture expansion</td>
<td>$\beta_2$</td>
<td>m/m/kg/kg</td>
<td>0.60</td>
<td>0.60</td>
<td>0.60</td>
</tr>
</tbody>
</table>
Macromechanical Analysis of a Lamina

7. STRESS-STRAIN RELATIONS FOR A LAMINA OF ARBITRARY ORIENTATION

1-2 coordinate system: local axes: material axes
1-direction: Longitudinal direction $L$
2-direction: Transverse direction $T$

x–y coordinate system: global axes: off-axes
Macromechanical Analysis of a Lamina

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = [T]^{-1}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
\cos^2\theta & \sin^2\theta & -2\sin\theta\cos\theta \\
\sin^2\theta & \cos^2\theta & 2\sin\theta\cos\theta \\
\sin\theta\cos\theta & -\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]

\[7-1\]

\[
[T] =
\begin{bmatrix}
c^2 & s^2 & 2sc \\
s^2 & c^2 & -2sc \\
-sc & sc & c^2 - s^2
\end{bmatrix}
\]

The transformation matrix

\[7-2\]

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = [T]^{-1}[Q]
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
\]

\[7-3\]

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12} / 2
\end{bmatrix} = [T]
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy} / 2
\end{bmatrix}
\]

Reuter Matrix

\[7-4\]

\[
[R] =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix} = [R][T][R]^{-1}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

\[7-5\]
The elements of the transformed reduced stiffness matrix are given by:

\[
egin{align*}
\bar{Q}_{11} &= Q_{11} \cos^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \sin^4 \theta \\
\bar{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{12}(\sin^4 \theta + \cos^4 \theta) \\
\bar{Q}_{22} &= Q_{11} \sin^4 \theta + 2(Q_{12} + 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{22} \cos^4 \theta \\
\bar{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin \theta \cos^3 \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin^3 \theta \cos \theta \\
\bar{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66}) \sin^3 \theta \cos \theta + (Q_{12} - Q_{22} + 2Q_{66}) \sin \theta \cos^3 \theta \\
\bar{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66}) \sin^2 \theta \cos^2 \theta + Q_{66}(\sin^4 \theta + \cos^4 \theta)
\end{align*}
\]

Note: The transformed reduced stiffness matrix has terms in all nine positions in contrast to the presence of zeros in the reduced stiffness matrix. However, there are still only four independent material constants because the lamina is orthotropic.
Macromechanical Analysis of a Lamina

\[
\begin{bmatrix}
\epsilon_1 \\
\epsilon_2 \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & 0 \\
S_{12} & S_{22} & 0 \\
0 & 0 & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]

\[
[R][T][R]^{-1} = [T]^{-T}
\]

\[
\begin{bmatrix}
\epsilon_x \\
\epsilon_y \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{16} \\
S_{12} & S_{22} & S_{26} \\
S_{16} & S_{26} & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix}
\]

Transformed compliance matrix:

\[
\begin{align*}
S_{11} &= S_{11} \cos^4 \theta + (2S_{12} + S_{66}) \sin^2 \theta \cos^2 \theta + S_{22} \sin^4 \theta \\
S_{12} &= S_{12}( \sin^4 \theta + \cos^4 \theta ) + (S_{11} + S_{22} - S_{66}) \sin^2 \theta \cos^2 \theta \\
S_{22} &= S_{11} \sin^4 \theta + (2S_{12} + S_{66}) \sin^2 \theta \cos^2 \theta + S_{22} \cos^4 \theta \\
S_{16} &= (2S_{11} - 2S_{12} - S_{66}) \sin \theta \cos^3 \theta - (2S_{22} - 2S_{12} - S_{66}) \sin^3 \theta \cos \theta \\
S_{26} &= (2S_{11} - 2S_{12} - S_{66}) \sin^3 \theta \cos \theta - (2S_{22} - 2S_{12} - S_{66}) \sin \theta \cos^3 \theta \\
S_{66} &= 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66}) \sin^2 \theta \cos^2 \theta + S_{66}( \sin^4 \theta + \cos^4 \theta )
\end{align*}
\]

Note:

- From Eqs. (6-3) & (6-4), for a unidirectional lamina loaded in the material axes directions, no coupling occurs between the normal and shearing terms of strains and stresses (specially orthotropic).
- However, for an angle lamina, from Eqs. (7-6) & (7-8), coupling takes place between the normal and shearing terms of strains and stresses.
- Therefore, Equation (7-6) & (7-8) are stress–strain equations for what is called a generally orthotropic lamina.
Macromechanical Analysis of a Lamina

Note:
- The only advantage associated with generally orthotropic as opposed to anisotropic laminae is that generally orthotropic laminae are easier to characterize experimentally.
- In fact, there is no difference between solutions for generally orthotropic laminae and those for anisotropic laminae whose stress-strain relations, under conditions of plane stress, can be written as:

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} =
\begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} \\
Q_{12} & Q_{22} & Q_{26} \\
Q_{16} & Q_{26} & Q_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
\]

and

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{16} \\
S_{12} & S_{22} & S_{26} \\
S_{16} & S_{26} & S_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
\]

where the anisotropic compliances in terms of the engineering constants are:

\[
S_{11} = \frac{1}{E_1}, \quad S_{22} = \frac{1}{E_2}, \quad S_{16} = \frac{\eta_{12,1}}{E_1} = \frac{\eta_{1,12}}{G_{12}}, \quad S_{26} = \frac{\eta_{12,2}}{E_2} = \frac{\eta_{2,12}}{G_{12}},
\]

or

\[
S_{12} = -\frac{v_{12}}{E_1} = -\frac{v_{21}}{E_2}, \quad S_{66} = \frac{1}{G_{12}}.
\]
Macromechanical Analysis of a Lamina

Anisotropic elasticity relations:

1. Lekhnitskii coefficients or mutual influence coefficients or shear-extension coupling coefficients:

\[ \eta_{i,j} = \frac{\varepsilon_i}{\gamma_{ij}} \]

for \( \tau_{ij} = \tau \) and all other stresses are zero.

\[ \eta_{ij,i} = \frac{\gamma_{ij}}{\varepsilon_i} \]

for \( \sigma_i = \sigma \) and all other stresses are zero.

2. Chentsov coefficients or shear-shear coefficients (not for in-plane loading):

\[ \mu_{ij,kl} = \text{Chentsov coefficient that characterizes the shearing strain in the kl-plane due to shearing stress in the ij-plane, i.e.,} \]

\[ \mu_{ij,kl} = \frac{\gamma_{kl}}{\gamma_{ij}} \]

for \( \tau_{ij} = \tau \) and all other stresses are zero.

\[ \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \begin{bmatrix} S_{14} & S_{15} & S_{16} \\ S_{24} & S_{25} & S_{26} \\ S_{34} & S_{35} & S_{36} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \tau_{12} \end{bmatrix} \]
Macromechanical Analysis of a Lamina

Reciprocal relation for Chentsov coeff. (like Betti’s law):

\[
\frac{\mu_{kl, ij}}{G_{kl}} = \frac{\mu_{ij, kl}}{G_{ij}}
\]

3. The out-of-plane shearing strains of an anisotropic lamina due to in-plane shearing stress and normal stresses are:

\[
\gamma_{13} = \frac{n_1 \sigma_{11} + n_2 \sigma_{22} + \mu_{12} \tau_{12}}{G_{13}}
\]
\[
\gamma_{23} = \frac{n_1 \sigma_{11} + n_2 \sigma_{22} + \mu_{12} \tau_{12}}{G_{23}}
\]

**Note:** Neither of these shear strains arise in an orthotropic material unless it is stressed in coordinates other than the principal material coordinates.
Macromechanical Analysis of a Lamina

The *apparent engineering constants* for an orthotropic lamina that is stressed in non-principal x-y coordinates (from Eqs. 7-9 & 7-11 & 7-12):

\[
\begin{align*}
\frac{1}{E_x} &= \frac{1}{E_1} \cos^4 \theta + \left[ \frac{1}{G_{12}} \frac{2v_{12}}{E_1} \right] \sin^2 \theta \cos^2 \theta + \frac{1}{E_2} \sin^4 \theta \\
\nu_{xy} &= E_x \left[ \frac{v_{12}}{E_1} \left( \sin^4 \theta + \cos^4 \theta \right) - \left( \frac{1}{E_1} + \frac{1}{E_2} - \frac{1}{G_{12}} \right) \sin^2 \theta \cos^2 \theta \right] \\
\frac{1}{E_y} &= \frac{1}{E_1} \sin^4 \theta + \left[ \frac{1}{G_{12}} \frac{2v_{12}}{E_1} \right] \sin^2 \theta \cos^2 \theta + \frac{1}{E_2} \cos^4 \theta \\
\frac{1}{G_{xy}} &= 2 \left[ \frac{2}{E_1} + \frac{2}{E_2} + \frac{4v_{12}}{E_1} - \frac{1}{G_{12}} \right] \sin^2 \theta \cos^2 \theta + \frac{1}{G_{12}} \left( \sin^4 \theta + \cos^4 \theta \right) \\
\eta_{xy,x} &= E_x \left[ \frac{2}{E_1} + \frac{2v_{12}}{E_1} - \frac{1}{G_{12}} \right] \sin \theta \cos \theta - \left( \frac{2}{E_2} + \frac{2v_{12}}{E_1} - \frac{1}{G_{12}} \right) \sin^3 \theta \cos \theta \\
\eta_{xy,y} &= E_y \left[ \frac{2}{E_1} + \frac{2v_{12}}{E_1} - \frac{1}{G_{12}} \right] \sin^3 \theta \cos \theta - \left( \frac{2}{E_2} + \frac{2v_{12}}{E_1} - \frac{1}{G_{12}} \right) \sin \theta \cos^3 \theta
\end{align*}
\]

**Question:** Obtain Eqs. (7-18)?
Macromechanical Analysis of a Lamina

From Eqs. (7-18):

These results were summarized by Jones as a simple theorem: the extremum (largest and smallest) material properties do not necessarily occur in principal material (1-2) coordinates: $V_{xy}$ or $G_{xy}$.
Macromechanical Analysis of a Lamina

Visual interpretation:
Why $G_{xy} > G_{12}$ & $E_1 > E_{45}$ for a composite material with fiber modulus much greater than matrix modulus?
Macromechanical Analysis of a Lamina

Know more...

- **Auxetic (Negative Poisson’s Ratio) Structures**

  ![Chiral Auxetic Structure](image)

- **Higher Poisson’s Ratio**

  Dickerson & Di Martino have published data for cross-plied boron-epoxy composites in which the Poisson's ratios range from 0.024 to 0.878 in the orthotropic case and from -0.414 to 1.97 for a +/-25° laminate

![Poisson's Ratio vs Orientation](image)

Variation of Poisson's ratio with orientation for the woven roving laminate panel

P. D. Craig & J. Summerscalest Experiments
Example 7-1: Find the following for a $60^\circ$ angle lamina of graphite/epoxy. Use the properties of unidirectional graphite/epoxy lamina from the last table.

1. Transformed compliance matrix
2. Transformed reduced stiffness matrix

If the applied stress is $\sigma_x=2 \text{ MPa}$, $\sigma_y=-3 \text{ MPa}$, and $\tau_{xy}=4 \text{ MPa}$, also find

3. Global strains
4. Local strains
5. Local stresses
6. Principal stresses
7. Maximum shear stress
8. Principal strains
9. Maximum shear strain
Macromechanical Analysis of a Lamina

Solution:
1: From example 5-2, then using Eq.(7-9):

\[ S_{11} = 0.5525 \times 10^{-11} \frac{1}{Pa} \quad S_{12} = -0.1547 \times 10^{-11} \frac{1}{Pa} \]

\[ S_{22} = 0.9709 \times 10^{-10} \frac{1}{Pa} \quad S_{66} = 0.1395 \times 10^{-9} \frac{1}{Pa} \]

\[ \bar{S}_{11} = 0.5525 \times 10^{-11}(0.500)^4 + [2(-0.1547 \times 10^{-11}) + 0.1395 \times 10^{-9}] (0.866)^2 (0.5)^2 + 0.9709 \times 10^{-10} (0.866)^4 \]

\[ = 0.8053 \times 10^{-10} \frac{1}{Pa} \]

\[ \bar{S}_{12} = -0.7878 \times 10^{-11} \frac{1}{Pa} \quad \bar{S}_{66} = -0.4696 \times 10^{-10} \frac{1}{Pa} \]

\[ \bar{S}_{16} = -0.3234 \times 10^{-10} \frac{1}{Pa} \quad \bar{S}_{66} = 0.1141 \times 10^{-9} \frac{1}{Pa}. \]

\[ \bar{S}_{22} = 0.3475 \times 10^{-10} \frac{1}{Pa}. \]
Macromechanical Analysis of a Lamina

2: by inverting Transformed compliance matrix or using Eq.(7-7):

\[
[\bar{Q}] = \begin{bmatrix}
0.8053 \times 10^{-10} & -0.7878 \times 10^{-11} & -0.3234 \times 10^{-10} \\
-0.7878 \times 10^{-11} & 0.3475 \times 10^{-10} & -0.4696 \times 10^{-10} \\
-0.3234 \times 10^{-10} & -0.4696 \times 10^{-10} & 0.1141 \times 10^{-9}
\end{bmatrix}^{-1}
= \begin{bmatrix}
0.2365 \times 10^{11} & 0.3246 \times 10^{11} & 0.2005 \times 10^{11} \\
0.3246 \times 10^{11} & 0.1094 \times 10^{12} & 0.5419 \times 10^{11} \\
0.2005 \times 10^{11} & 0.5419 \times 10^{11} & 0.3674 \times 10^{11}
\end{bmatrix} \text{ Pa}
\]

3: from Eq.(7-8):

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
\bar{S}_{11} & \bar{S}_{12} & \bar{S}_{16} \\
\bar{S}_{12} & \bar{S}_{22} & \bar{S}_{26} \\
\bar{S}_{16} & \bar{S}_{26} & \bar{S}_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} =
\begin{bmatrix}
0.8053 \times 10^{-10} & -0.7878 \times 10^{-11} & -0.3234 \times 10^{-10} \\
-0.7878 \times 10^{-11} & 0.3475 \times 10^{-10} & -0.4696 \times 10^{-10} \\
-0.3234 \times 10^{-10} & -0.4696 \times 10^{-10} & 0.1141 \times 10^{-9}
\end{bmatrix}
\begin{bmatrix}
2 \times 10^6 \\
-3 \times 10^6 \\
4 \times 10^6
\end{bmatrix}
= \begin{bmatrix}
0.5534 \times 10^{-4} \\
-0.3078 \times 10^{-3} \\
0.5328 \times 10^{-3}
\end{bmatrix}
\]
Macromechanical Analysis of a Lamina

4: using Eq. (7-4):

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}/2
\end{bmatrix}
= \begin{bmatrix}
0.2500 & 0.7500 & 0.8660 \\
0.7500 & 0.2500 & -0.8660 \\
-0.4330 & 0.4330 & -0.500
\end{bmatrix}
\begin{bmatrix}
0.5534 \times 10^{-4} \\
-0.3078 \times 10^{-3} \\
0.5328 \times 10^{-3}/2
\end{bmatrix}
= \begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
= \begin{bmatrix}
0.1367 \times 10^{-4} \\
-0.2662 \times 10^{-3} \\
-0.5809 \times 10^{-3}
\end{bmatrix}
\]

5: using Eq. (7-1):

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
= \begin{bmatrix}
0.2500 & 0.7500 & 0.8660 \\
0.7500 & 0.2500 & -0.8660 \\
-0.4330 & 0.4330 & -0.500
\end{bmatrix}
\begin{bmatrix}
2 \times 10^6 \\
-3 \times 10^6 \\
4 \times 10^6
\end{bmatrix}
= \begin{bmatrix}
0.1714 \times 10^7 \\
-0.2714 \times 10^7 \\
-0.4165 \times 10^7
\end{bmatrix}
Pa
\]
Macromechanical Analysis of a Lamina

6:

\[ \sigma_{\text{max/min}} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \]

\[ = \frac{2 \times 10^6 - 3 \times 10^6}{2} \pm \sqrt{\left(\frac{2 \times 10^6 + 3 \times 10^6}{2}\right)^2 + (4 \times 10^6)^2} = 4.217, -5.217 \text{ MPa} \]

\[ \theta_p = \frac{1}{2} \tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = \frac{1}{2} \tan^{-1}\left(\frac{2(4 \times 10^6)}{2 \times 10^6 + 3 \times 10^6}\right) = 29.00^\circ \]

7:

\[ \tau_{\text{max}} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{2 \times 10^6 - 3 \times 10^6}{2}\right)^2 + (4 \times 10^6)^2} = 4.717 \text{ MPa} \]

\[ \theta_s = \frac{1}{2} \tan^{-1}\left(-\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right) = \frac{1}{2} \tan^{-1}\left(-\frac{2 \times 10^6 + 3 \times 10^6}{2(4 \times 10^6)}\right) = 16.00^\circ \]
Macromechanical Analysis of a Lamina

8:

\[ \varepsilon_{\text{max, min}} = \frac{\varepsilon_x + \varepsilon_y}{2} \pm \sqrt{ \left( \frac{\varepsilon_x - \varepsilon_y}{2} \right)^2 + \left( \frac{\gamma_{xy}}{2} \right)^2 } \]

\[ = \frac{0.5534 \times 10^{-4} + 0.3078 \times 10^{-3}}{2} \pm \sqrt{ \left( \frac{0.5534 \times 10^{-4} + 0.3078 \times 10^{-3}}{2} \right)^2 + \left( \frac{0.532 \times 10^{-3}}{2} \right)^2 } \]

\[ = 1.962 \times 10^{-4}, -4.486 \times 10^{-4} \]

**Note:** The axes of principal normal stresses and principal normal strains do not match, unlike in isotropic materials.

9:

\[ \gamma_{\text{max}} = \sqrt{(\varepsilon_x - \varepsilon_y)^2 + \gamma_{xy}^2} = \sqrt{(0.5534 \times 10^{-4} + 0.3078 \times 10^{-3})^2 + (0.532 \times 10^{-3})^2} = 6.448 \times 10^{-4} \]

\[ \theta_s = \frac{1}{2} \tan^{-1} \left( -\frac{\varepsilon_x - \varepsilon_y}{\gamma_{xy}} \right) = -17.14^\circ \]
Macromechanical Analysis of a Lamina

Example 7-2: A uniaxial load is applied to a $10^\circ$ ply. The linear stress–strain curve along the line of load is related as $\sigma_x = 123\varepsilon_x$, where the stress is measured in GPa and strain in m/m. Given $E_1 = 180 \text{ GPa}$, $E_2 = 10 \text{ GPa}$ and $\nu_{12} = 0.25$, find the value of

1. shear modulus, $G_{12}$
2. modulus $E_x$ for a $60^\circ$ ply.

Solution:
8. Strength and Failure Criteria for An Orthotropic Lamina

- A successful design of a structure requires efficient and safe use of materials.
- Theories need to be developed to compare the state of stress in a material to failure criteria.
- It should be noted that failure theories are only stated and their application is validated by experiments.
- The first step in such a definition is the establishment of allowable stresses or strengths in the principal material directions.

- For a lamina stressed in its own plane, there are five fundamental strengths:

  \[ \begin{align*}
  X_t &= \text{axial or longitudinal strength in tension} \\
  X_c &= \text{axial or longitudinal strength in compression} \\
  Y_t &= \text{transverse strength in tension} \\
  Y_c &= \text{transverse strength in compression} \\
  S &= \text{shear strength}
  \end{align*} \]
Macromechanical Analysis of a Lamina

$\sigma_{ult}^T = \text{Ultimate longitudinal tensile strength (in direction 1)},$
$\sigma_{ult}^C = \text{Ultimate longitudinal compressive strength (in direction 1)},$
$\sigma_{ult}^T = \text{Ultimate transverse tensile strength (in direction 2)},$
$\sigma_{ult}^C = \text{Ultimate transverse compressive strength (in direction 2)},$ and
$\tau_{ult} = \text{Ultimate in-plane shear strength (in plane 12)}.$

**Experimental Determination of Strength and Stiffness:**

- $E_1 = \text{Young's modulus in the 1-direction}$
- $E_2 = \text{Young's modulus in the 2-direction}$
- $\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1}$ for $\sigma_1 = \sigma$ and all other stresses are zero
- $\nu_{21} = -\frac{\varepsilon_1}{\varepsilon_2}$ for $\sigma_2 = \sigma$ and all other stresses are zero
- $G_{12} = \text{shear modulus in 1-2 coordinates}$

**Strength**
- $X = \text{axial or longitudinal strength (1-direction)}$
- $Y = \text{transverse strength (2-direction)}$
- $S = \text{shear strength (1-2 coordinates)}$
Macromechanical Analysis of a Lamina

Measuring $E_1$ & $V_{12}$ & $X$:
Uniaxial tension loading in the 1-direction

\[ \sigma_1 = \frac{P}{A} \quad E_1 = \frac{\sigma_1}{\varepsilon_1} \quad \nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} \quad X = \frac{P_{\text{ult}}}{A} \]

ASTM D 638 tension test specimen
Macromechanical Analysis of a Lamina

Measuring $E_2$ & $V_{21}$ & $Y$:
Uniaxial tension loading in the 2-direction

\[
\sigma_2 = \frac{P}{A} \quad E_2 = \frac{\sigma_2}{\varepsilon_2} \quad V_{21} = -\frac{\varepsilon_1}{\varepsilon_2} \quad Y = \frac{P_{\text{ult}}}{A}
\]

\[
\frac{V_{12}}{E_1} = \frac{V_{21}}{E_2}
\]
Macromechanical Analysis of a Lamina

Measuring $G_{12}$:
Uniaxial tension loading at $45^\circ$ to the 1-direction

$$E_x = \frac{P}{A \varepsilon_x}$$

$$\frac{1}{E_x} = \frac{1}{4} \left[ \frac{1}{E_1} - \frac{2v_{12}}{E_1} + \frac{1}{G_{12}} + \frac{1}{E_2} \right]$$

$G_{12} = \frac{4}{E_x} - \frac{1}{E_1} - \frac{1}{E_2} + \frac{2v_{12}}{E_1}$
Macromechanical Analysis of a Lamina

Measuring $G_{12}$ & $S$:
Torsion tube test

\[
\tau_{12} = \frac{T}{2\pi r^2 t} \quad S = \tau_{12,ut} = \frac{T_{ult}}{2\pi r^2 t}
\]

\[
G_{12} = \frac{\tau_{12}}{\gamma_{12}}
\]

Sandwich cross-beam test
(By Shockey and described by Waddoups)

Rail shear test
(By Whitney, Stansbarger, & Howell)
“It’s widely used in aerospace industry”
The stresses acting on a lamina are resolved into the normal and shear stresses in the local axes. Failure is predicted in a lamina, if any of the normal or shear stresses in the local axes of a lamina is equal to or exceeds the corresponding ultimate strengths of the unidirectional lamina. Given the stresses or strains in the global axes of a lamina, one can find the stresses in the material axes by using Equation (7-1). Theses 5 criteria act independently and don’t interact with the others.
Macromechanical Analysis of a Lamina

**Example 8-1:** Find the max. value of $S > 0$ if a stress of $\sigma_x = 2S$, $\sigma_y = -3S$, and $\tau_{xy} = 4S$ is applied to the 60° lamina of graphite/epoxy. Use maximum stress failure theory and the properties of a unidirectional graphite/epoxy lamina given in Table of page 97.

**Solution:**

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
= \begin{bmatrix}
0.2500 & 0.7500 & 0.8660 \\
0.7500 & 0.2500 & -0.8660 \\
-0.4330 & 0.4330 & -0.5000
\end{bmatrix}
\begin{bmatrix}
2S \\
-3S \\
4S
\end{bmatrix}
= \begin{bmatrix}
0.1714 \times 10^1 \\
-0.2714 \times 10^1 \\
-0.4165 \times 10^1
\end{bmatrix}
\]

\[
(\sigma_1^T)_{ult} = 1500 \text{ MPa} \quad (\sigma_2^T)_{ult} = 40 \text{ MPa} \quad (\tau_{12})_{ult} = 68 \text{ MPa}
\]

\[
\begin{align*}
-1500 \times 10^6 < & 0.1714 \times 10^1 S < 1500 \times 10^6 \\
-875.1 \times 10^6 < & S < 875.1 \times 10^6 \\
-246 \times 10^6 < & -0.2714 \times 10^1 S < 40 \times 10^6 \\
-14.73 \times 10^6 < & S < 90.64 \times 10^6 \\
-68 \times 10^6 < & -0.4165 \times 10^1 S < 68 \times 10^6 \\
-16.33 \times 10^6 < & S < 16.33 \times 10^6
\end{align*}
\]

- The preceding inequalities also show that the angle lamina will fail in shear.
- The maximum stress that can be applied before failure is:

\[
\sigma_x = 32.66 \text{ MPa}, \sigma_y = -48.99 \text{ MPa}, \tau_{xy} = 65.32 \text{ MPa}
\]
Macromechanical Analysis of a Lamina

**Strength Ratio:**
- In a failure theory such as the maximum stress failure theory of Section, it can be determined whether a lamina has failed if any of the inequalities of Equation (8-1) are violated.
- However, this does not give the information about how much the load can be increased if the lamina is safe or how much the load should be decreased if the lamina has failed.

\[
SR = \frac{\text{Maximum Load Which Can Be Applied}}{\text{Load Applied}}
\]
Macromechanical Analysis of a Lamina

**Examples 8-2:** Assume that one is applying a load of \( \sigma_x = 2 \text{MPa}, \sigma_y = -3 \text{MPa}, \tau_{xy} = 4 \text{MPa} \) to a 60° angle lamina of graphite/epoxy. Find the strength ratio using the maximum stress failure theory.

**Solution:**
If the strength ratio is \( R \), then the maximum stress that can be applied is
\[
\sigma_x = 2R, \quad \sigma_y = -3R, \quad \tau_{xy} = 4R
\]
Following Example 8-1 for finding the local stresses gives
\[
\sigma_1 = 0.1714 \times 10 R, \quad \sigma_2 = -0.2714 \times 10 R, \quad \tau_{12} = -0.4165 \times 10 R
\]
Using the maximum stress failure theory as given by Equation (8-1) yields
\[ R = 16.33 \]
Thus, the load that can be applied just before failure is
\[
\sigma_x = 16.33 \times 2 = 32.66 \text{ MPa}, \quad \sigma_y = 16.33 \times (-3) = -48.99 \text{ MPa}, \quad \tau_{xy} = 16.33 \times 4 = 65.32 \text{ MPa}
\]

Note that all the components of the stress vector must be multiplied by the strength ratio.
If \( SR > 1 \): then the lamina is safe and the applied stress can be increased by a factor of \( SR \).
If \( SR < 1 \): the lamina is unsafe and the applied stress needs to be reduced by a factor of \( SR \).
\( SR = 1 \): implies the failure load.
Macromechanical Analysis of a Lamina

Failure Envelopes:
A failure envelope is a three-dimensional plot of the combinations of the normal and shear stresses that can be applied to an angle lamina just before failure.

Drawing three dimensional graphs can be time consuming: developing failure envelopes for constant shear stress $\tau_{xy}$ (2 normal stresses $\sigma_x$ and $\sigma_y$ as the two axes): Then, if the applied stress is within the failure envelope, the lamina is safe; otherwise, it has failed.

Failure envelope for the 60° lamina of graphite/epoxy for a constant shear stress of $\tau_{xy} = 24$ Mpa (Example p.145, Kaw)
Macromechanical Analysis of a Lamina

- Maximum Strain Failure Theory:
  - This theory is based on the maximum normal strain theory by St. Venant and the maximum shear stress theory by Tresca as applied to isotropic materials.
  - The strains applied to a lamina are resolved to strains in the local axes.
  - Failure is predicted in a lamina, if any of the normal or shearing strains in the local axes of a lamina equal or exceed the corresponding ultimate strains of the unidirectional lamina.

\[
\begin{align*}
- (\varepsilon_1^C)_{\text{ult}} &< \varepsilon_1 < (\varepsilon_1^T)_{\text{ult}}, \text{ or} \\
- (\varepsilon_2^C)_{\text{ult}} &< \varepsilon_2 < (\varepsilon_2^T)_{\text{ult}}, \text{ or} \\
- (\gamma_{12})_{\text{ult}} &< \gamma_{12} < (\gamma_{12})_{\text{ult}}
\end{align*}
\]

- \((\varepsilon_1^T)_{\text{ult}}\) = ultimate longitudinal tensile strain (in direction 1)
- \((\varepsilon_1^C)_{\text{ult}}\) = ultimate longitudinal compressive strain (in direction 1)
- \((\varepsilon_2^T)_{\text{ult}}\) = ultimate transverse tensile strain (in direction 2)
- \((\varepsilon_2^C)_{\text{ult}}\) = ultimate transverse compressive strain (in direction 2)
- \((\gamma_{12})_{\text{ult}}\) = ultimate in-plane shear strain (in plane 1-2)

**Note:** In fact, if the Poisson’s ratio is zero in the unidirectional lamina, the two failure theories will give identical results.
Macromechanical Analysis of a Lamina

**Example 8-3:** Find the maximum value of \( S>0 \) if a stress, \( \sigma_x=2S \), \( \sigma_y=-3S \), and \( \tau_{xy}=4S \), is applied to a 60° graphite/epoxy lamina. Use maximum strain failure theory. Use the properties of the graphite/epoxy unidirectional lamina given in Table p.97

**Solution:**
Considering “[S]” and “local stresses” from example 2.6 (Kaw) and example 8-1 (here):

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}
\end{bmatrix}
= [S]
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix}
= \begin{bmatrix}
0.5525 \times 10^{-11} & -0.1547 \times 10^{-11} & 0 \\
-0.1547 \times 10^{-11} & 0.9709 \times 10^{-10} & 0 \\
0 & 0 & 0.1395 \times 10^{-9}
\end{bmatrix}
\begin{bmatrix}
0.1714 \times 10^1 \\
-0.2714 \times 10^1 \\
0.4165 \times 10^1
\end{bmatrix}
\begin{bmatrix}
0.1367 \times 10^{-10} \\
-0.2662 \times 10^{-9} \\
-0.5809 \times 10^{-9}
\end{bmatrix}
\]
Macromechanical Analysis of a Lamina

Assume a linear relationship between all the stresses and strains until failure; then the ultimate failure strains are

\[
(e^T_1)_{ult} = \frac{(\sigma^T_1)_{ult}}{E_1} = \frac{1500 \times 10^6}{181 \times 10^9} = 8.287 \times 10^{-3} \\
(e^T_2)_{ult} = \frac{(\sigma^T_2)_{ult}}{E_2} = \frac{40 \times 10^6}{10.3 \times 10^9} = 3.883 \times 10^{-3}
\]

\[
(e^C_1)_{ult} = \frac{(\sigma^C_1)_{ult}}{E_1} = \frac{1500 \times 10^6}{181 \times 10^9} = 8.287 \times 10^{-3} \\
(e^C_2)_{ult} = \frac{(\sigma^C_2)_{ult}}{E_2} = \frac{246 \times 10^6}{10.3 \times 10^9} = 2.388 \times 10^{-2}
\]

\[
(\gamma_{12})_{ult} = \frac{(\tau_{12})_{ult}}{G_{12}} = \frac{68 \times 10^6}{7.17 \times 10^6} = 9.483 \times 10^{-3}
\]

Using the inequalities (8-2) and recognizing that \(S > 0\):

\[
-8.287 \times 10^{-3} < 0.1367 \times 10^{-10} S < 8.287 \times 10^{-3} \\
-2.388 \times 10^{-2} < -0.2662 \times 10^{-9} S < 3.883 \times 10^{-3} \\
-9.483 \times 10^{-3} < -0.5809 \times 10^{-9} S < 9.483 \times 10^{-3}
\]

\[
-606.2 \times 10^6 < S < 606.2 \times 10^6 \\
-14.58 \times 10^6 < S < 89.71 \times 10^6 \\
-16.33 \times 10^6 < S < 16.33 \times 10^6
\]

\[\Rightarrow \quad 0 < S < 16.33 \text{ MPa}\]
Macromechanical Analysis of a Lamina

- The maximum value of $S$ before failure is 16.33 MPa. The same maximum value of $S = 16.33$ MPa is also found using maximum stress failure theory.

- There is no difference between the two values because the mode of failure is shear. However, if the mode of failure were other than shear, a difference in the prediction of failure loads would have been present due to the Poisson’s ratio effect, which couples the normal strains and stresses in the local axes.

- Neither the maximum stress failure theory nor the maximum strain failure theory has any coupling among the five possible modes of failure.

- The following theories are based on the interaction failure theory:
  - Tsai-Hill
  - Tsai-Wu
Macromechanical Analysis of a Lamina

- **Tsai–Hill Failure Theory**

- This theory is based on the distortion energy failure theory of Von-Mises’ distortional energy yield criterion for isotropic materials as applied to anisotropic materials.

- The strain energy in a body consists of two parts:
  1. Due to a change in volume and is called the dilation energy
  2. Due to a change in shape and is called the distortion energy

- It is assumed that failure in the material takes place only when the distortion energy is greater than the failure distortion energy of the material.

“Hill” adopted the Von-Mises’ distortional energy yield criterion to anisotropic materials. “Tsai” adopted it to a unidirectional lamina. Based on the distortion energy theory, he proposed that a lamina has failed if below inequality is violated:

\[
(G_2 + G_3)\sigma_1^2 + (G_1 + G_3)\sigma_2^2 + (G_1 + G_2)\sigma_3^2 - 2G_3\sigma_1\sigma_2 - 2G_2\sigma_1\sigma_3 - 2G_1\sigma_2\sigma_3 + 2G_4\tau_{23}^2 + 2G_5\tau_{13}^2 + 2G_6\tau_{12}^2 < 1
\]
Macromechanical Analysis of a Lamina

The components $G_1$, $G_2$, $G_3$, $G_4$, $G_5$, and $G_6$ of the strength criterion depend on the failure strengths and are found as follows:

1. Apply $\sigma_1 = (\sigma_{1}^{T})_{ult}$ to a unidirectional lamina; then, the lamina will fail. Thus, Equation (8-3) reduces to

$$ (G_2 + G_3)(\sigma_{1}^{T})_{ult}^2 = 1 $$

2. Apply $\sigma_2 = (\sigma_{2}^{T})_{ult}$ to a unidirectional lamina; then, the lamina will fail. Thus, Equation (8-3) reduces to

$$ (G_1 + G_3)(\sigma_{2}^{T})_{ult}^2 = 1 $$

3. Apply $\sigma_3 = (\sigma_{2}^{T})_{ult}$ to a unidirectional lamina and, assuming that the normal tensile failure strength is same in directions (2) and (3), the lamina will fail. Thus, Equation (8-3) reduces to

$$ (G_1 + G_2)(\sigma_{2}^{T})_{ult}^2 = 1 $$

4. Apply $\tau_{12} = (\tau_{12})_{ult}$ to a unidirectional lamina; then, the lamina will fail. Thus, Equation (8-3) reduces to

$$ 2G_6(\tau_{12})_{ult}^2 = 1 $$
Macromechanical Analysis of a Lamina

From Equation (8-4) to Equation (8-7):

\[
G_1 = \frac{1}{2} \left( \frac{2}{[(\sigma_2^T)_{ult}]^2} - \frac{1}{[(\sigma_1^T)_{ult}]^2} \right) \quad G_2 = \frac{1}{2} \left( \frac{1}{[(\sigma_1^T)_{ult}]^2} \right) \quad G_3 = \frac{1}{2} \left( \frac{1}{[(\sigma_1^T)_{ult}]^2} \right) \quad G_6 = \frac{1}{2} \left( \frac{1}{[(\tau_{12})_{ult}]^2} \right)
\]

Because the unidirectional lamina is assumed to be under plane stress ($\sigma_3=\tau_{31}=\tau_{23}=0$), then Equation (8-3) reduces through Equation (8-8) to:

\[
\left[ \frac{\sigma_1}{(\sigma_1^T)_{ult}} \right]^2 - \left[ \frac{\sigma_1 \sigma_2}{(\sigma_1^T)^2_{ult}} \right] + \left[ \frac{\sigma_2}{(\sigma_2^T)_{ult}} \right]^2 + \left[ \frac{\tau_{12}}{(\tau_{12})_{ult}} \right]^2 < 1
\]
Macromechanical Analysis of a Lamina

**Example 8-4:** Find the maximum value of $S > 0$ if a stress of $\sigma_x = 2S$, $\sigma_y = -3S$, and $\tau_{xy} = 4S$ is applied to a 60° graphite/epoxy lamina. Use Tsai–Hill failure theory. Use the unidirectional graphite/epoxy lamina properties given in Table slide 97.

**Solution:**
From example 8-1:

\[ \sigma_1 = 1.714S \quad \sigma_2 = -2.714S \quad \tau_{12} = -4.165S \]

Using the Tsai–Hill failure theory from Equation (8-9):

\[
\left( \frac{1.714S}{1500 \times 10^6} \right)^2 + \left( \frac{-2.714S}{1500 \times 10^6} \right)^2 + \left( \frac{-4.165S}{68 \times 10^6} \right)^2 < 1
\]

\[ S < 10.94 \text{ MPa} \]
Macromechanical Analysis of a Lamina

Remarks:
1. Unlike the maximum strain and maximum stress failure theories, the Tsai–Hill failure theory considers the interaction among the three unidirectional lamina strength parameters.

2. The Tsai–Hill failure theory does not distinguish between the compressive and tensile strengths in its equations. This can result in underestimation of the maximum loads that can be applied when compared to other failure theories. For the load of \( \sigma_x = 2 \text{ MPa}, \sigma_y = -3 \text{ MPa}, \) and \( \tau_{xy} = 4 \text{ MPa}, \) as found in Example 8-2, Example 8-3, and Example 8-4, the strength ratios are given by

\[
SR = 16.33 \text{ (maximum stress failure theory)}
\]

\[
SR = 16.33 \text{ (maximum strain failure theory)}
\]

\[
SR = 10.94 \text{ (Tsai–Hill failure theory)}
\]

Tsai–Hill failure theory underestimates the failure stress because the transverse tensile strength of a unidirectional lamina is generally much less than its transverse compressive strength and the compressive strengths are not used in the Tsai–Hill failure theory, but it can be modified to use corresponding tensile or compressive strengths in the failure theory as follows

\[
\left( \frac{\sigma_1}{X_1} \right)^2 - \left( \frac{\sigma_1}{X_2} \right)^2 + \left( \frac{\sigma_2}{Y} \right)^2 + \left( \frac{\tau_{12}}{S} \right)^2 < 1
\]

\[
\begin{align*}
\left( \frac{\sigma_1}{(\sigma_1)_{ult}} \right)^2 - \left( \frac{\sigma_1}{(\sigma_2)_{ult}} \right)^2 + \left( \frac{\sigma_2}{(\tau_{12})_{ult}} \right)^2 < 1
\end{align*}
\]
Macromechanical Analysis of a Lamina

Where:

\[ X_1 = (\sigma_1^T)_{ult} \text{ if } \sigma_1 > 0 \]
\[ X_2 = (\sigma_1^T)_{ult} \text{ if } \sigma_2 > 0 \]
\[ Y = (\sigma_2^T)_{ult} \text{ if } \sigma_2 > 0 \]
\[ S = (\tau_{12})_{ult} \]
\[ = (\sigma_1^C)_{ult} \text{ if } \sigma_1 < 0; \]
\[ = (\sigma_1^C)_{ult} \text{ if } \sigma_2 < 0; \]
\[ = (\sigma_2^C)_{ult} \text{ if } \sigma_2 < 0 \]

For Example 8-4, the modified Tsai–Hill failure theory given by Equation (2.10) now gives:

\[
\left( \frac{1.714\sigma}{1500 \times 10^6} \right)^2 - \left( \frac{1.714\sigma}{1500 \times 10^6} \right) \left( \frac{-2.714\sigma}{1500 \times 10^6} \right) + \left( \frac{-2.714\sigma}{246 \times 10^6} \right)^2 + \left( \frac{-4.165\sigma}{68 \times 10^6} \right)^2 < 1
\]

\[ \sigma < 16.06 \text{ MPa}, \]

the strength ratio is \( SR = 16.06 \) (modified Tsai–Hill failure theory)
Macromechanical Analysis of a Lamina

3. The Tsai–Hill failure theory is a unified theory and thus does not give the mode of failure like the maximum stress and maximum strain failure theories do. However, one can make a reasonable guess of the failure mode by calculating $\left| \sigma_1 / (\sigma_1)^{ult} \right|$, $\left| \sigma_2 / (\sigma_2)^{ult} \right|$ and $\left| \tau_{12} / (\tau_{12})^{ult} \right|$. The maximum of these three values gives the associated mode of failure.

In the modified Tsai–Hill failure theory, calculate the maximum of $|\sigma_1 / X_1|$, $|\sigma_2 / Y|$, and $|\tau_{12} / S|$ for the associated mode of failure.
Macromechanical Analysis of a Lamina

- **Tsai–Wu Failure Theory**
  - This failure theory is based on the total strain energy failure theory of Beltrami.
  - Tsai-Wu applied the failure theory to a lamina in plane stress.
  - A lamina is considered to be failed if below inequality is violated:

\[
H_1 \sigma_1 + H_2 \sigma_2 + H_6 \tau_{12} + H_{11} \sigma_1^2 + H_{22} \sigma_2^2 + H_{66} \tau_{12}^2 + 2H_{12} \sigma_1 \sigma_2 < 1
\]  

- This failure theory is **more general** than the Tsai–Hill failure theory because it distinguishes between the **compressive** and **tensile strengths** of a lamina.

- The components $H_1$, $H_2$, $H_6$, $H_{11}$, $H_{22}$, and $H_{66}$ of the failure theory are found using the five strength parameters of a unidirectional lamina as follows:

1- Apply $\sigma_1 = (\sigma_{1})_{ult}, \sigma_2 = 0, \tau_{12} = 0$ to a unidirectional lamina; the lamina will fail. Equation (8-11) reduces to:

\[
H_1 (\sigma_{1})_{ult} + H_{11} (\sigma_{1})_{ult}^2 = 1
\]
Macromechanical Analysis of a Lamina

2- Apply \( \sigma_1 = -(\sigma_1^C)_{ult} \), \( \sigma_2 = 0 \), \( \tau_{12} = 0 \) to a unidirectional lamina; the lamina will fail. Equation (8-11) reduces to

\[
-H_1(\sigma_1^C)_{ult} + H_{11}(\sigma_1^C)^2_{ult} = 1
\]  

From Equation (8-12) and (8-13):

\[
H_1 = \frac{1}{(\sigma_1^T)_{ult}} - \frac{1}{(\sigma_1^C)_{ult}}
\]  

\[
H_{11} = \frac{1}{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}
\]

3- Apply \( \sigma_1 = 0 \), \( \sigma_2 = (\sigma_2^T)_{ult} \), \( \tau_{12} = 0 \) to a unidirectional lamina; the lamina will fail. Equation (8-11) reduces to

\[
H_2(\sigma_2^T)_{ult} + H_{22}(\sigma_2^T)^2_{ult} = 1
\]
Macromechanical Analysis of a Lamina

4- Apply \( \sigma_1 = 0, \sigma_2 = -(\sigma_2^c)^{ult}, \tau_{12} = 0 \) to a unidirectional lamina; the lamina will fail. Equation (8-11) reduces to

\[
-H_2(\sigma_2^c)^{ult} + H_{22}(\sigma_2^c)^2 = 1
\]  

From Equation (8-16) and (8-17):

\[
H_2 = \frac{1}{(\sigma_2^T)^{ult}} - \frac{1}{(\sigma_2^C)^{ult}}
\]

\[
H_{22} = \frac{1}{(\sigma_2^T)^{ult}(\sigma_2^C)^{ult}}
\]

5- Apply \( \sigma_1 = 0, \sigma_2 = 0, \) and \( \tau_{12} = (\tau_{12})^{ult} \) to a unidirectional lamina; it will fail. Equation (8-11) reduces to

\[
H_6(\tau_{12})^{ult} + H_{66}(\tau_{12})^2 = 1
\]
6- Apply $\sigma_1 = 0$, $\sigma_2 = 0$, and $\tau_{12} = -(\tau_{12})_{ult}$ to a unidirectional lamina; the lamina will fail. Equation (8-11) reduces to

$$-H_6(\tau_{12})_{ult} + H_{66}(\tau_{12})_{ult}^2 = 1$$  \hspace{1cm} 8-21

From Equation (8-20) and (8-21):

$$H_6 = 0$$  \hspace{1cm} 8-22

$$H_{66} = \frac{1}{(\tau_{12})_{ult}^2}$$  \hspace{1cm} 8-23

And, from experiments:

$$H_{12} = -\frac{1}{2(\sigma_1^T)^2_{ult}}$$  \hspace{1cm} Tsai–Hill failure theory  \hspace{1cm} 8-24

$$H_{12} = -\frac{1}{2(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}}$$  \hspace{1cm} Hoffman criterion

$$H_{12} = \frac{1}{2\sqrt{(\sigma_1^T)_{ult}(\sigma_1^C)_{ult}(\sigma_2^T)_{ult}(\sigma_2^C)_{ult}}}$$  \hspace{1cm} Mises–Hendry criterion
Macromechanical Analysis of a Lamina

**Example 8-5**: Find the maximum value of $S > 0$ if a stress $\sigma_x=2S$, $\sigma_y=-3S$, and $T_{xy} = 4S$ are applied to a $60^\circ$ lamina of graphite/epoxy. Use Tsai–Wu failure theory.

**Solution:**
From example 8-1:

\[
\begin{align*}
\sigma_1 &= 1.714S & \sigma_2 &= -2.714S & \tau_{12} &= -4.165S \\
\end{align*}
\]

From above equations:

\[
\begin{align*}
H_1 &= \frac{1}{1500 \times 10^6} - \frac{1}{1500 \times 10^6} = 0 \text{ Pa}^{-1} & H_{66} &= \frac{1}{(68 \times 10^6)^2} = 2.1626 \times 10^{-16} \text{ Pa}^{-2} \\
H_2 &= \frac{1}{40 \times 10^6} - \frac{1}{246 \times 10^6} = 2.093 \times 10^{-8} \text{ Pa}^{-1} & H_6 &= 0 \text{ Pa}^{-1} \\
H_{11} &= \frac{1}{(1500 \times 10^6)(1500 \times 10^6)} = 4.4444 \times 10^{-19} \text{ Pa}^{-2} \\
H_{22} &= \frac{1}{(40 \times 10^6)(246 \times 10^6)} = 1.0162 \times 10^{-16} \text{ Pa}^{-2} \\
\end{align*}
\]
Macromechanical Analysis of a Lamina

Using the Mises–Hencky criterion for evaluation of $H_{12}$:

$$H_{12} = -\frac{1}{2} \sqrt{\frac{1}{(1500 \times 10^6)(1500 \times 10^6)(40 \times 10^6)(246 \times 10^6)}} = 3.360 \times 10^{-18} Pa^{-2}$$

Substituting these values in Equation (8-11):

$$S < 22.39 \text{ MPa}$$

Summarizing the four failure theories:

- $S = 16.33$ (maximum stress failure theory)
- $S = 16.33$ (maximum strain failure theory)
- $S = 10.94$ (Tsai–Hill failure theory)
- $S = 16.06$ (modified Tsai–Hill failure theory)
- $S = 22.39$ (Tsai–Wu failure theory)
Macromechanical Analysis of a Lamina

Problems Chapter 2:

1. Derive Equation (7-6).

2. Prove $[R][T][R]^{-1} = [T]^{-T}$

3. Derive Equation (7-8).

4. Prove $[R][T]^{-1}[R]^{-1} = [T]^{T}$

5. Plot the apparent engineering constants $E_x$, $E_y$, $G_{xy}$, $V_{xy}$, $\eta_{xy,y}$ & $\eta_{xy,x}$ as functions of $\theta$ from $\theta=0$ to $\theta=90$ for high-modulus graphite-epoxy, an orthotropic material with $E_1=30 \times 10^6$ psi (207 GPa), $E_2=0.75 \times 10^6$ psi (5.2 GPa), $G_{12}=0.375 \times 10^6$ psi (2.59 GPa), and $V_{12}=0.25$.

6. Find the engineering constants of a 60° graphite/epoxy lamina. Use the properties of a unidirectional graphite/epoxy lamina from Table of slide 95.
Macromechanical Analysis of a Lamina

7. A 60° angle graphite/epoxy lamina is subjected only to a shear stress \( \tau_{xy} = 2 \text{ MPa in the global axes.} \) What would be the value of the strains measured by the strain gage rosette — that is, what would be the normal strains measured by strain gages A, B, and C? Use the properties of unidirectional graphite/epoxy lamina from table of slide 97.

8. What are the values of stiffness matrix elements \( C_{11} \) and \( C_{12} \) in terms of the Young’s modulus and Poisson’s ratio for an isotropic material?
9. Consider an orthotropic material with the stiffness matrix given by

\[
[C] = \begin{bmatrix}
-0.67308 & -1.8269 & -1.0577 & 0 & 0 & 0 \\
-1.8269 & -0.67308 & -1.4423 & 0 & 0 & 0 \\
-1.0577 & -1.4423 & 0.48077 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1.5 \\
\end{bmatrix} \text{ GPa}
\]

Find:

a. The stresses in the principal directions of symmetry if the strains in the principal directions of symmetry at a point in the material are \( \varepsilon_1 = 1 \ \mu m/m, \varepsilon_2 = 3 \ \mu m/m, \varepsilon_3 = 2 \ \mu m/m; \gamma_{23} = 0, \gamma_{31} = 5 \ \mu m/m, \gamma_{12} = 6 \ \mu m/m \)

b. The compliance matrix \([S]\)

c. The engineering constants \(E_1, E_2, E_3, \nu_{12}, \nu_{23}, \nu_{31}, G_{12}, G_{23}, G_{31}\)

10. Find the strains in the 1–2 coordinate system (local axes) in a unidirectional boron/epoxy lamina, if the stresses in the 1–2 coordinate system applied to are \( \sigma_1 = 4 \ \text{MPa}, \sigma_2 = 2 \ \text{MPa}, \) and \( \tau_{12} = -3 \ \text{MPa}. \) Use the properties of a unidirectional boron/epoxy lamina from Table of slide 97.
10. Find the reduced stiffness $[Q]$ and the compliance $[S]$ matrices for a unidirectional lamina of boron/epoxy. Use the properties of a unidirectional boron/epoxy lamina from Table page 97.

11. The reduced stiffness matrix $[Q]$ is given for a unidirectional lamina is given as follows:

$$[Q] = \begin{bmatrix} 5.681 & 0.3164 & 0 \\ 0.3164 & 1.217 & 0 \\ 0 & 0 & 0.6006 \end{bmatrix} \text{ Msi}$$

What are the four engineering constants, $E_1, E_2, \nu_{12}, \text{ and } G_{12}$, of the lamina?

12. Find the off-axis shear strength and mode of failure of a 60° boron/epoxy lamina. Use the properties of a unidirectional boron/epoxy lamina from Table page 97. Apply the maximum stress failure and maximum strain failure theories.
13. Find the maximum biaxial stress, $\sigma_x = -\sigma$, $\sigma_y = -\sigma$, $\sigma > 0$, that one can apply to a 60° lamina of graphite/epoxy. Use the properties of a unidirectional graphite/epoxy lamina from Table of page 97. Use maximum strain failure theories.
• Concepts of volume and weight fraction
• Elastic moduli for the composite
Micromechanical Analysis of a Lamina

1. Introduction, Definition:

• Macromechanics of a lamina discussed about the “apparent” properties of a lamina.
• In previous chapter, a large enough piece of the lamina was considered so that the fact that the lamina is made of two or more constituent materials cannot be detected.
• In previous chapter, we were able to say that a unidirectional lamina (boron fibers in epoxy) has certain stiffnesses and strengths that we measured in various directions. but, we didn’t find out how can the stiffnesses & strengths of a boron-epoxy composite material be varied by changing the proportion of boron fibers to epoxy matrix?

Basic Question of Micromechanics
Micromechanical Analysis of a Lamina

Unlike in isotropic materials, experimental evaluation of parameters for unidirectional lamina is quite costly and time consuming because they are functions of several variables like:
• The individual constituents of the composite material
• Fiber volume fraction
• Packing geometry
• Processing
• Etc

Thus

“The need and motivation for developing analytical models to find these parameters are very important”
Micromechanical Analysis of a Lamina

Some important definitions:

• **Volume fraction:**

Consider a composite consisting of fiber and matrix:

\[ V_f = \frac{v_f}{v_c} \quad 1-1 \]

\[ V_m = \frac{v_m}{v_c} \quad 1-2 \]

\[ V_f + V_m = 1 \quad 1-3 \]

\[ v_f + v_m = v_c \quad 1-4 \]

• **Mass fraction (weight fraction):**

\[ w_{c,f,m} = \text{mass of composite, fiber, and matrix, respectively} \]
Micromechanical Analysis of a Lamina

mass fraction of the fibers

\[ W_f = \frac{w_f}{w_c} \quad 1-5 \]

\[ w_f + w_m = w_c \]

mass fraction of the matrix

\[ W_m = \frac{w_m}{w_c} \quad 1-6 \]

\[ W_m = \frac{\rho_m}{\rho_c} V_m \quad 1-8 \]

\[ W_f = \frac{\rho_f}{\rho_c} V_f \]

\[ W_f + W_m = 1 \quad 1-7 \]

\[ \rho_c V_c = \rho_f V_f + \rho_m V_m \quad 1-9 \]

\[ \rho_c = \rho_f \frac{V_f}{V_c} + \rho_m \frac{V_m}{V_c} \quad 1-10 \]

\[ \rho_c = \rho_f V_f + \rho_m V_m \quad 1-11 \]

\[ \frac{1}{\rho_c} = \frac{W_f}{\rho_f} + \frac{W_m}{\rho_m} \quad 1-12 \]
### Micromechanical Analysis of a Lamina

Table 3-1

Typical Properties of Fibers (SI System of Units)

<table>
<thead>
<tr>
<th>Property</th>
<th>Units</th>
<th>Graphite</th>
<th>Glass</th>
<th>Aramid</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial modulus</td>
<td>GPa</td>
<td>230</td>
<td>85</td>
<td>124</td>
</tr>
<tr>
<td>Transverse modulus</td>
<td>GPa</td>
<td>22</td>
<td>85</td>
<td>8</td>
</tr>
<tr>
<td>Axial Poisson’s ratio</td>
<td>—</td>
<td>0.30</td>
<td>0.20</td>
<td>0.36</td>
</tr>
<tr>
<td>Transverse Poisson’s ratio</td>
<td>—</td>
<td>0.35</td>
<td>0.20</td>
<td>0.37</td>
</tr>
<tr>
<td>Axial shear modulus</td>
<td>GPa</td>
<td>22</td>
<td>35.42</td>
<td>3</td>
</tr>
<tr>
<td>Axial coefficient of thermal expansion</td>
<td>μm/m/°C</td>
<td>−1.3</td>
<td>5</td>
<td>−5.0</td>
</tr>
<tr>
<td>Transverse coefficient of thermal expansion</td>
<td>μm/m/°C</td>
<td>7.0</td>
<td>5</td>
<td>4.1</td>
</tr>
<tr>
<td>Axial tensile strength</td>
<td>MPa</td>
<td>2067</td>
<td>1550</td>
<td>1379</td>
</tr>
<tr>
<td>Axial compressive strength</td>
<td>MPa</td>
<td>1999</td>
<td>1550</td>
<td>276</td>
</tr>
<tr>
<td>Transverse tensile strength</td>
<td>MPa</td>
<td>77</td>
<td>1550</td>
<td>7</td>
</tr>
<tr>
<td>Transverse compressive strength</td>
<td>MPa</td>
<td>42</td>
<td>1550</td>
<td>7</td>
</tr>
<tr>
<td>Shear strength</td>
<td>MPa</td>
<td>36</td>
<td>35</td>
<td>21</td>
</tr>
<tr>
<td>Specific gravity</td>
<td>—</td>
<td>1.8</td>
<td>2.5</td>
<td>1.4</td>
</tr>
</tbody>
</table>
Micromechanical Analysis of a Lamina

Table 3-2

Typical Properties of Matrices (SI System of Units)

<table>
<thead>
<tr>
<th>Property</th>
<th>Units</th>
<th>Epoxy</th>
<th>Aluminum</th>
<th>Polyamide</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axial modulus</td>
<td>GPa</td>
<td>3.4</td>
<td>71</td>
<td>3.5</td>
</tr>
<tr>
<td>Transverse modulus</td>
<td>GPa</td>
<td>3.4</td>
<td>71</td>
<td>3.5</td>
</tr>
<tr>
<td>Axial Poisson’s ratio</td>
<td>—</td>
<td>0.30</td>
<td>0.30</td>
<td>0.35</td>
</tr>
<tr>
<td>Transverse Poisson’s ratio</td>
<td>—</td>
<td>0.30</td>
<td>0.30</td>
<td>0.35</td>
</tr>
<tr>
<td>Axial shear modulus</td>
<td>GPa</td>
<td>1.308</td>
<td>27</td>
<td>1.3</td>
</tr>
<tr>
<td>Coefficient of thermal expansion</td>
<td>$\mu m/m/^{\circ}C$</td>
<td>63</td>
<td>23</td>
<td>90</td>
</tr>
<tr>
<td>Coefficient of moisture expansion</td>
<td>m/m/kg/kg</td>
<td>0.33</td>
<td>0.00</td>
<td>0.33</td>
</tr>
<tr>
<td>Axial tensile strength</td>
<td>MPa</td>
<td>72</td>
<td>276</td>
<td>54</td>
</tr>
<tr>
<td>Axial compressive strength</td>
<td>MPa</td>
<td>102</td>
<td>276</td>
<td>108</td>
</tr>
<tr>
<td>Transverse tensile strength</td>
<td>MPa</td>
<td>72</td>
<td>276</td>
<td>54</td>
</tr>
<tr>
<td>Transverse compressive strength</td>
<td>MPa</td>
<td>102</td>
<td>276</td>
<td>108</td>
</tr>
<tr>
<td>Shear strength</td>
<td>MPa</td>
<td>34</td>
<td>138</td>
<td>54</td>
</tr>
<tr>
<td>Specific gravity</td>
<td>—</td>
<td>1.2</td>
<td>2.7</td>
<td>1.2</td>
</tr>
</tbody>
</table>
Micromechanical Analysis of a Lamina

**Example 1:** A glass/epoxy lamina consists of a 70% fiber volume fraction. Use properties of glass and epoxy from Table 3.1 & Table 3.2, respectively, to determine the:
1. Density of lamina
2. Mass fractions of the glass and epoxy
3. Volume of composite lamina if the mass of the lamina is 4 kg
4. Volume and mass of glass and epoxy in part (3)

**Solution:**

1. Table 3-1
   \[ \rho_f = 2500 \text{ kg} / \text{m}^3 \]

2. Table 3-2
   \[ \rho_m = 1200 \text{ kg} / \text{m}^3 \]

3. \[ \rho_c = (2500)(0.7) + (1200)(0.3) = 2110 \text{ kg} / \text{m}^3 \]
Micromechanical Analysis of a Lamina

2. 

\[ W_f = \frac{\omega_f}{\omega_c} = \frac{\rho_f}{\rho_c} V_f \quad \Rightarrow \quad W_f = \frac{2500}{2110} \times 0.7 = 0.8294 \]

\[ W_m = \frac{\omega_m}{\omega_c} = \frac{\rho_m}{\rho_c} V_m \quad \Rightarrow \quad W_m = \frac{1200}{2110} \times 0.3 = 0.1706 \]

\[ W_f + W_m = 0.8294 + 0.1706 = 1.000 \]

3. 

\[ \nu_c = \frac{\omega_c}{\rho_c} = \frac{4}{2110} = 1.896 \times 10^{-3} m^3 \]

4. 

\[ \nu_f = V_f \nu_c = (0.7)(1.896 \times 10^{-3}) = 1.327 \times 10^{-3} m^3 \quad \Rightarrow \quad w_f = \rho_f \nu_f = 3.318 \text{ kg} \]

\[ \nu_m = V_m \nu_c = (0.3)(0.1896 \times 10^{-3}) = 0.5688 \times 10^{-3} \text{ m}^3 \quad \Rightarrow \quad w_m = \rho_m \nu_m = 0.6826 \text{ kg} \]
2. Evaluation of the Four Elastic Moduli:

there are four elastic moduli of a unidirectional lamina:

- Longitudinal Young’s modulus, $E_1$
- Transverse Young’s modulus, $E_2$
- Major Poisson’s ratio, $\nu_{12}$
- In-plane shear modulus, $G_{12}$
Micromechanical Analysis of a Lamina

The objective of all micromechanics approaches is to determine the elastic moduli or stiffnesses or compliances of a composite material in terms of the elastic moduli of the constituent materials:

$$C_{ij} = C_{ij}(E_f, \nu_f, V_f, E_m, \nu_m, V_m)$$

Three approaches to determine the four elastic moduli:

A- Strength of materials approach

B-Semi-empirical models

C-Elasticity approach
Micromechanical Analysis of a Lamina

3. Strength of materials approach:

- From a unidirectional lamina, take a representative volume element that consists of the fiber surrounded by the matrix.
- The fiber, matrix, and the composite are assumed to be of the same width, $h$, but of thicknesses $t_f$, $t_m$, and $t_c$, respectively.
- The area of the fiber, the matrix, & the composite:

$$
\begin{align*}
A_f &= t_f h \\
A_m &= t_m h \\
A_c &= t_c h
\end{align*}
$$

$$
\begin{align*}
V_f &= \frac{A_f}{A_c} = \frac{t_f}{t_c} \\
V_m &= \frac{A_m}{A_c} = \frac{t_m}{t_c} = 1 - V_f
\end{align*}
$$

3-1
Micromechanical Analysis of a Lamina

The following assumptions are made in the strength of materials approach model:

- The bond between fibers and matrix is perfect.
- The elastic moduli, diameters, and space between fibers are uniform.
- The fibers are continuous and parallel.
- The fibers and matrix follow Hook’s law (linearly elastic).
- The fibers possess uniform strength.
- The composite is free of voids.
Micromechanical Analysis of a Lamina

3.1. Longitudinal Young’s Modulus

Assuming that the fibers, matrix, and composite follow Hooke’s law and that the fibers and the matrix are isotropic, the stress–strain relationship for each component and the composite is:

\[ \sigma_c = E_1 \varepsilon_c \quad \sigma_f = E_f \varepsilon_f \quad \sigma_m = E_m \varepsilon_m \]  (3-4)

(3-3) & (3-4) in (3-2)

\[ E_1 \varepsilon_c A_c = E_f \varepsilon_f A_f + E_m \varepsilon_m A_m \]

\[ \varepsilon_c = \varepsilon_f = \varepsilon_m \]

\[ E_1 = E_f \frac{A_f}{A_c} + E_m \frac{A_m}{A_c} \]  (3-6)
Micromechanical Analysis of a Lamina

\[ E_1 = E_f V_f + E_m V_m \]

The rule of mixtures

Fraction of load of composite carried by fibers as a function of fiber volume fraction for constant fiber to matrix moduli ratio.
Micromechanical Analysis of a Lamina

Example 2: Find the longitudinal elastic modulus of a unidirectional glass/epoxy lamina with a 70% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively. Also, find the ratio of the load taken by the fibers to that of the composite.

Solution:

From Table 3.1 & 3.2: 

\[ E_f = 85 \text{ GPa} \quad E_m = 3.4 \text{ GPa} \]

From Eq. (3-7):

\[ E_1 = (85)(0.7) + (3.4)(0.3) = 60.52 \text{ GPa} \]

From Eq. (3-8):

\[ \frac{F_f}{F_c} = \frac{85}{60.52}(0.7) = 0.9831 \]
Micromechanical Analysis of a Lamina

Longitudinal Young’s modulus as function of fiber volume fraction and comparison with experimental data points for a typical glass/polyester lamina. (Experimental data points reproduced with permission of ASM International.)
Micromechanical Analysis of a Lamina

3.2. Transverse Young’s Modulus

\[ \sigma_c = \sigma_f = \sigma_m \] \hspace{1cm} (3-9)

\[ \Delta_c = \Delta_f + \Delta_m \] \hspace{1cm} (3-10)

\[ \Delta_c = t_c \varepsilon_c \hspace{1cm} \Delta_f = t_f \varepsilon_f \hspace{1cm} \Delta_m = t_m \varepsilon_m \] \hspace{1cm} (3-11)

by using Hook’s law

\[ \varepsilon_c = \frac{\sigma_c}{E_c} \hspace{1cm} \varepsilon_f = \frac{\sigma_f}{E_f} \hspace{1cm} \varepsilon_m = \frac{\sigma_m}{E_m} \] \hspace{1cm} (3-12)

(3-11) & (3-12) in (3-10) & using (3-9)

\[ \frac{1}{E_2} = \frac{1}{E_f} \frac{t_f}{t_c} + \frac{1}{E_m} \frac{t_m}{t_c} = \frac{V_f}{E_f} + \frac{V_m}{E_m} \] \hspace{1cm} (3-13)
**Example 3:** Find the transverse Young’s modulus of a glass/epoxy lamina with a fiber volume fraction of 70%. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively.

**Solution:**

\[
\frac{1}{E_2} = \frac{0.7}{85} + \frac{0.3}{3.4}
\]

\[E_2 = 10.37 \text{ GPa}\]

Transverse Young’s modulus as a function of fiber volume fraction for constant fiber to matrix moduli ratio.
Micromechanical Analysis of a Lamina

Theoretical values of transverse Young’s modulus as a function of fiber volume fraction for a Boron/Epoxy unidirectional lamina ($E_f = 414$ GPa, $v_f = 0.2$, $E_m = 4.14$ GPa, $v_m = 0.35$) and comparison with experimental values.
3.3. Major Poisson’s Ratio

The major Poisson’s ratio is defined as the negative of the ratio of the normal strain in the transverse direction to the normal strain in the longitudinal direction, when a normal load is applied in the longitudinal direction...

Assume a composite is loaded in the direction parallel to the fibers:

\[
\delta_c^T = \delta_f^T + \delta_m^T
\]  

The deformations in the transverse direction of the composite
Micromechanical Analysis of a Lamina

\[
\varepsilon_f^T = \frac{\delta_f^T}{t_f} \quad \varepsilon_m^T = \frac{\delta_m^T}{t_m} \quad \varepsilon_c^T = \frac{\delta_c^T}{t_c}
\]

(3-15)

\[
t_c \varepsilon_c^T = t_f \varepsilon_f^T + t_m \varepsilon_m^T
\]

(3-16)

\[
v_f = -\frac{\varepsilon_f^T}{\varepsilon_f^L} \quad v_m = -\frac{\varepsilon_m^T}{\varepsilon_m^L} \quad v_{12} = -\frac{\varepsilon_c^T}{\varepsilon_c^L}
\]

(3-17)

\[
\dot{\varepsilon}_c^L = \dot{\varepsilon}_f^L = \dot{\varepsilon}_m^L \quad t_c v_{12} = t_f v_f + t_m v_m \quad v_{12} = v_f \frac{t_f}{t_c} + v_m \frac{t_m}{t_c}
\]

(3-19)
3.4. In-Plane Shear Modulus

\[ \delta_c = \delta_f + \delta_m \]  
3-20

The resulting shear deformations

\[ \delta_c = \gamma_c t_c \quad \delta_f = \gamma_f t_f \quad \delta_m = \gamma_m t_m \]  
3-21

\[ \gamma_c = \frac{\tau_c}{G_{12}} \quad \gamma_f = \frac{\tau_f}{G_f} \quad \gamma_m = \frac{\tau_m}{G_m} \]  
3-22

\[
\frac{\tau_c}{G_{12}} t_c = \frac{\tau_f}{G_f} t_f + \frac{\tau_m}{G_m} t_m \quad \tau_c = \tau_f = \tau_m
\]  
assumption

\[
\frac{1}{G_{12}} = \frac{1}{G_f} \frac{t_f}{t_c} + \frac{1}{G_m} \frac{t_m}{t_c}
\]
3-23
Example 4: Find the major and minor Poisson’s ratio and the in-plane shear modulus of a glass/epoxy lamina with a 70% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively.

Solution:

From Table 3.1 & 3.2:

\[ v_f = 0.2 \quad v_m = 0.3 \]

From Eq. 3-19:

\[ v_{12} = (0.2)(0.7) + (0.3)(0.3) = 0.230 \]

\[ v_{21} = v_{12} \frac{E_2}{E_1} = 0.230 \left( \frac{10.37}{60.52} \right) = 0.03941 \]

\[
G_f = \frac{E_f}{2(1 + v_f)} = \frac{85}{2(1 + 0.2)} = 35.42 \text{ GPa}
\]

\[
G_m = \frac{E_m}{2(1 + v_m)} = \frac{3.40}{2(1 + 0.3)} = 1.308 \text{ GPa}
\]

\[
\begin{align*}
G_{12} &= \frac{1}{G_{12}} = \frac{0.70}{35.42} + \frac{0.30}{1.308} \\
&= 4.014 \text{ GPa}
\end{align*}
\]
Micromechanical Analysis of a Lamina

Theoretical values of in-plane shear modulus as a function of fiber volume fraction and comparison with experimental values for a unidirectional glass/epoxy lamina ($G_f = 30.19$ GPa, $G_m = 1.83$ GPa).
Micromechanical Analysis of a Lamina

4. Semi-empirical models

- Poor agreement of transverse Young’s modulus (E2) and in-plane shear modulus (G12) with experimental results
- Better modeling techniques: finite element, finite difference and boundary element methods, elasticity solution, and variational principal models
- Semi-empirical models: Halpin and Tsai Model (simple equations by curve fitting to results that are based on elasticity)
  - Unfortunately, these models are available only as complicated equations or in graphical form
4.1. Longitudinal Young’s Modulus
same as that obtained through the strength of materials approach

\[ E_1 = E_f V_f + E_m V_m \]

4.2. Transverse Young’s Modulus

\[ \frac{E_2}{E_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f} \]

\[ \eta = \frac{(E_f / E_m) - 1}{(E_f / E_m) + \xi} \]

- Fiber geometry
- Packing geometry
- Loading conditions

reinforcing factor

\( \xi = 2 \)

For circular fiber

square packing geometry

hexagonal packing geometry

For rectangular fiber

\( \xi = 2(a/b) \)

(a: length/ b: width)
(b is in the direction of loading)
Micromechanical Analysis of a Lamina

Concept of direction of loading for calculation of transverse Young’s modulus by Halpin-Tsai equations
Micromechanical Analysis of a Lamina

Example 5: Find the transverse Young’s modulus for a glass/epoxy lamina with a 70% fiber volume fraction. Use the properties for glass and epoxy from Table 3.1 and Table 3.2 (slide 147-148), respectively. Use Halpin–Tsai equations for a circular fiber in a square array packing geometry.

Solution
Because the fibers are circular and packed in a square array, the reinforcing factor $\xi = 2$.
From Table 3.1: $E_f = 85$ GPa.
From Table 3.2: $E_m = 3.4$ GPa.
From Equation (4-2)
$$\eta = \frac{(85/3.4) - 1}{(85/3.4) + 2} = 0.8889$$

From Equation (4-1), the transverse Young’s modulus of the unidirectional lamina is:
$$\frac{E_2}{3.4} = \frac{1 + 2(0.8889)(0.7)}{1 - (0.8889)(0.7)} \quad E_2 = 20.20 \text{ GPa}$$

(This modulus using the mechanics of materials approach for the same problem, from Example 3, was found to be 10.37 Gpa)
Micromechanical Analysis of a Lamina

Theoretical values of transverse Young’s modulus as a function of fiber volume fraction and comparison with experimental values for boron/epoxy unidirectional lamina ($E_f = 414 \text{ GPa}$, $v_f = 0.2$, $E_m = 4.14 \text{ GPa}$, $v_m = 0.35$)
Micromechanical Analysis of a Lamina

4.3. Major Poisson’s Ratio
same as that obtained through the strength of materials approach

4.4. In-Plane Shear Modulus

\[
\frac{G_{12}}{G_m} = \frac{1 + \xi \eta V_f}{1 - \eta V_f} \quad 4-3
\]

\[
\eta = \frac{(G_f / G_m) - 1}{(G_f / G_m) + \xi} \quad 4-4
\]

Reinforcing factor depends on:
- Fiber geometry
- Packing geometry
- Loading conditions

For circular fiber \( \xi = 1 \)

For rectangular fiber \( \xi = \sqrt{3} \log_e (a / b) \)

(square packing geometry)

(hexagonal packing geometry)
Micromechanical Analysis of a Lamina

Note: The value of $\xi = 1$ for circular fibers in a square array gives reasonable results only for fiber volume fractions of up to 0.5.

Hewitt and Malherbe function $\xi = 1 + 40V_f^{10}$

Concept of direction of loading to calculate in-plane shear modulus by Halphin-Tsai equations.
Micromechanical Analysis of a Lamina

Example 6: Using Halphin–Tsai equations, find the shear modulus of a glass/epoxy composite with a 70% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively. Assume that the fibers are circular and are packed in a square array. Also, get the value of the shear modulus by using Hewitt and Malherbe’s formula for the reinforcing factor.

Solution
For Halphin–Tsai’s equations with circular fibers in a square array $\xi=1$. From Example 4, $G_f=35.42 \text{ GPa}$, $G_m=1.308 \text{ GPa}$.

From Eq. (4-4): $\eta = \frac{(35.42/1.308) - 1}{(35.42/1.308) + 1} = 0.9288$

From Eq. (4-3): $\frac{G_{12}}{1.308} = \frac{1+(1)(0.9288)(0.7)}{1-(0.9288)(0.7)} \Rightarrow G_{12} = 6.169 \text{ GPa}$

(This modulus using the mechanics of materials approach for the same problem, from Example 4, was found to be 4.013 Gpa)

Because the volume fraction is greater than 50%, Hewitt and Malherbe suggested a reinforcing factor $\xi = 1 + 40V_f^{10} = 1 + 40(0.7)^{10} = 2.130$

$\eta = 0.8928 \Rightarrow G = 8.130 \text{ GPa}$
Micromechanical Analysis of a Lamina

Theoretical values of in-plane shear modulus as a function of fiber volume fraction compared with experimental values for unidirectional glass/epoxy lamina \((G_f = 30.19 \text{ GPa}, G_m = 1.83 \text{ GPa})\).
Micromechanical Analysis of a Lamina

5. Elasticity approach

- Elasticity accounts for equilibrium of forces, compatibility, and Hooke’s law relationships in three dimensions.
- The strength of materials approach may not satisfy compatibility and/or account for Hooke’s law in three dimensions.
- Semi-empirical approaches are just as the name implies — partly empirical.

The elasticity models described here are called composite cylinder assemblage (CCA) models:

In a CCA model, one assumes the fibers are circular in cross-section, spread in a periodic arrangement, and continuous.
The composite can be considered to be made of repeating elements called the representative volume elements (RVEs):

- The RVE is considered to represent the composite and respond the same as the whole composite does.
- Appropriate boundary conditions are applied to this composite cylinder based on the elastic moduli being evaluated.

The representative volume elements (RVE)

\[ V_f = \frac{a^2}{b^2} \]

Composite cylinder assemblage (CCA) model used for predicting elastic moduli of unidirectional composites.
5.1 Longitudinal Young’s Modulus

Apply an axial load, $P$, in direction 1

$$\sigma_1 = \frac{P}{\pi b^2}$$

$$E_1 = \frac{P}{\pi b^2 \varepsilon_1}$$

To find $E_1$ in terms of elastic moduli of the fiber and the matrix, and the geometrical parameters such as fiber volume fraction, we need to relate the axial load, $P$, and the axial strain, $\varepsilon_1$, in these terms:

Assuming the response of a cylinder is axisymmetric, the equilibrium equation in the radial direction is given by

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$
Micromechanical Analysis of a Lamina

The normal stress–normal strain relationships in polar coordinates, $r$–$\theta$–$z$, for an isotropic material with Young’s modulus, $E$, and Poisson’s ratio, $\nu$, are given by:

$$\begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \sigma_z \end{bmatrix} = \begin{bmatrix} \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} \\ \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{E(1-\nu)}{(1-2\nu)(1+\nu)} \end{bmatrix} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \end{bmatrix}$$

**Note:** The shear stresses and shear strains are zero in the $r$–$\theta$–$z$ coordinate system for axisymmetric response.
Micromechanical Analysis of a Lamina

The strain-displacement equations for axisymmetric response are

\[
\varepsilon_r = \frac{du}{dr} \quad \varepsilon_\theta = \frac{u}{r} \quad \varepsilon_z = \frac{dw}{dz}
\]

(5-5)

\[u = \text{displacement in radial direction,} \quad w = \text{displacement in axial direction.}\]

(5-5) in (5-4)

& \varepsilon_z = \varepsilon_1

\[
\begin{bmatrix}
\sigma_r \\ \sigma_\theta \\ \sigma_z
\end{bmatrix} =
\begin{bmatrix}
\frac{E(1-v)}{(1-2v)(1+v)} & \frac{vE}{(1-2v)(1+v)} & \frac{vE}{(1-2v)(1+v)} \\
\frac{vE}{(1-2v)(1+v)} & \frac{E(1-v)}{(1-2v)(1+v)} & \frac{vE}{(1-2v)(1+v)} \\
\frac{vE}{(1-2v)(1+v)} & \frac{vE}{(1-2v)(1+v)} & \frac{E(1-v)}{(1-2v)(1+v)}
\end{bmatrix}
\begin{bmatrix}
\frac{du}{dr} \\ \frac{u}{r} \\ \varepsilon_1
\end{bmatrix}
\]

5-6
Substituting Equation (5-6) in the equilibrium equation (5-3) gives:

\[ \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} - \frac{u}{r^2} = 0 \]  

The solution to the linear ordinary differential equation is found by assuming that

\[ u = \sum_{n=-\infty}^{\infty} A_n r^n \]

(5-7) in (5-8)

\[
\sum_{n=\infty}^{\infty} n(n-1)A_n r^{n-2} + \frac{1}{r} \sum_{n=\infty}^{\infty} nA_n r^{n-1} - \frac{1}{r^2} \sum_{n=\infty}^{\infty} A_n r^n = 0 \quad \Rightarrow \quad \sum_{n=-\infty}^{\infty} \left[ n(n-1) + n - 1 \right] A_n r^{n-2} = 0
\]

\[
\sum_{n=\infty}^{\infty} (n^2 - 1)A_n r^{n-2} = 0 \quad \Rightarrow \quad \sum_{n=-\infty}^{\infty} (n-1)(n+1)A_n r^{n-2} = 0
\]
Micromechanical Analysis of a Lamina

The preceding expression (5-9) requires that

\[ A_n = 0, \quad n = -\infty, \ldots, \infty, \text{ except for } n = 1 \text{ and } n = -1 \]  

Therefore, the form of the radial displacement is

\[ u = A_1 r + \frac{A_{-1}}{r} \quad \Rightarrow \quad u = Ar + \frac{B}{r} \]  

The preceding equations are valid for a cylinder with an axisymmetric response. Thus, the radial displacement, \( u_f \) and \( u_m \), in the fiber and matrix cylinders, respectively, can be assumed of the form

\[ u_f = A_f r + \frac{B_f}{r}, 0 \leq r \leq a \] \hspace{1cm} (5-12) \hspace{1cm} \[ u_m = A_m r + \frac{B_m}{r}, a \leq r \leq b \] \hspace{1cm} (5-13)

However, because the fiber is a solid cylinder and the radial displacement \( u_f \) is finite, \( B_f = 0 \); otherwise, the radial displacement of the fiber \( u_f \) would be infinite. Thus

\[ u_f = A_f r, 0 \leq r \leq a \] \hspace{1cm} (5-14)
Differentiating Equation (5-13) and Equation (5-14) gives:

\[
\frac{du_f}{dr} = A_f \quad \frac{du_m}{dr} = A_m - \frac{B_m}{r^2}
\]

Using Equation (5-15) in Equation (5-6), the stress–strain relationships for the fiber and matrix are:

\[
\begin{bmatrix}
\sigma_{f}^r \\
\sigma_{f}^\theta \\
\sigma_{f}^z
\end{bmatrix} =
\begin{bmatrix}
C_{11}^f & C_{12}^f & C_{12}^f \\
C_{12}^f & C_{11}^f & C_{12}^f \\
C_{12}^f & C_{12}^f & C_{11}^f
\end{bmatrix}
\begin{bmatrix}
A_f \\
A_f \\
e_1
\end{bmatrix}
\]

\[
C_{11}^f = \frac{E_f (1-v_f)}{(1-2v_f)(1+v_f)} \quad C_{12}^f = \frac{v_f E_f}{(1-2v_f)(1+v_f)}
\]

\[
\begin{bmatrix}
\sigma_{m}^r \\
\sigma_{m}^\theta \\
\sigma_{m}^z
\end{bmatrix} =
\begin{bmatrix}
C_{11}^m & C_{12}^m & C_{12}^m \\
C_{12}^m & C_{11}^m & C_{12}^m \\
C_{12}^m & C_{12}^m & C_{11}^m
\end{bmatrix}
\begin{bmatrix}
A_m - \frac{B_m}{r^2} \\
A_m + \frac{B_m}{r^2} \\
e_1
\end{bmatrix}
\]

\[
C_{11}^m = \frac{E_m (1-v_m)}{(1-2v_m)(1+v_m)} \quad C_{12}^m = \frac{v_mE_m}{(1-2v_m)(1+v_m)}
\]
Micromechanical Analysis of a Lamina

How do we now solve for the unknown constants $A_f$, $A_m$, $B_m$, and $\epsilon_1$?

The following four boundary and interface conditions will allow us to do that:

1- The radial displacement is continuous at the interface, $r = a$,

$$u_f(r = a) = u_m(r = a) \quad \text{From (5-13) & (5-14)}$$

$$A_f a = A_m a + \frac{B_m}{a} \quad \text{5-20}$$

2- The radial stress is continuous at $r = a$:

$$\left(\sigma^f_r\right)(r = a) = \left(\sigma^m_r\right)(r = a) \quad \text{From (5-16) & (5-18)}$$

$$C_{11}^f A_f + C_{12}^f A_f + C_{12}^f \epsilon_1 = C_{11}^m \left(\frac{A_m - \frac{B_m}{a^2}}{a^2}\right) + C_{12}^m \left(\frac{A_m + \frac{B_m}{a^2}}{a^2}\right) + C_{12}^m \epsilon_1 \quad \text{5-21}$$
Micromechanical Analysis of a Lamina

3- Because the surface at $r = b$ is traction free, the radial stress on the outside of matrix, $r = b$, is zero:

$$\left(\sigma^m_r\right)_{r = b} = 0$$

Then, (5-16) gives

$$C_{11}^m \left( A_m \frac{B_m}{b^2} \right) + C_{12}^m \left( A_m + \frac{B_m}{b^2} \right) + C_{12}^m \varepsilon_1 = 0 \quad 5-22$$

4- The overall axial load over the fiber-matrix cross-sectional area in direction 1 is the applied load, $P$, then:

$$\int_A \sigma_z dA = P \quad 5-23$$

$$\int_0^b \int_0^{2\pi} \sigma_z r dr d\theta = P$$

Because the axial normal stress, $\sigma_z$, is independent of $\theta$,

$$\int_0^b \sigma_z 2\pi r dr = P$$

$$\begin{cases} \sigma_z = \sigma_z^f, & 0 \leq r \leq a \\ \sigma_z = \sigma_z^m, & a \leq r \leq b. \end{cases} \quad 5-24$$
Then, from Equation (5-16) and (5-18):

\[
\int_{0}^{a} (C_{12}^{f} A_f + C_{12}^{m} A_m + C_{11}^{f} \varepsilon_1) 2\pi r dr + \int_{a}^{b} \left( C_{12}^{m} \left( A_m - \frac{B_m}{r^2} \right) + C_{12}^{m} \left( A_m + \frac{B_m}{r^2} \right) + C_{11}^{m} \varepsilon_1 \right) 2\pi r dr = P \tag{5-25}
\]

Solving Equation (5-20), Equation (5-21), Equation (5-22), and Equation (5-25), we get the solution to \( A_f, A_m, B_m, \text{ and } \varepsilon_1 \).

Using the resulting solution for \( \varepsilon_1 \), and using Equation (5-2):

\[
E_1 = \frac{P}{\pi b^2 \varepsilon_1} = E_f V_f + E_m (1 - V_f) - \frac{2E_m E_f V_f (v_f - v_m)^2 (1 - V_f)}{E_f (2v_m^2 V_f - v_m + V_f v_m - V_f - 1) + E_m (-1 - 2V_f v_f^2 + V_f v_f + 2v_f^2 + V_f)} \tag{5-26}
\]
Example 7: Find the longitudinal Young’s modulus for a glass/epoxy lamina with a 70% fiber volume fraction. Use the properties for glass and epoxy from Table 3.1 and Table 3.2, respectively. Use equations obtained using the elasticity model.

Solution:
From Table 3.1 & 3.2

\[ E_f = 85 \text{ GPa} \quad \nu_f = 0.2 \quad E_m = 3.4 \text{ GPa} \quad \nu_m = 0.3 \]

Using Equation (5-26), the longitudinal Young’s modulus:

\[
E_1 = (85 \times 10^9)(0.7) + (3.4 \times 10^9)(1-0.7) - \frac{2(3.4 \times 10^9)(85 \times 10^9)(0.7)(0.2-0.3)^2(1-0.7)}{(85 \times 10^9)(2(0.3)^2(0.7) - 0.3 + (0.7)(0.3) - 0.7 - 1) + (3.4 \times 10^9)(-1 - 2(0.7)(0.2)^2 + 0.2 - (0.7)(0.2) + 2(0.2)^2 + 0.7)} = 60.53 \times 10^9 \text{ Pa} = 60.53 \text{ GPa}
\]

For the same problem, the longitudinal Young’s modulus was found to be 60.52 GPa from the mechanics of materials approach as well as the Halphin–Tsai equations.
Micromechanical Analysis of a Lamina

5.2 Major Poisson’s Ratio

5.3 Transverse Young’s Modulus

5.4 Axial Shear Modulus

Study from page 239, Mechanics of Composite Materials, Autar K. Kaw
Micromechanical Analysis of a Lamina

Problems Chapter 3:

1. A hybrid lamina uses glass and graphite fibers in a matrix of epoxy for its construction. The fiber volume fractions of glass and graphite are 40 and 20%, respectively. The specific gravity of glass, graphite, and epoxy is 2.6, 1.8, and 1.2, respectively. Find
   a. Mass fractions
   b. Density of the composite

2. Determine the expression for the modulus of a composite material that consists of matrix material reinforced by a slab of constant thickness in the direction in which the modulus is desired as in below Figure

3. A resin hybrid lamina is made by reinforcing graphite fibers in two matrices: resin A and resin B. The fiber weight fraction is 40%; for resin A and resin B, the weight fraction is 30% each. If the specific gravity of graphite, resin A, and resin B is 1.2, 2.6, and 1.7, respectively, find
   a. Fiber volume fraction
   b. Density of composite
Micromechanical Analysis of a Lamina

4. Show that \( G_{12} = \frac{G_m}{1-V_f} \), if the fibers are much stiffer than the matrix — that is, \( G_f >> G_m \).

5. A unidirectional glass/epoxy lamina with a fiber volume fraction of 70% is replaced by a graphite/epoxy lamina with the same longitudinal Young’s modulus. Find the fiber volume fraction required in the graphite/epoxy lamina. Use properties of glass, graphite, and epoxy from Table 3.1 and Table 3.2.

6. Sometimes, the properties of a fiber are determined from the measured properties of a unidirectional lamina. As an example, find the experimentally determined value of the Poisson’s ratio of an isotropic fiber from the following measured properties of a unidirectional lamina:
   a. Major Poisson’s ratio of composite = 0.27
   b. Poisson’s ratio of the matrix = 0.35
   c. Fiber volume fraction = 0.65
Micromechanical Analysis of a Lamina

7. Using elasticity model equations, find the elastic moduli of a glass/epoxy unidirectional lamina with 40% fiber volume fraction. Use the properties of glass and epoxy from Table 3.1 and Table 3.2, respectively (slide 144). Compare your results with those obtained by using the strength of materials approach and the Halphin–Tsai approach. Assume that the fibers are circularly shaped and are in a square array for the Halphin–Tsai approach.
• Code for laminate stacking sequence
• Relationships of mechanical loads applied to a laminate to strains and stresses in each lamina.
1. Introduction:

The two basic questions of laminate analysis are:
1. what are the conditions that the laminae must meet to be a laminate?
2. how will a laminate respond to loading, i.e., imposed forces and moments?

The main reason for lamination is to combine laminae to create a laminate for achieving the largest possible bending stiffness for the materials used.

\[ \Delta = \frac{PL^3}{48EI} \quad \Delta = \frac{PL^3}{48E(2)\frac{1}{12}bh^3} = \frac{PL^3}{8Eb^3} \quad \Delta = \frac{PL^3}{48E} = \frac{PL^2}{12} \frac{1}{b(2h)^3} = \frac{PL^3}{32Eb^3} \]

\[ \Delta_{\text{unbonded}} = 4\Delta_{\text{bonded}} \]

a. Unbonded Beams  b. Bonded Beams
2. Laminate Code:

$[0/-45/90/60/30]:$ It consists of five plies, each of which has a different angle to the reference $x$-axis. A slash separates each lamina. The code also implies that each ply is made of the same material and is of the same thickness. Sometimes, $[0/-45/90/60/30]_T$ may also denote this laminate, where the subscript $T$ stands for a total laminate.
Macromechanical Analysis of a Laminate

amiratrian@gmail.com

\begin{align*}
\begin{array}{c|c|c|c|c|c|c|c}
 & 0 & -45 & 90 & 90 & 60 & 0 \\
\hline
0 & & & & & & \\
-45 & & & & & & \\
90 & & & & & & \\
90 & & & & & & \\
60 & & & & & & \\
0 & & & & & & \\
\end{array} \\
[0/ -45/ 90_2/ 60/ 0]
\end{align*}

\begin{align*}
\begin{array}{c|c|c|c|c|c|c|c}
 & 0 & -45 & 60 & 60 & -45 & 0 \\
\hline
0 & & & & & & \\
-45 & & & & & & \\
60 & & & & & & \\
60 & & & & & & \\
-45 & & & & & & \\
0 & & & & & & \\
\end{array} \\
[0/ -45/ 60]_s
\end{align*}

\begin{align*}
\begin{array}{c|c|c|c|c|c|c|c}
 & 0 & -45 & 60 & -45 & 0 \\
\hline
0 & & & & & & \\
-45 & & & & & & \\
60 & & & & & & \\
-45 & & & & & & \\
0 & & & & & & \\
\end{array} \\
[0/ -45/ 60]_s
\end{align*}

\begin{align*}
\begin{array}{c|c|c|c|c|c|c|c}
 Graphite/epoxy & 0 & & & & & \\
 Boron/epoxy & 45 & & & & & \\
 Boron/epoxy & -45 & & & & & \\
 Boron/epoxy & -45 & & & & & \\
 Boron/epoxy & 45 & & & & & \\
 Graphite/epoxy & 0 & & & & & \\
\end{array} \\
[0^{Gr}/ \pm 45^B]_s
\end{align*}
Macromechanical Analysis of a Laminate

3. Classical Lamination Theory (CLT):

3.1. 1D Linearly Elastic & Isotropic Beam Stress–Strain Relation

\[ \sigma_x = \frac{P}{A}, \quad \varepsilon_x = \frac{P}{AE} \]  \hspace{1cm} (4-1)

Distance \( z \), from the centroidal line

\[ \varepsilon_{xx} = \frac{z}{\rho} \]  \hspace{1cm} (4-2)

the radius of curvature of the beam

\[ \sigma_{xx} = \frac{Ez}{\rho}, \quad \sigma_{xx} = \frac{Mz}{I} \]  \hspace{1cm} (4-3)
Macromechanical Analysis of a Laminate

This shows that, under a combined uniaxial and bending load, the strain varies linearly through the thickness of the beam.

\[ \varepsilon_{xx} = \left( \frac{1}{AE} \right) P + \left( \frac{z}{EI} \right) M \]

\[ \varepsilon_{xx} = \varepsilon_0 + z \left( \frac{1}{\rho} \right) \]

\[ \varepsilon_{xx} = \varepsilon_0 + z \kappa \]

\( \varepsilon_0 \) is the strain at \( z = 0 \)

curvature of the beam
3.2. Strain-Displacement Equations

Consider a plate under in-plane loads such as shear and axial forces, and bending and twisting moments:

\[ u_0, v_0, w_0: \] displacements in the \( x, y \) & \( z \) directions, respectively, at the mid-plane.

\( u, v, w: \) the displacements at any point in the \( X, y, \) & \( z \) directions, respectively.
Macromechanical Analysis of a Laminate

Assumptions:

1. Each lamina is orthotropic.
2. Each lamina is homogeneous.
3. A line straight and perpendicular to the middle surface remains straight and perpendicular to the middle surface during deformation \((\gamma_{xz} = \gamma_{yz} = 0)\)
4. The laminate is thin and is loaded only in its plane (plane stress) \((\sigma_z = \tau_{xz} = \tau_{yz} = 0)\)
5. Displacements are continuous and small throughout the laminate \((|u|, |v|, |w| << |h|)\), where \(h\) is the laminate thickness
6. Each lamina is elastic
7. No slip occurs between the lamina interfaces
Macromechanical Analysis of a Laminate

At any point other than the mid-plane, the two displacements in the \( x-y \) plane will depend on the axial location of the point and the slope of the laminate mid-plane with the \( x \) and \( y \) directions:

\[
\begin{align*}
    u &= u_0 - z\alpha \\
    \alpha &= \frac{\partial w_0}{\partial x}
\end{align*}
\]

the displacement \( u \) in the \( x \)-direction is:

\[
\begin{align*}
    u &= u_0 - z \frac{\partial w_0}{\partial x} \\
    \varepsilon_x &= \frac{\partial u}{\partial x} - z \frac{\partial^2 w_0}{\partial x^2}
\end{align*}
\]

Similarly, taking a cross-section in the \( y-z \) plane would give the displacement in the \( y \)-direction as:

\[
\begin{align*}
    v &= v_0 - z \frac{\partial w_0}{\partial y} \\
    \varepsilon_y &= \frac{\partial v}{\partial y} - z \frac{\partial^2 w_0}{\partial y^2}
\end{align*}
\]

\[
\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} - 2z \frac{\partial^2 w_0}{\partial x \partial y}
\]
Macromechanical Analysis of a Laminate

All these Eqs. can be written in matrix form as:

\[
\begin{pmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial u_0}{\partial x} \\
\frac{\partial v_0}{\partial y} \\
\frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x}
\end{pmatrix} + z \begin{pmatrix}
-\frac{\partial^2 w_0}{\partial x^2} \\
-\frac{\partial^2 w_0}{\partial y^2} \\
-2 \frac{\partial^2 w_0}{\partial x \partial y}
\end{pmatrix}
\]

This eq. shows:
- the linear relationship of the strains in a laminate to the curvatures of the laminate.
- the strains are independent of the x and y coordinates.
- Also, note the similarity between Eq. (4.10) & Eq.(4.4), which was developed for the one dimensional beam.
Macromechanical Analysis of a Laminate

**Example 4.1:** A 0.010 in. thick laminate is subjected to in-plane loads. If the midplane strains and curvatures are given as follows, find the global strains at the top surface of the laminate?

\[
\begin{align*}
\{\varepsilon_x^0\} &= \begin{bmatrix} 2751 \\ -1331 \\ -1125 \end{bmatrix} \text{ in/in} \\
\{\varepsilon_y^0\} &= \begin{bmatrix} -1331 \\ -1125 \end{bmatrix} \\
\gamma_{xy}^0 &= -1125 \\
\kappa_x &= 1.965 \\
\kappa_y &= 0.2385 \\
\kappa_{xy} &= -1.773
\end{align*}
\]

**Solution:**

\[
\begin{align*}
\{\varepsilon_x\} &= \begin{bmatrix} 2751 \\ -1331 \\ -1125 \end{bmatrix} \times 10^{-6} - 0.005 \\
\{\varepsilon_y\} &= \begin{bmatrix} 0.2385 \\ -1.773 \end{bmatrix} \\
\gamma_{xy} &= \begin{bmatrix} 7740 \end{bmatrix} \mu \text{ in/in}
\end{align*}
\]
Macromechanical Analysis of a Laminate

3.3. Strain and Stress in a Laminate:

If the strains are known at any point along the thickness of the laminate, the stress–strain eq. calculates the global stresses in each lamina

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

The reduced transformed stiffness matrix, \([\bar{Q}]\) corresponds to that of the ply located at the point along the thickness of the laminate...

\[(4-10) \text{ in } (4-11)\]

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} =
\begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
+ \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}
\begin{bmatrix}
K_x \\
K_y \\
K_{xy}
\end{bmatrix}
\]

\[(4-12)\]
Macromechanical Analysis of a Laminate

It’s clear from eq. (4-12):
• The stresses vary linearly only through the thickness of each lamina.
• The stresses, however, may jump from lamina to lamina because the transformed reduced-stiffness matrix changes from ply to ply because this matrix \([\mathbf{Q}]\) depends on the material and orientation of the ply.
Macromechanical Analysis of a Laminate

3.4. Force and Moment Resultants Related to Midplane Strains & Curvatures (how to find the midplane strains and curvatures if the loads applied to the laminate are known)

Consider a laminate made of $n$ plies:
Macromechanical Analysis of a Laminate

The z-coordinate of each ply k surface (top and bottom) is given by

Ply 1:

\[ h_0 = -\frac{h}{2} \text{ (top surface)} \quad h_1 = -\frac{h}{2} + t_1 \text{ (bottom surface)} \]

Ply k: (k = 2, 3,...n – 2, n – 1):

\[ h_{k-1} = -\frac{h}{2} + \sum_{i=1}^{k-1} t \text{ (top surface)} \quad h_k = -\frac{h}{2} + \sum_{i=1}^{k} t \text{ (bottom surface)} \]

Ply n:

\[ h_{n-1} = \frac{h}{2} - t_n \text{ (top surface)} \quad h_n = \frac{h}{2} \text{ (bottom surface)} \]
Macromechanical Analysis of a Laminate

Integrating the global stresses in each lamina gives the resultant forces per unit length in the \(x\–y\) plane through the laminate thickness as

\[
\begin{align*}
N_x &= \int_{-h/2}^{h/2} \sigma_x \, dz \\
N_y &= \int_{-h/2}^{h/2} \sigma_y \, dz \\
N_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} \, dz
\end{align*}
\]

Similarly, integrating the global stresses in each lamina gives the resulting moments per unit length in the \(x\–y\) plane through the laminate thickness as

\[
\begin{align*}
M_x &= \int_{-h/2}^{h/2} \sigma_z \, dz \\
M_y &= \int_{-h/2}^{h/2} \sigma_y \, dz \\
M_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} \, dz
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} &= \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \, dz \\
\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} &= \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_z \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \, dz
\end{align*}
\]

\[\text{4-14}\]

\[\text{4-15}\]
Macromechanical Analysis of a Laminate

Which gives

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} \, dz
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} \, zdz
\]

\[4-16\]

\[4-12\) in \(4-16\)

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} \, dz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix} \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} \, zdz
\]

\[4-17\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix} \, zdz + \sum_{k=1}^{n} \int_{h_{k-1}}^{h_k} \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix} \begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} \, z^2 \, dz
\]

\[4-18\]
Macromechanical Analysis of a Laminate

Due to independency of midplane strains and plate curvatures to z coordinate & to be constant of the transformed reduced stiffness matrix for each ply:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} = \left\{ \sum_{k=1}^{n} \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}_{k} h_{k} \int_{h_{k-1}}^{h_{k}} d\bar{z} \right\} \begin{bmatrix}
\varepsilon_{x}^{0} \\
\varepsilon_{y}^{0} \\
\gamma_{xy}^{0}
\end{bmatrix} + \left\{ \sum_{k=1}^{n} \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}_{k} h_{k-1} \int_{h_{k-1}}^{h_{k}} z d\bar{z} \right\} \begin{bmatrix}
\kappa_{x} \\
\kappa_{y} \\
\kappa_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} = \left\{ \sum_{k=1}^{n} \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}_{k} h_{k} \int_{h_{k-1}}^{h_{k}} z d\bar{z} \right\} \begin{bmatrix}
\varepsilon_{x}^{0} \\
\varepsilon_{y}^{0} \\
\gamma_{xy}^{0}
\end{bmatrix} + \left\{ \sum_{k=1}^{n} \begin{bmatrix}
\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\
\bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\
\bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}
\end{bmatrix}_{k} h_{k-1} \int_{h_{k-1}}^{h_{k}} z^2 d\bar{z} \right\} \begin{bmatrix}
\kappa_{x} \\
\kappa_{y} \\
\kappa_{xy}
\end{bmatrix}
\]

Knowing that:

\[
\int_{h_{k-1}}^{h_{k}} d\bar{z} = (h_{k} - h_{k-1})
\]

\[
\int_{h_{k-1}}^{h_{k}} z d\bar{z} = \frac{1}{2}(h_{k}^2 - h_{k-1}^2)
\]

\[
\int_{h_{k-1}}^{h_{k}} z^2 d\bar{z} = \frac{1}{3}(h_{k}^3 - h_{k-1}^3)
\]
It gives:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy}
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} & A_{16} \\
A_{12} & A_{22} & A_{26} \\
A_{16} & A_{26} & A_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix}
+ 
\begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

\[
\begin{bmatrix}
M_x \\
M_y \\
M_{xy}
\end{bmatrix} =
\begin{bmatrix}
B_{11} & B_{12} & B_{16} \\
B_{12} & B_{22} & B_{26} \\
B_{16} & B_{26} & B_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0
\end{bmatrix}
+ 
\begin{bmatrix}
D_{11} & D_{12} & D_{16} \\
D_{12} & D_{22} & D_{26} \\
D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]

**Extensional Matrix**
relating the resultant in-plane forces to the in-plane strains

\[
A_{ij} = \sum_{k=1}^{n} [(\overline{Q}_{ik}) h_k - (\overline{Q}_{ik}) h_{k-1}], \quad i = 1, 2, 6; \quad j = 1, 2, 6,
\]

**Coupling Matrix**
coupling the force and moment terms to the midplane strains and midplane curvatures.

\[
B_{ij} = \frac{1}{2} \sum_{k=1}^{n} [(\overline{Q}_{ik}) (h_k^2 - h_{k-1}^2)], \quad i = 1, 2, 6; \quad j = 1, 2, 6,
\]

**Bending Stiffness Matrix**
relating the resultant bending moments to the plate curvatures

\[
D_{ij} = \frac{1}{3} \sum_{k=1}^{n} [(\overline{Q}_{ik}) h_k^3 - (\overline{Q}_{ik}) h_{k-1}^3], \quad i = 1, 2, 6; \quad j = 1, 2, 6.
\]
Macromechanical Analysis of a Laminate

Combining these eqs. gives 6 simultaneous linear eqs. & six unknown as:

\[
\begin{bmatrix}
N_x \\
N_y \\
N_{xy} \\
M_x \\
M_y \\
M_{xy}
\end{bmatrix}
= 
\begin{bmatrix}
A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\
A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\
A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{56} \\
B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\
B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\
B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0 \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix}
\]
Macromechanical Analysis of a Laminate

Steps for analyzing a laminated composite subjected to the applied forces and moments:

1. Find the value of the reduced stiffness matrix [Q] for each ply using its four elastic moduli, E1, E2, v12 & G12 (Eq.6-7 chap.2).

2. Find the value of the transformed reduced stiffness matrix for each ply using the [Q] matrix calculated in step 1 and the angle of the ply (Eq.7-7 Chap.2).

3. Knowing the thickness, tk, of each ply, find the coordinate of the top and bottom surface, hi, i = 1..., n, of each ply, using Eqs. (4-13).

4. Use the matrices from step 2 and the location of each ply from step 3 to find the three stiffness matrices [A], [B], and [D] from Eqs. (4-23).

5. Substitute the stiffness matrix values found in step 4 and the applied forces and moments in Eq. (4-24).
Macromechanical Analysis of a Laminate

6. Solve the six simultaneous Eqs. (4-24) to find the midplane strains and curvatures.

7. Now that the location of each ply is known, find the global strains in each ply using Eq. (4-10).

8. For finding the global stresses, use the stress–strain (Eq. 7-6 chap.2).

9. For finding the local strains, use the transformation (Eq. 7-4 chap.2).

10. For finding the local stresses, use the transformation (Eq. 7-1 chap.2).
Example 4.2: Find the three stiffness matrices \([A]\), \([B]\), and \([D]\) for a three-ply \([0/30/-45]\) graphite/epoxy laminate. Use the unidirectional properties from Table of slide 95 of graphite/epoxy. Assume that each lamina has a thickness of 5 mm.

Solution:

the reduced stiffness matrix for the \(0^\circ\) graphite/epoxy ply is:

\[
[Q] = \begin{bmatrix}
181.8 & 2.897 & 0 \\
2.897 & 10.35 & 0 \\
0 & 0 & 7.17
\end{bmatrix} \left(10^9\right) \text{ Pa}
\]
Macromechanical Analysis of a Laminate

the transformed reduced stiffness matrix $[\bar{Q}]$ for each of the three plies is (Eq. 7-7 chap.2):

$$[\bar{Q}]_0 = \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} \times (10^9) \text{ Pa}$$

$$[\bar{Q}]_{30} = \begin{bmatrix} 109.4 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix} \times (10^9) \text{ Pa}$$

$$[\bar{Q}]_{45} = \begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix} \times (10^9) \text{ Pa}$$
Macromechanical Analysis of a Laminate

the locations of the ply surfaces are (Eq. 4-13):

\[ h_0 = -0.0075 \text{ m} \]
\[ h_1 = -0.0025 \text{ m} \]
\[ h_2 = 0.0025 \text{ m} \]
\[ h_3 = 0.0075 \text{ m} \]

\[ A_{ij} = \sum_{k=1}^{3} [\bar{Q}_{jk}]_{k}(h_k - h_{k-1}) \]

\[
[A] = \begin{bmatrix}
181.8 & 2.897 & 0 \\
2.897 & 10.35 & 0 \\
0 & 0 & 7.17
\end{bmatrix} (10^9)[(-0.0025) - (-0.0075)] + \begin{bmatrix}
109.4 & 32.46 & 54.19 \\
32.46 & 23.65 & 20.05 \\
54.19 & 20.05 & 36.74
\end{bmatrix} (10^9)[0.0025 - (-0.0025)]
\]

\[
\begin{bmatrix}
56.66 & 42.32 & -42.87 \\
42.32 & 56.66 & -42.87 \\
-42.87 & -42.87 & 46.59
\end{bmatrix} (10^8)(0.0075 - 0.0025) = \begin{bmatrix}
1.739 \times 10^6 & 3.884 \times 10^8 & 5.663 \times 10^7 \\
3.884 \times 10^8 & 4.533 \times 10^8 & -1.141 \times 10^8 \\
5.663 \times 10^7 & -1.141 \times 10^8 & 4.525 \times 10^8
\end{bmatrix} \text{ Pa m} \]
Macromechanical Analysis of a Laminate

\[ B_{ij} = \frac{1}{2} \sum_{k=1}^{3} [Q_{ij}]_{k} (h_{k}^2 - h_{k-1}^2) \]

\[ [B] = \frac{1}{2} \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} \begin{bmatrix} (10^9)[(-0.0025)^2 - (-0.0075)^2] + \frac{1}{2} \begin{bmatrix} 109.4 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix} \begin{bmatrix} 10^9[(0.0025)^2 - (-0.0025)^2] \end{bmatrix} \]

\[ + \frac{1}{2} \begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix} \begin{bmatrix} (10^9)[(0.0075)^2 - (0.0025)^2] \end{bmatrix} = \begin{bmatrix} -3.129 \times 10^6 & 9.855 \times 10^5 & -1.972 \times 10^6 \\ 9.855 \times 10^5 & 1.158 \times 10^6 & -1.972 \times 10^6 \\ -1.072 \times 10^6 & -1.072 \times 10^6 & 9.855 \times 10^5 \end{bmatrix} Pa \cdot m^2 \]

\[ D_{ij} = \frac{1}{3} \sum_{k=1}^{3} [Q_{ij}]_{k} (h_{k}^3 - h_{k-1}^3) \]

\[ [D] = \frac{1}{3} \begin{bmatrix} 181.8 & 2.897 & 0 \\ 2.897 & 10.35 & 0 \\ 0 & 0 & 7.17 \end{bmatrix} \begin{bmatrix} (10^9)[(-0.0025)^3 - (-0.0075)^3] + \frac{1}{3} \begin{bmatrix} 109.4 & 32.46 & 54.19 \\ 32.46 & 23.65 & 20.05 \\ 54.19 & 20.05 & 36.74 \end{bmatrix} \begin{bmatrix} (10^9)[(0.0025)^3 - (-0.0025)^3] \end{bmatrix} \]

\[ + \frac{1}{3} \begin{bmatrix} 56.66 & 42.32 & -42.87 \\ 42.32 & 56.66 & -42.87 \\ -42.87 & -42.87 & 46.59 \end{bmatrix} \begin{bmatrix} (10^9)[(0.0075)^3 - (0.0025)^3] \end{bmatrix} = \begin{bmatrix} 3.343 \times 10^4 & 6.461 \times 10^3 & -5.240 \times 10^3 \\ 6.461 \times 10^3 & 9.320 \times 10^3 & -5.596 \times 10^3 \\ -5.240 \times 10^3 & -5.596 \times 10^3 & 7.663 \times 10^3 \end{bmatrix} Pa \cdot m^3 \]
Macromechanical Analysis of a Laminate

Example 4.3: For previous example if the laminate is subjected to a load of $N_x = N_y = 1000$ N/m, find:
(a) Mid-plane strains and curvatures
(b) Global and local stresses on top surface of $30^\circ$ ply
(c) Percentage of load, $N_x$, taken by each ply

Solution:

Solving this system of eqs. gives
Macromechanical Analysis of a Laminate

The strains at the top surface of the 30° ply \((z=h_1=-0.0025 \text{ m})\) (Eq. 4-10):

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}^{30^\circ, \text{top}}
\end{bmatrix} =
\begin{bmatrix}
3.123 \times 10^{-7} \\
3.492 \times 10^{-6} \\
-7.598 \times 10^{-7}
\end{bmatrix} + (-0.0025)
\begin{bmatrix}
2.971 \times 10^{-5} \\
-3.285 \times 10^{-4} \\
4.101 \times 10^{-4}
\end{bmatrix} =
\begin{bmatrix}
2.380 \times 10^{-7} \\
4.313 \times 10^{-6} \\
-1.785 \times 10^{-6}
\end{bmatrix} \text{ } \text{m/m}
\]

The global stresses from Eq. 7-6 chap.2:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}^{30^\circ, \text{top}}
\end{bmatrix} =
\begin{bmatrix}
109.4 & 32.46 & 54.19 \\
32.46 & 23.65 & 20.05 \\
54.19 & 20.05 & 36.74
\end{bmatrix} (10^9)
\begin{bmatrix}
2.380 \times 10^{-7} \\
4.313 \times 10^{-6} \\
-1.785 \times 10^{-6}
\end{bmatrix} =
\begin{bmatrix}
6.930 \times 10^4 \\
7.391 \times 10^4 \\
3.381 \times 10^4
\end{bmatrix} \text{Pa}
\]

The local strains (Eq. 7-4 chap.2) and local stresses (Eq. 7-1 chap.2) in the 30° ply at the top surface:

\[
\begin{bmatrix}
\varepsilon_1 \\
\varepsilon_2 \\
\gamma_{12}/2
\end{bmatrix} =
\begin{bmatrix}
0.7500 & 0.2500 & 0.8660 \\
0.2500 & 0.7500 & -0.8660 \\
-4.330 & 0.4330 & 0.5000
\end{bmatrix}
\begin{bmatrix}
2.380 \times 10^{-7} \\
4.313 \times 10^{-6} \\
-1.785 \times 10^{-6}/2
\end{bmatrix} =
\begin{bmatrix}
4.837 \times 10^{-7} \\
4.067 \times 10^{-6} \\
2.636 \times 10^{-6}
\end{bmatrix} \text{m/m}
\]
### Macromechanical Analysis of a Laminate

\[
\begin{bmatrix}
\sigma_1 \\
\sigma_2 \\
\tau_{12}
\end{bmatrix} = \begin{bmatrix}
0.7500 & 0.2500 & 0.8660 \\
0.2500 & 0.7500 & -0.8660 \\
-0.4330 & 0.4330 & 0.5000
\end{bmatrix} \begin{bmatrix}
6.930 \times 10^4 \\
7.391 \times 10^4 \\
3.381 \times 10^4
\end{bmatrix} = \begin{bmatrix}
9.973 \times 10^4 \\
4.348 \times 10^4 \\
1.890 \times 10^4
\end{bmatrix} Pa
\]

### Global Strains (m/m)

<table>
<thead>
<tr>
<th>Ply no.</th>
<th>Position</th>
<th>(\varepsilon_x)</th>
<th>(\varepsilon_y)</th>
<th>(\gamma_{xy})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (0°)</td>
<td>Top</td>
<td>(8.944 \times 10^{-8})</td>
<td>(5.955 \times 10^{-6})</td>
<td>(-3.836 \times 10^{-6})</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>(1.637 \times 10^{-7})</td>
<td>(5.134 \times 10^{-6})</td>
<td>(-2.811 \times 10^{-6})</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>(2.380 \times 10^{-7})</td>
<td>(4.313 \times 10^{-6})</td>
<td>(-1.785 \times 10^{-6})</td>
</tr>
<tr>
<td>2 (30°)</td>
<td>Top</td>
<td>(2.380 \times 10^{-7})</td>
<td>(4.313 \times 10^{-6})</td>
<td>(-1.785 \times 10^{-6})</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>(3.123 \times 10^{-7})</td>
<td>(3.492 \times 10^{-6})</td>
<td>(-7.598 \times 10^{-7})</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>(3.866 \times 10^{-7})</td>
<td>(2.670 \times 10^{-6})</td>
<td>(2.655 \times 10^{-7})</td>
</tr>
<tr>
<td>3 (−45)</td>
<td>Top</td>
<td>(3.866 \times 10^{-7})</td>
<td>(2.670 \times 10^{-6})</td>
<td>(2.655 \times 10^{-7})</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>(4.609 \times 10^{-7})</td>
<td>(1.849 \times 10^{-6})</td>
<td>(1.291 \times 10^{-6})</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>(5.352 \times 10^{-7})</td>
<td>(1.028 \times 10^{-6})</td>
<td>(2.316 \times 10^{-6})</td>
</tr>
</tbody>
</table>
# Macromechanical Analysis of a Laminate

## Global Stresses (Pa)

<table>
<thead>
<tr>
<th>Ply no.</th>
<th>Position</th>
<th>$\sigma_x$</th>
<th>$\sigma_y$</th>
<th>$\tau_{xy}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ($0^\circ$)</td>
<td>Top</td>
<td>$3.351 \times 10^4$</td>
<td>$6.188 \times 10^4$</td>
<td>$-2.750 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>$4.464 \times 10^4$</td>
<td>$5.359 \times 10^4$</td>
<td>$-2.015 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>$5.577 \times 10^4$</td>
<td>$4.531 \times 10^4$</td>
<td>$-1.280 \times 10^4$</td>
</tr>
<tr>
<td>2 ($30^\circ$)</td>
<td>Top</td>
<td>$6.930 \times 10^4$</td>
<td>$7.391 \times 10^4$</td>
<td>$3.381 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>$1.063 \times 10^5$</td>
<td>$7.747 \times 10^4$</td>
<td>$5.903 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>$1.434 \times 10^5$</td>
<td>$8.102 \times 10^4$</td>
<td>$8.426 \times 10^4$</td>
</tr>
<tr>
<td>3 ($-45^\circ$)</td>
<td>Top</td>
<td>$1.235 \times 10^5$</td>
<td>$1.563 \times 10^5$</td>
<td>$-1.187 \times 10^5$</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>$4.903 \times 10^4$</td>
<td>$6.894 \times 10^4$</td>
<td>$-3.888 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>$-2.547 \times 10^4$</td>
<td>$-1.840 \times 10^4$</td>
<td>$4.091 \times 10^4$</td>
</tr>
</tbody>
</table>

## Local Strains (m/m)

<table>
<thead>
<tr>
<th>Ply no.</th>
<th>Position</th>
<th>$\epsilon_1$</th>
<th>$\epsilon_2$</th>
<th>$\gamma_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ($0^\circ$)</td>
<td>Top</td>
<td>$8.944 \times 10^{-8}$</td>
<td>$5.955 \times 10^{-6}$</td>
<td>$-3.836 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>$1.637 \times 10^{-7}$</td>
<td>$5.134 \times 10^{-6}$</td>
<td>$-2.811 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>$2.380 \times 10^{-7}$</td>
<td>$4.313 \times 10^{-6}$</td>
<td>$-1.785 \times 10^{-6}$</td>
</tr>
<tr>
<td>2 ($30^\circ$)</td>
<td>Top</td>
<td>$4.837 \times 10^{-7}$</td>
<td>$4.067 \times 10^{-6}$</td>
<td>$2.636 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>$7.781 \times 10^{-7}$</td>
<td>$3.026 \times 10^{-6}$</td>
<td>$2.374 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>$1.073 \times 10^{-6}$</td>
<td>$1.985 \times 10^{-6}$</td>
<td>$2.111 \times 10^{-6}$</td>
</tr>
<tr>
<td>3 ($-45^\circ$)</td>
<td>Top</td>
<td>$1.396 \times 10^{-6}$</td>
<td>$1.661 \times 10^{-6}$</td>
<td>$-2.284 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>$5.096 \times 10^{-7}$</td>
<td>$1.800 \times 10^{-6}$</td>
<td>$-1.388 \times 10^{-6}$</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>$-3.766 \times 10^{-7}$</td>
<td>$1.940 \times 10^{-6}$</td>
<td>$-4.928 \times 10^{-7}$</td>
</tr>
</tbody>
</table>
Macromechanical Analysis of a Laminate

Local Stresses (Pa)

<table>
<thead>
<tr>
<th>Ply no.</th>
<th>Position</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\sigma_{12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ($0^\circ$)</td>
<td>Top</td>
<td>$3.351 \times 10^4$</td>
<td>$6.188 \times 10^4$</td>
<td>$-2.750 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>$4.464 \times 10^4$</td>
<td>$5.359 \times 10^4$</td>
<td>$-2.015 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>$5.577 \times 10^4$</td>
<td>$4.531 \times 10^4$</td>
<td>$-1.280 \times 10^4$</td>
</tr>
<tr>
<td>2 ($30^\circ$)</td>
<td>Top</td>
<td>$9.973 \times 10^4$</td>
<td>$4.348 \times 10^4$</td>
<td>$1.890 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>$1.502 \times 10^5$</td>
<td>$3.356 \times 10^4$</td>
<td>$1.702 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>$2.007 \times 10^5$</td>
<td>$2.364 \times 10^4$</td>
<td>$1.513 \times 10^4$</td>
</tr>
<tr>
<td>3 ($-45^\circ$)</td>
<td>Top</td>
<td>$2.586 \times 10^5$</td>
<td>$2.123 \times 10^4$</td>
<td>$-1.638 \times 10^4$</td>
</tr>
<tr>
<td></td>
<td>Middle</td>
<td>$9.786 \times 10^4$</td>
<td>$2.010 \times 10^4$</td>
<td>$-9.954 \times 10^3$</td>
</tr>
<tr>
<td></td>
<td>Bottom</td>
<td>$-6.285 \times 10^4$</td>
<td>$1.898 \times 10^4$</td>
<td>$-3.533 \times 10^3$</td>
</tr>
</tbody>
</table>

The portion of the load, $N_x$, taken by each ply can be calculated by integrating the stress, $\sigma_{xx}$, through the thickness of each ply. However, because the stress varies linearly through each ply, the portion of the load, $N_x$, taken is simply the product of the stress, $\sigma_{xx}$, at the middle of each ply and the thickness of the ply:

- Portion of load $N_x$ taken by $0^\circ$ ply = $(4.464 \times 10^4)(5 \times 10^{-3}) = 223.2$ N/m
- Portion of load $N_x$ taken by $30^\circ$ ply = $(1.063 \times 10^5)(5 \times 10^{-3}) = 531.5$ N/m
- Portion of load $N_x$ taken by $-45^\circ$ ply = $(4.903 \times 10^4)(5 \times 10^{-3}) = 245.2$ N/m
Macromechanical Analysis of a Laminate

The sum total of the loads shared by each ply is 1000 N/m, \((223.2 + 531.5 + 245.2)\), which is the applied load in the \(x\)-direction, \(N_x\).

\[
\text{Percentage of load } N_x \text{ taken by } 0^\circ \text{ ply} = \frac{223.2}{1000} \times 100 = 22.32\%
\]

\[
\text{Percentage of load } N_x \text{ taken by } 30^\circ \text{ ply} = \frac{531.5}{1000} \times 100 = 53.15\%
\]

\[
\text{Percentage of load } N_x \text{ taken by } -45^\circ \text{ ply} = \frac{245.2}{1000} \times 100 = 24.52\%
\]
Macromechanical Analysis of a Laminate

Problems of Chap.4:

1. Condense (summarize) the following expanded laminate codes:
   a. $[0/45/-45/90]$
   b. $[0/45/-45/-45/45/0]$
   c. $[0/90/60/60/90/0]$
   d. $[0/45/60/45/0]$
   e. $[45/-45/45/-45/-45/45/-45/45]$

2. Expand the following laminate codes:
   - $[45/-45]_5$
   - $[45/-45_2/90]_5$
   - $[45/0]_{3s}$
   - $[45/\pm30]_2$
   - $[45/\pm30]_2$
Macromechanical Analysis of a Laminate

3. A laminate of 0.015 in. thickness under a complex load gives the following midplane strains and curvatures:

\[
\begin{bmatrix}
\varepsilon_x^0 \\
\varepsilon_y^0 \\
\gamma_{xy}^0 \\
\kappa_x \\
\kappa_y \\
\kappa_{xy}
\end{bmatrix} =
\begin{bmatrix}
2 \times 10^{-6} \\
3 \times 10^{-6} \\
4 \times 10^{-6} \\
1.2 \times 10^{-4} \\
1.5 \times 10^{-4} \\
2.6 \times 10^{-4}
\end{bmatrix} \text{ in./in.}
\]

Find the global strains at the top, middle and bottom surface of the laminate.

4. Show that, for a symmetric laminate, the coupling stiffness matrix is equal to zero.
### Macromechanical Analysis of a Laminate

5. The global stresses in a three-ply laminate are given at the top and bottom surface of each ply. Each ply is 0.005 in. thick. Find the resultant forces and moments on the laminate if it has a top cross section of 4 in. × 4 in.

<table>
<thead>
<tr>
<th>Ply no.</th>
<th>$\sigma_{xx}$ (psi)</th>
<th>$\sigma_{yy}$ (psi)</th>
<th>$\tau_{xy}$ (psi)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Top</td>
<td>Bottom</td>
<td>Top</td>
</tr>
<tr>
<td>1</td>
<td>-3.547 × 10^4</td>
<td>-2.983 × 10^3</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-9.267 × 10^3</td>
<td>1.658 × 10^4</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>7.201 × 10^3</td>
<td>2.435 × 10^4</td>
<td>3</td>
</tr>
</tbody>
</table>

6. Demonstrate that the force per unit width on a two-layered laminate with orthotropic laminae of equal thickness oriented at $+\alpha$ and $-\alpha$ to the applied force is

$$N_x = A_{11} \varepsilon_x^0 + A_{12} \varepsilon_y^0 + B_{16} \kappa_{xy}$$

What are $A_{11}$, $A_{12}$ & $B_{16}$ in terms of the transformed reduced stiffnesses $(\overline{Q}_{ij})_{+\alpha}$ of a lamina and the lamina thickness, $t$?
The End